On the Rational Polytopes with Chvátal Rank 1

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December 14, 2016

Abstract

In this paper, we study the following problem: given a polytope P with Chvátal rank 1, does P contain an integer point? Boyd and Pulleyblank observed that this problem is in the complexity class NP ∩ co-NP, an indication that it is probably not NP-complete. We solve this problem in polynomial time for polytopes arising from the satisfiability problem of a given formula with at least three literals per clause, for simplices whose integer hull can be obtained by adding at most a constant number of Chvátal inequalities, and for rounded polytopes. We prove that any closed convex set whose Chvátal closure is empty has an integer width of at most n, and we give an example showing that this bound is tight within an additive constant of 1. The promise that a polytope has Chvátal rank 1 seems hard to verify though. We prove that deciding emptiness of the Chvátal closure of a given rational polytope P is NP-complete, even when P is contained in the unit hypercube or is a rational simplex, and even when P does not contain any integer point. This has two implications: (i) It is NP-hard to decide whether a given rational polytope P has Chvátal rank 1, even when P is contained in the unit cube or is a rational simplex; (ii) The optimization and separation problems over the Chvátal closure of a given rational polytope contained in the unit hypercube or of a given rational simplex are NP-hard. These results improve earlier complexity results of Cornuéjols and Li and Eisenbrand. Finally, we prove that, for any positive integer k, it is NP-hard to decide whether adding at most k Chvátal inequalities is sufficient to describe the integer hull of a given rational polytope.

1 Introduction

Let $P \subset \mathbb{R}^n$ be a rational polyhedron, and let P_I denote its integer hull, namely $P_I := \text{conv}(P \cap \mathbb{Z}^n)$, the convex hull of all the integer points in P. If an inequality $cx \le d$ is valid for P for some $c \in \mathbb{Z}^n$, then $cx \le \lfloor d \rfloor$ is a valid inequality for P_I. Chvátal [6] introduced the following beautiful notion of closure.

$$P' := \bigcap_{c \in \mathbb{Z}^n} \{x \in \mathbb{R}^n : \; cx \le \lfloor \max\{cx : \; x \in P\} \rfloor\}$$

It follows from the definition that $P_I \subseteq P' \subseteq P$. The set P' is called the *Chvátal closure* of P. Although P' is defined as the intersection of infinitely many half-spaces, P' turns out to be a rational polytope when P is a rational polytope [6]. Schrijver [30] later extended this result to rational polyhedra. Bockmayr and Eisenbrand [3] proved that the number of inequalities needed to describe P' is polynomially bounded in fixed dimension.

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The problem of deciding whether a rational polytope contains an integer point is NP-complete [4]. The main motivation of this paper is the following question: for a rational polytope with Chvátal rank 1, can the integer feasibility problem be solved in polynomial time? Boyd and Pulleyblank [5] observed that this problem belongs to the complexity class NP $\cap$ co-NP. This is an indication that this problem is not NP-complete (unless NP = co-NP). In Section 2 we give polynomial algorithms for several special cases of this problem.

Although researchers have investigated the Chvátal closure of polytopes associated with several combinatorial optimization problems, such as the fractional stable set [18] and matching polytopes [15], a constructive characterization of the Chvátal closure of an arbitrary polytope is not well studied. Giles and Pulleyblank [19] showed that, for every rational polyhedron, there exists a totally dual integral (TDI) system $Ax \leq b$ where A is an integral matrix. The Chvátal closure can be obtained by just rounding the right hand side of the integral TDI system [30]. However, finding an integral TDI system is not an easy task, so this approach is not efficient in characterizing the Chvátal closure.

In Section 3, we prove that any closed convex set in $\mathbb{R}^n$ whose Chvátal closure is empty has an integer width of at most n, and we give an example showing that this bound is tight within an additive constant of 1. We remark that the upper bound on the width implies the existence of a deterministic $2^{O(n)}n^n$ Lenstra-type algorithm for the integer feasibility problem over a given rational polyhedron with Chvátal rank 1. On the other hand, the lower bound indicates that we cannot improve this time complexity if we use a Lenstra-type procedure.

Letchford, Pokutta and Schulz [26] asked whether the separation problem for the Chvátal closure remains NP-hard for rational polytopes contained in the unit hypercube. We prove this in Section 4.1. In fact, we prove that deciding emptiness of the Chvátal closure of a rational polytope P contained in the unit hypercube $[0, 1]^n$ is NP-complete, even when P contains no integer point. This result implies that optimizing over the Chvátal closure of a rational polytope contained in the unit hypercube is NP-hard by the equivalence between optimization and separation [20].

We also study the same computational complexity question for a full-dimensional rational simplex in $\mathbb{R}^n$, which is the convex hull of $n + 1$ affinely independent points. Sebö [32] observed that integer programming over rational simplices is NP-complete. In Section 4.2, we show that the problem of deciding emptiness of the Chvátal closure of a given rational simplex is NP-complete. Strong NP-completeness is still an open problem.

In Section 5, we investigate the problem of deciding whether a given rational polytope has Chvátal rank 1. Cornuéjols and Li [9] showed that the problem is NP-hard. We prove that the problem remains NP-hard even when the polytope is contained in the unit hypercube or is a simplex. This result answers an open question raised in [9].

In Section 6, we study the problem of deciding whether we can obtain the integer hull of a polyhedron P by adding at most k Chvátal inequalities to the description of P for a given nonnegative integer k. In the case of $k = 0$, we know that it is strongly co-NP-complete to decide whether a given polyhedron is integral [14, 28]. In the case of $k \geq 1$, we prove that deciding whether the integer hull of a polyhedron P can be obtained by adding at most k Chvátal inequalities is NP-hard. Previously, Mahajan and Ralphs [27] showed that deciding whether a given rational polytope is contained in a strip, where a strip is a set $\{x \in \mathbb{R}^n : \pi_0 < x < \pi_0 + 1\}$ for some $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$, is NP-complete. We borrow some of their ideas to prove our NP-hardness result.

2 Polynomially solvable cases and open problems

In this section, we consider the problem of deciding whether a rational polyhedron P contains an integer point under the promise that P has Chvátal rank 1. This promise on the input P very likely modifies the computational complexity of the integer feasibility problem.

Theorem 1 (Boyd and Pulleyblank [5])**.** *Let $P \subset \mathbb{R}^n$ be a rational polyhedron with Chvátal rank 1. The problem of deciding whether P contains an integer point is in the complexity class NP$\cap$ co-NP.*

The problems in NP $\cap$ co-NP are probably not NP-complete (since otherwise NP = co-NP), so we have the following question:

Open Question 1. Let $P \subset \mathbb{R}^n$ be a rational polyhedron with Chvátal rank 1. Can we decide in polynomial time whether P contains an integer point?

It does not seem straightforward to use the Chvátal rank 1 condition, because it is NP-hard to decide whether a rational polytope has Chvátal rank 1 (Cornuéjols and Li [9]). We also note that the Chvátal rank of a polyhedron is not directly related to its geometry. In particular, the Chvátal rank is not invariant under translation. The following example shows that the Chvátal rank of a polyhedron may vary significantly under translation.

Example 2. Let $Q_1 := \{x \in [0,1]^n : \sum_{j=1}^n v_j(1-x_j) + (1-v_j)x_j \geq \frac{1}{2}\ \forall v \in \{0,1\}^n\}$. Notice that Q_1 contains no integer point. Chvátal, Cook and Hartmann [7] proved that the Chvátal rank of Q_1 is exactly n. Now, let us translate Q_1 so that its center point is at the origin. We denote by Q_2 the resulting polytope. Since $Q_2 \subseteq [-\frac{1}{2}, \frac{1}{2}]^n$, the only integer point contained in Q_2 is the origin. We can obtain both $x_i \geq 0$ and $x_i \leq 0$ as Chvátal inequalities for Q_2 for all $i = 1, \cdots, n$. Hence, the Chvátal rank of Q_2 is exactly 1.

The difficulty in understanding the Chvátal rank 1 condition is an indication that the above open question might not be easy to answer in general. Next, we present several special cases of Open Question 1 where the answer is positive.

2.1 Satisfiability problem with Chvátal rank 1

The satisfiability problem is NP-complete (see [17]), and it can be transformed to a binary integer program. Given a formula in conjunctive normal form with m clauses that consist of literals $x_1, \cdots, x_n$ and their negations, we can turn this formula into a polytope in the unit hypercube. For example, the formula $(x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_3 \vee x_4)$ can be turned into the polytope $\{x \in [0,1]^4 : x_1 + (1-x_2) + x_3 \geq 1,\ (1-x_3) + x_4 \geq 1\}$. The linear description of such a polytope consists of *generalized set covering inequalities* and the bounds $0 \leq x \leq 1$. We call it a *SAT polytope*. Notice that the satisfiability problem is equivalent to the integer feasibility problem over a *SAT polytope*.

Open Question 2. Given a SAT polytope P whose Chvátal rank is 1, can we decide in polynomial time whether P contains an integer point?

The k-satisfiability problem is a variant of the satisfiability problem where each clause in a given formula has at most k literals. It remains NP-complete for $k \geq 3$ (see [17]). There is a simple polynomial algorithm for the 2-satisfiability problem. We consider Open Question 2 for the case where each clause contains at least 3 literals.

Theorem 3. *Let P be a SAT polytope such that each generalized set covering inequality has at least 3 variables. If P has Chvátal rank 1, then P always contains an integer point.*

Proof. Observe that setting any variable to 0 or 1, and all other $n-1$ variables to 1/2 satisfies all the constraints of P (because every generalized set covering inequality involves at least three variables). In other words, the middle point of each facet of the hypercube $[0,1]^n$ is contained in P. A theorem of Chvátal, Cook and Hartmann [7] implies that the Chvátal closure of P contains the middle point $(\frac{1}{2},\cdots,\frac{1}{2})$ of the hypercube, so the Chvátal closure of P is always nonempty. Because the Chvátal rank of P is 1, P contains an integer point. □

The gap between Open Question 2 and Theorem 3 is on the SAT polytopes defined by the boolean formulas involving both clauses with 2 literals and clauses with at least 3 literals. SAT polytopes whose generalized set covering inequalities have at most 2 variables are well understood by Gerards and Schrijver [18]. They gave a characterization of the Chvátal closure of a SAT polytope in such a case, and they provided a polynomial algorithm to separate over it. Furthermore, we remark that the Chvátal rank of a SAT polytope in that case is always 1 whenever it contains no integer point. However, the Chvátal closure of a SAT polytope which includes both generalized set covering inequalities with 2 variables and 3 variables has not been studied.

2.2 When a few Chvátal cuts are sufficient

In this section, we consider another special case of Open Question 1, where we assume that the integer hull of a given polyhedron can be obtained by adding a constant number of Chvátal inequalities.

Open Question 3. Let $P \subset \mathbb{R}^n$ be a rational polyhedron such that P_I can be obtained by adding at most k Chvátal inequalities to the description of P, for some constant k. Can we solve the integer feasibility problem of P in polynomial time?

In fact, Open Question 3 is open even when $k = 1$. The difficulty might be due to the result that it is NP-complete to decide whether P_I can be obtained from a polytope P by adding one Chvátal inequality, which will be proved in Section 6.

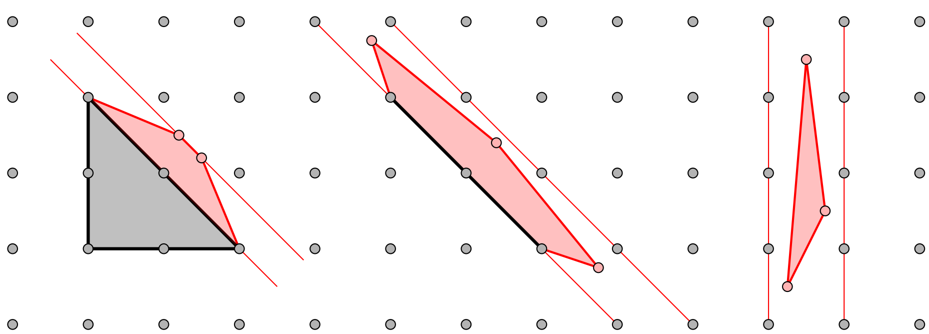

Figure 1: When one Chvátal inequality is sufficient in $\mathbb{R}^2$

Remark 4. *Let $P = \{x \in \mathbb{R}^n : Ax \le b\}$ be a rational polytope such that addition of one Chvátal inequality to P gives P_I. Then there exists an algorithm for the integer feasibility problem over P*

which runs in time bounded by $m^n n^3 poly(L)$ where m and L denote the number of constraints in P and the encoding size of P, respectively.

This is easy to show because if v is a fractional vertex of P, then v should be removed by the Chvátal inequality. Therefore P contains an integer point if and only if an extreme point of P is integral. In this case, there is a trivial algorithm for the integer feasibility problem: check every vertex of P and conclude that $P_I \neq \emptyset$ if there exists an integral vertex or $P_I = \emptyset$ otherwise. Since there are $O(m^n)$ extreme points of P and the time complexity of the Gaussian elimination method is bounded by $n^3\text{poly}(L)$, the algorithm runs in time bounded by $m^n n^3\text{poly}(L)$. In fact, a byproduct of Theorem 18 shows the existence of a $2^{O(n)}\text{poly}(L)$ time algorithm for the case of $k = 1$.

In the following, we are able to prove that Open Question 3 is true in a special case.

Theorem 5. *Let k be a positive constant in $\mathbb{Z}$. Given a rational simplex $P \subset \mathbb{R}^n$ such that P_I can be obtained from P by adding at most k Chvátal inequalities, and a vector $w \in \mathbb{Q}^n$, there is a polynomial-time algorithm to optimize the linear function wx over P_I.*

Proof. Suppose that the dimension of P is ℓ for some $\ell \leq n$. Let P be written as $\{x \in \mathbb{R}^n : \ Ax = b,\ Cx \leq d\}$, where $Ax = b$ is the system of all implicit equalities. We denote by $Ex \leq f$ the set of k Chvátal inequalities such that $P_I = \{x \in \mathbb{R}^n : \ Ax = b,\ Cx \leq d,\ Ex \leq f\}$. So the inequalities $Ex \leq f + \epsilon\mathbf{1}$ are valid for P, where $\epsilon \in (0, 1)$ and $\mathbf{1}$ denotes the all-1 vector, and $P \subseteq S$, where $S := \{x \in \mathbb{R}^n : \ Ax = b,\ Ex \leq f + \epsilon\mathbf{1}\}$.

We can find in polynomial time a unimodular matrix U such that $AU = (D, 0)$ is a Hermite normal form of A. If $D^{-1}b$ is not integral, we can just conclude that P does not contain an integer point. Thus, we may assume that $D^{-1}b$ is integral. Let U_1 and U_2 denote the two submatrices of U which consist of the first $n - \ell$ columns of U and the last ℓ columns of U, respectively. Let $u : \mathbb{R}^n \to \mathbb{R}^n$ be a unimodular transformation defined by $u(x) = U^{-1}x$. Consider the images of P, P_I, and S under u:

$$\begin{aligned}
u(P) &= \left\{(y_1, y_2) \in \mathbb{R}^{(n-\ell)+\ell} : \ y_1 = D^{-1}b,\ CU_2y_2 \leq d - CU_1D^{-1}b\right\},\\
u(P_I) &= \left\{(y_1, y_2) \in \mathbb{R}^{(n-\ell)+\ell} : \ y_1 = D^{-1}b,\ CU_2y_2 \leq d - CU_1D^{-1}b,\ EU_2y_2 \leq f - EU_1D^{-1}b\right\},\\
u(S) &= \left\{(y_1, y_2) \in \mathbb{R}^{(n-\ell)+\ell} : \ y_1 = D^{-1}b,\ EU_2y_2 \leq f + \epsilon\mathbf{1} - EU_1D^{-1}b\right\}.
\end{aligned}$$

Note that $u(P)$ is an ℓ-dimensional simplex in $\mathbb{R}^n$, so $Q := \left\{y_2 \in \mathbb{R}^\ell : \ CU_2y_2 \leq d - CU_1D^{-1}b\right\}$ is an ℓ-dimensional simplex in $\mathbb{R}^\ell$. Furthermore, $u(P_I)$ is integral. Since $D^{-1}b$ is integral, $\{y_2 \in \mathbb{R}^\ell : CU_2y_2 \leq d - CU_1D^{-1}b,\ EU_2y_2 \leq f - EU_1D^{-1}b\}$ is integral and thus $Q \cap \{y_2 \in \mathbb{R}^\ell : EU_2y_2 \leq f - EU_1D^{-1}b\}$ is integral. We claim that the inequalities in the system $EU_2y_2 \leq f - EU_1D^{-1}b$ are Chvátal inequalities for Q. In fact, we know that $u(P) \subseteq u(S)$, so $Q \subseteq \left\{y_2 \in \mathbb{R}^\ell : \ EU_2y_2 \leq f + \epsilon\mathbf{1} - EU_1D^{-1}b\right\}$. That means the inequalities in $EU_2y_2 \leq f + \epsilon\mathbf{1} - EU_1D^{-1}b$ are all valid for Q, so those in the system $EU_2y_2 \leq f - EU_1D^{-1}b$ are Chvátal inequalities for Q. Now, we have obtained a full-dimensional rational simplex Q in $\mathbb{R}^\ell$ such that its integer hull Q_I can be described by adding at most k Chvátal inequalities.

Q has $\ell+1$ constraints in its description, so Q_I can be described by $\ell+k+1$ linear inequalities. When $\ell \leq k$, the dimension of Q is fixed and we can optimize a linear function over Q_I in polynomial time by Lenstra's algorithm [25]. Thus, we may assume that $\ell > k$. Suppose that Q_I is not empty. Then let $z \in \mathbb{Z}^\ell$ be an extreme point of Q_I. So there are ℓ linearly independent inequalities in the description of Q_I that are active at z. This means that at least $\ell - k$ inequalities among the $\ell + 1$ inequalities in the original description of Q are active at z. Thus, z belongs to a k-dimensional face of Q. Hence, if no k-dimensional face of Q contains an integer point, Q_I is empty. Since k is fixed, we can optimize a linear function over the integer hull of each k-dimensional face of Q. Notice that there are exactly $\binom{\ell+1}{k+1}$ k-dimensional faces of Q. Therefore, we can optimize a linear function over Q_I in polynomial time. □

Theorem 5 suggests the following open question.

Open Question 4. Given a rational simplex $P \subset \mathbb{R}^n$ with Chvátal rank 1, can we decide in polynomial time whether P contains an integer point?

2.3 Rounded polytopes

A full-dimensional polytope $P \subset \mathbb{R}^n$ is *rounded with factor* $\ell > 1$ if $B_2^n(a,r) \subseteq P \subseteq B_2^n(a,\ell r)$, where $B_2^n(p,q)$ denotes an Euclidean ball $\{x \in \mathbb{R}^n : \|x-p\|_2 \le q\}$ which is centered at p with radius q. In this section, we prove the following theorem.

Theorem 6. *Let $\ell > 1$ be a constant, and let $P \subset \mathbb{R}^n$ be a rounded polytope with factor ℓ. If P_I can be obtained by adding one Chvátal inequality to the description of P, then we can decide in polynomial time whether P contains an integer point.*

To prove this theorem, we use the notion of *integer width* of a convex set.

Definition 7. *Let $K \subset \mathbb{R}^n$ be a convex set and $d \in \mathbb{Z}^n$. The integer width of K along d is*

$$w(K,d) := \lfloor \sup\{dx : x \in K\} \rfloor - \lceil \inf\{dx : x \in K\} \rceil + 1.$$

The integer width of K, $w(K,\mathbb{Z}^n)$, is the infimum of the values $w(K,d)$ over all $d \in \mathbb{Z}^n \setminus \{0\}$.

$$w(K,\mathbb{Z}^n) := \inf_{d \in \mathbb{Z}^n \setminus \{0\}} w(K,d).$$

Lemma 8. *Let $P \subset \mathbb{R}^n$ be a rounded polytope with factor $\ell > 1$. If there exists an integral vector c such that $w(P,c) \le k$ for some nonnegative integer k, then either $\|c\|_2 \le (k+1)\ell$ or $w(P,e^i) \le 1$ for all $i = 1, \cdots, n$.*

Proof. Since P is rounded with factor ℓ, P satisfies $B_2^n(a,r) \subseteq P \subseteq B_2^n(a,\ell r)$ for some $r > 0$ and $a \in \mathbb{R}^n$. Assume that $\|c\|_2 > (k+1)\ell$. Since $w(P,c) \le k$, there exists $c_0 \in \mathbb{Z}$ such that $c_0 < cx < c_0 + k + 1$ for all $x \in P$. Since $B_2^n(a,r) \subseteq P \subseteq \{x \in \mathbb{R}^n : c_0 < cx < c_0 + k + 1\}$ and the distance between the two hyperplanes $\{x \in \mathbb{R}^n : cx = c_0\}$ and $\{x \in \mathbb{R}^n : cx = c_0 + k + 1\}$ is exactly $(k+1)/\|c\|_2$, $2r$ is at most $(k+1)/\|c\|_2$. Hence, we get $r \le \frac{k+1}{2\|c\|_2} < \frac{1}{2\ell}$, *i.e.*, $2\ell r < 1$. Suppose that there is some i such that $w(P,e^i) \ge 2$. Then there are two points $u, v \in P$ such that $u_i \le b$ and $v_i \ge b+1$ for some $b \in \mathbb{Z}$. So $\|u-v\|_2 \ge |u_i - v_i| \ge 1$. Since $B_2^n(a,\ell r)$ contains P, the distance between any two points in P is at most $2\ell r$ and thus we get $2\ell r \ge 1$. However, this contradicts the above result that $2\ell r < 1$. Therefore, $w(P,e^i) \le 1$ for all $i = 1, \cdots, n$. □

Proof of Theorem 6. Consider the following algorithm:

(1) For each $c \in \mathbb{Z}^n$ with $\|c\|_2 \le \ell$, compute $w(P,c)$. If $w(P,c) = 0$ for some c with $\|c\|_2 \le \ell$, then $P_I = \emptyset$. Otherwise, go to step (2).

(2) Compute $w(P, e^i)$ for $i \in [n]$. If there exists $i \in [n]$ such that $w(P, e^i) \geq 2$, then $P_I \neq \emptyset$. If there exists $i \in [n]$ such that $w(P, e^i) = 0$, then $P_I = \emptyset$. Otherwise, go to step (3).

(3) Let $z_j := \lfloor \max\{x_j : x \in P\} \rfloor$ for $j = 1, \cdots, n$. If $(z_1, \cdots, z_n) \in P$, then $P_I \neq \emptyset$. Otherwise, $P_I = \emptyset$.

Step (1) can be done in polynomial time, because there are at most $\binom{n}{\ell} 2^\ell \binom{2\ell-1}{\ell}$ integral vectors c with $\|c\|_2 \leq \ell$.

By assumption, there exists a Chvátal inequality $dx \leq d_0$ such that $\{x \in P : dx \leq d_0\} = P_I$. Note that P_I is empty if and only if $w(P, d) = 0$. Going into Step (2), we have $w(P, c) \geq 1$ for all $c \in \mathbb{Z}^n$ with $\|c\|_2 \leq \ell$. If $w(P, e^i) \geq 2$ for some $i \in [n]$, then $w(P, d) \geq 1$ by Lemma 8 (when $k = 0$) and thus $P_I \neq \emptyset$. If $w(P, e^i) = 0$ for some $i \in [n]$, then P_I is trivially empty.

Therefore, going into Step (3), we have $w(P, e^i) = 1$ for all $i = 1, \cdots, n$, and P can have at most one integer point. This point z can be computed by solving n linear programs, therefore, in polynomial time. □

Remark 9. *The integer feasibility problem can be solved in quasi-polynomial time, when $P \subseteq \mathbb{R}^n$ is a rounded polytope with factor $O(\log n)$ and its integer hull can be obtained by adding one Chvátal inequality. In fact, the algorithm given in the proof of Theorem 6 solves $n^{O(\log n)}$ linear programs in this case.*

3 Flatness theorem for closed convex sets with empty Chvátal closure

Recall the definition of integer width of a convex set K given in Definition 7. When K is unbounded or has a large volume, there exists a direction $d \in \mathbb{Z}^n \setminus \{0\}$ where $w(K, d)$ is large. On the other hand, it is possible that there is a direction $d \in \mathbb{Z}^n \setminus \{0\}$ such that $w(K, d)$ is relatively small if K does not contain any integer point. In fact, the famous flatness theorem by Khinchine [24] states that $w(K, \mathbb{Z}^n)$ for any compact convex set K containing no integer point is bounded by $f(n)$, which is only a function of the dimension n. Khinchine's flatness theorem [24] shows that $f(n) = (n+1)!$. A crucial component of Lenstra's algorithm [25] is to find a flat direction $d \in \mathbb{Z}^n \setminus \{0\}$ of a polyhedron $P \subset \mathbb{R}^n$ containing no integer point. Lenstra [25] gave a polynomial algorithm to find a direction $d \in \mathbb{Z}^n \setminus \{0\}$ such that $w(K, d) \leq 2^{O(n^2)}$ for a given lattice-free compact convex set K. Then, it generates $2^{O(n^2)}$ subproblems in $\mathbb{R}^{n-1}$ by intersecting K with $2^{O(n^2)}$ parallel hyperplanes orthogonal to d. Hence, the algorithm works recursively, and the number of total steps required is $2^{O(n^3)}$.

Over the last few decades there have been huge improvements on the upper bound $f(n)$ (see [1, 2, 23, 24, 29]). The current best known asymptotic upper bound is $f(n) = O(n^{4/3}\text{polylog}(n))$ given by Banaszczyk, Litvak, Pajor, and Szarek [2] and Rudelson [29]. It has been even conjectured that $f(n) = O(n)$. However, the existence of a polynomial algorithm to find a direction $d \in \mathbb{Z}^n$ such that $w(K, d) = O(n^{4/3}\text{polylog}(n))$ for a convex set K containing no integer point is not known. Dadush, Peikert and Vempala [12] and Dadush and Vempala [13] developed an algorithm to find all vectors $d \in \mathbb{Z}^n \setminus \{0\}$ such that $w(K, d) = w(K, \mathbb{Z}^n)$ in $2^{O(n)}\text{poly}(L)$ time and space.

In this section, we prove that $f(n)$ is indeed n if K is a closed convex set with empty Chvátal closure. The Chvátal closure of a closed convex set is defined similarly to that of a polyhedron.

Definition 10. *Let $K \subset \mathbb{R}^n$ be a closed convex set. The Chvátal closure of K is defined as*

$$K' := \bigcap_{c \in \mathbb{Z}^n} \{x \in \mathbb{R}^n : \ cx \leq \lfloor \sup\{cx : \ x \in K\} \rfloor\}.$$

If K is a compact convex set, its Chvátal closure is a rational polytope [11]. The following is the main result of this section.

3.1 Flatness result

Theorem 11. *The integer width of any closed convex set in $\mathbb{R}^n$ whose Chvátal closure is empty is at most n.*

The upper bound given by Theorem 11 turns out to be quite tight as shown in the following proposition.

Proposition 12. *There exists a polytope in $\mathbb{R}^n$ such that its Chvátal closure is empty and its integer width is $n-1$.*

Proof. Let $P_n : \{x \in \mathbb{R}^n : \ x \geq \frac{1}{n+1}\mathbf{1}, \ \sum_{i=1}^n x_i \leq n - 1 + \frac{n}{n+1}\}$. Then P_n is the convex hull of $(n-1)e^i + \frac{1}{n+1}\mathbf{1}$ for $i = 1, \cdots, n$ and $\frac{1}{n+1}\mathbf{1}$. Since $x_i \geq 1$ is valid for P'_n for each i, $\sum_{i=1}^n x_i \geq n$ is valid for P'_n. Together with $\sum_{i=1}^n x_i \leq n - 1 + \frac{n}{n+1}$, this shows the emptiness of P'_n.

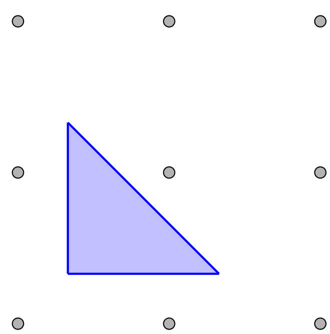

Figure 2: P_2 in $\mathbb{R}^2$

Now we show that the integer width of P_n is $n-1$. Let $d \in \mathbb{Z}^n \setminus \{0\}$. Since the integer width of P_n along d is the same as that along $-d$, we may assume $\sum_{i=1}^n d_i \geq 0$. Notice that $\max\{dx : \ x \in P_n\} = (n-1)\max\{d_1, \cdots, d_n\} + \frac{1}{n+1}\sum_{i=1}^n d_i$ and $\min\{dx : \ x \in P_n\} = (n-1)\min\{0, d_1, \cdots, d_n\} + \frac{1}{n+1}\sum_{i=1}^n d_i$. Then the integer width of P_n along d is either $(n-1)(\max\{d_1, \cdots, d_n\} - \min\{0, d_1, \cdots, d_n\})$ or $(n-1)(\max\{d_1, \cdots, d_n\} - \min\{0, d_1, \cdots, d_n\}) + 1$. Clearly, $\max\{d_1, \cdots, d_n\} - \min\{0, d_1, \cdots, d_n\}$ is at least 1. Hence, the integer width of P_n along d is at least $n-1$. It is easy to show that the integer width of P_n along $\mathbf{1}$ is exactly $n-1$. ☐

As a corollary of Theorem 11, we obtain an improved flatness theorem for polyhedra with Chvátal rank 1. The result will be used in developing an algorithm for solving the integer feasibility problem over the rational polyhedra with Chvátal rank 1 in the later part of this section.

Corollary 13. *Let $P \subset \mathbb{R}^n$ be a polyhedron with Chvátal rank 1. Then, either P contains an integer point or the integer width of P is at most n.*

3.2 Proof of Theorem 11

To prove Theorem 11, we introduce the concept of a *simplicial cylinder*. Let $P \subset \mathbb{R}^n$ be a full-dimensional rational polyhedron. We denote by L and $L^\perp$ the lineality space of P and its complement, respectively. We say that P is a *simplicial cylinder* if $P \cap L^\perp$ is a simplex.

Remark 14. *A simplicial cylinder $P \subset \mathbb{R}^n$ whose lineality space L has dimension $n - \ell$ can be described by $\ell + 1$ linear inequalities.*

Given a full-dimensional rational polyhedron $P \subset \mathbb{R}^n$ and its linear description $Ax \leq b$, where each entry in $[A; b]$ is an integer and each row of A has relatively prime integers, we call P a *thin simplicial cylinder* if it is a simplicial cylinder and $Ax \leq b - \mathbf{1}$, where $\mathbf{1}$ denotes the all 1 vector, is an infeasible system.

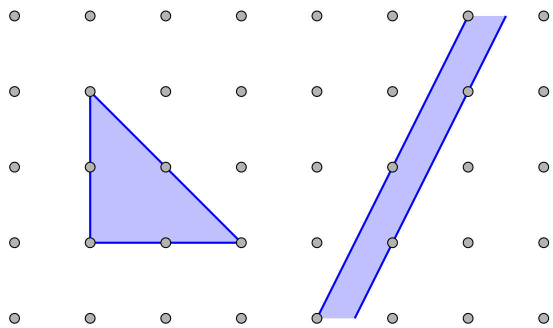

Figure 3: Thin simplicial cylinders in $\mathbb{R}^2$

Remark 15. *The Chvátal closure of any compact convex set contained in the interior of a thin simplicial cylinder is empty.*

Theorem 16. *Any closed convex set whose Chvátal closure is empty is contained in the interior of a thin simplicial cylinder.*

Proof. Let $K \subset \mathbb{R}^n$ be a closed convex set whose Chvátal closure is empty. Helly's theorem implies that there be $\ell + 1$ Chvátal inequalities of K for some $\ell \leq n$ such that the intersection of the corresponding linear half-spaces is empty. In other words, there exists a system $Ax \leq b - \epsilon\mathbf{1}$ of $\ell + 1$ linear inequalities valid for K

$$\begin{array}{rcl} a^1 x & \leq & b_1 - \epsilon \\ a^2 x & \leq & b_2 - \epsilon \\ & \vdots & \\ a^\ell x & \leq & b_\ell - \epsilon \\ a^{\ell+1} x & \leq & b_{\ell+1} - \epsilon \end{array}$$

where $0 < \epsilon < 1$, $a^1, \cdots, a^{\ell+1} \in \mathbb{Z}^n$, and $b_1, \cdots, b_{\ell+1} \in \mathbb{Z}$ for some $\ell \leq n$ such that the system $Ax \leq b - \mathbf{1}$ obtained by rounding down the right hand side values of the above is infeasible. Without loss of generality, we may assume that each a^i has relatively prime entries. We may also assume that the system is minimal in a sense that $\{x \in \mathbb{R}^n : a^i x \le b_i - 1 \text{ for } i \in I\}$ is not empty for any proper subset I of $[\ell + 1]$.

Consider the polyhedron $P := \{x \in \mathbb{R}^n : Ax \le b\}$. We claim that its recession cone $C := \{x : Ax \le 0\}$ has empty interior. Otherwise, the polyhedron P contains points in the form of $x + kr$ for some $x \in P$ and some ray vector $r \in \mathbb{R}^n$ in the interior of C, where $k \in \mathbb{R}_+$. For k large enough, the points of the form are also in the polyhedron $Q := \{x \in \mathbb{R}^n : Ax \le b - 1\}$, which is empty, a contradiction. Therefore, the linear space $C - C$ has dimension strictly less than n.

By the Minkowski-Weyl theorem, we can write the polyhedron P as $P = S + C$ where S is a polytope. Consider the cylinder $R := S + C - C$. Consider all the inequalities $a^i x \le b_i$, $i = 1, \cdots, t$, in the description of P that are valid for R. Then every ray vector $r \in C$ satisfies $a^i r < 0$ for $i = t + 1, \cdots, \ell + 1$. We claim that the linear system $a^i x \le b_i - 1$, $i = 1, \cdots, t$, is infeasible. If $a^i x \le b_i - 1$, $i = 1, \cdots, t$, were feasible, then, by the same argument as given in the above paragraph, Q would be nonempty, a contradiction. Thus $a^i x \le b_i - 1$, $i = 1, \cdots, t$, is infeasible. By the minimality of the system, this implies $t = \ell + 1$, and therefore S is a simplex of dimension ℓ. □

Proof of Theorem 11. Let $K \subset \mathbb{R}^n$ be a closed convex set whose Chvátal closure is empty. By Theorem 16, there exists a thin simplicial cylinder $P := \{x \in \mathbb{R}^n : Ax \le b\}$ containing K in its interior. Hence, there exists a small $\epsilon \in (0, 1)$ such that $K \subseteq \{x \in \mathbb{R}^n : Ax \le b - \epsilon \mathbf{1}\}$. We assume that the number of rows in A is $\ell + 1$ for some $\ell \le n$. We denote by $a^1, \cdots, a^{\ell+1}$ the rows of A. Notice that $P \cap L^\perp$ is an ℓ-dimensional simplex, where L and $L^\perp$ denote the lineality space of P and its orthogonal complement, respectively.

We will show that the integer width of P along some a^i is at most $\ell + 1$. Then the integer width of K is at most ℓ, because the hyperplane defined by $a^i x = b_i$ does not go through K. Suppose that the integer width of P along each a^i is at least $\ell + 2$ for the sake of contradiction. Then, the width of P along each a^i is at least $\ell + 1$. Using an affine transformation, we can transform P to $\{x \in \mathbb{R}^n : x_1, \cdots, x_\ell \ge 0, \sum_{i=1}^{\ell} x_i \le 1\}$. Under the same affine transformation, we know that $\{x \in \mathbb{R}^n : Ax \le b - \mathbf{1}\}$ is transformed to $\{x \in \mathbb{R}^n : x_i \ge \epsilon_i \ \forall\ i \in [\ell], \sum_{i=1}^{\ell} x_i \le 1 - \epsilon\}$ for some $\epsilon_i \le \frac{1}{l+1}$ for $i \in [\ell]$ and $\epsilon \le \frac{1}{l+1}$. Notice that $\frac{1}{l+1}\mathbf{1}$ is contained in $\{x \in \mathbb{R}^n : x_i \ge \epsilon_i \ \forall\ i \in [\ell], \sum_{i=1}^{\ell} x_i \le 1 - \epsilon\}$. However, $\{x \in \mathbb{R}^n : Ax \le b - \mathbf{1}\}$ is empty by the assumption that P is a thin simplicial cylinder, and it cannot be transformed to a nonempty set under any affine transformation. With this contradiction, we have proved that the integer width of K is at most $\ell \le n$. □

In the following, we give a simpler proof of the same flatness theorem for bounded closed convex sets.

Proposition 17. *Let $K \subset \mathbb{R}^n$ be a compact convex set whose Chvátal closure is empty. If $E(C, a) \subset K \subset E(\frac{1}{\ell}C, a)$, then the integer width of K is at most $\lceil \ell \rceil$.*

Proof. Since the Chvátal closure of K is empty, the center point a of the inner ellipsoid should be cut off by a Chvátal inequality of K. In other words, there exists $(d, d_0) \in \mathbb{Z}^{n+1}$ such that $\max\{dx : x \in K\} < d_0$ and $da > d_0 - 1$. Let $x^* := \arg\max\{dx : x \in E(C, a)\}$. Then $dx^* < d_0$. Geometrically, the minimum of dx over $E(\frac{1}{\ell}C, a)$ should be obtained at $a - \ell(x^* - a)$. Then $\min\{dx : x \in K\} \ge \min\{dx : x \in E(\frac{1}{\ell}C, a)\} = d(a - \ell(x^* - a)) > d_0 - \ell - 1$. Therefore, the integer width of K along d is at most $\lceil \ell \rceil$. □

The Löwner-John ellipsoid can be used to show that there always exists such a pair of ellipsoids with $\ell \leq n$. Hence, the proposition also implies that the integer width of every compact convex set in $\mathbb{R}^n$ whose Chvátal closure is empty is at most n.

3.3 A Lenstra-type algorithm

Recently Hildebrand and Köppe [21], Dadush, Peikert, and Vempala (see [10, 12, 13]) improved Lenstra-type algorithms for integer programming. Their algorithms are similar to Lenstra's algorithm in spirit in that a main step consists in finding a flat direction of a lattice-free convex body. In particular, Dadush, Peikert, and Vempala (see [10, 12, 13]) used a $2^{O(n)}\text{poly}(L)$ time algorithm to find a flattest direction for a convex body containing no integer point. Together with Rudelson's flatness theorem [29], they proved that the time complexity of the algorithm is bounded by $2^{O(n)}\left(n^{4/3}\text{polylog}(n)\right)^n \text{poly}(L)$. Theorem 11 implies that there exists a $2^{O(n)}n^n\text{poly}(L)$ time Lenstra-type algorithm for the integer feasibility problem over Chvátal rank 1 rational polyhedra. On the other hand, Proposition 12 indicates that we cannot improve this time complexity if we use a Lenstra-type procedure. Note that this does not improve the current best algorithm for integer programming. Dadush [10] provided a $2^{O(n)}n^n\text{poly}(L)$ time Kannan-type algorithm for integer programming over general convex compact sets in his doctoral dissertation, and we remark that it is the fastest algorithm for integer programming. Instead of finding one flat direction at a time, his algorithm finds many flat directions at each step, thereby reducing the number of recursive steps from $\left(n^{4/3}\text{polylog}(n)\right)^n$ to $(3n)^n$.

Based on Theorem 11 and Theorem 16, we can state the following theorem. We do not describe our algorithm in this paper, because it is similar to the earlier work done by Dadush, Peikert, and Vempala (see [10, 12, 13]).

Theorem 18. *Let $P \subset \mathbb{R}^n$ be rational polyhedron with Chvátal rank 1. Assume that if P contains no integer point, then P is contained in the interior of a thin simplicial cylinder defined by $\ell + 1$ inequalities. Then, there exists a $2^{O(n)}\ell^n\text{poly}(L)$ time Lenstra-type algorithm that decides whether P contains an integer point, where L is the encoding size of P.*

Since any rational polyhedron with empty Chvátal closure in $\mathbb{R}^n$ is always contained in the interior of a thin simplicial cylinder which is defined by at most $n + 1$ inequalities, Theorem 18 directly implies the following.

Corollary 19. *There is a Lenstra-type algorithm that can decide in $2^{O(n)}n^n\text{poly}(L)$ time, where L is the encoding size of P, whether a given rational Chvátal rank 1 polyhedron $P \subset \mathbb{R}^n$ contains an integer point.*

4 Deciding emptiness of the Chvátal closure

In this section, we prove that deciding whether a given polytope has empty Chvátal closure is NP-complete, even if the input polytope is either a polytope contained in the unit hypercube or a simplex. Consequently, the optimization and separation problems over the Chvátal closure remain NP-hard, when the input is either a rational polytope contained in the unit hypercube or a rational simplex. This extends the result of Eisenbrand [16] that the optimization and separation problems over the Chvátal closure of a polyhedron are NP-hard.

Equality Knapsack Problem (see [17]). Given positive integers $a_1, \cdots, a_n, b$, is there a set of nonnegative integers $\{x_i\}_{i=1}^n$ satisfying $\sum_{i=1}^n a_i x_i = b$.

Without loss of generality, we assume that $a_1, \cdots, a_n$ are relatively prime and that $a_1 < \cdots < a_n$. Throughout this section, we refer to the equality knapsack problem as "the knapsack problem" for simplicity. Using the input data for an instance of the knapsack problem, we construct a rational polytope in the unit hypercube and a rational simplex. By doing so, we are able to provide a polynomial-time reduction from the knapsack problem to the problem of deciding whether the Chvátal closure is empty.

4.1 The case of polytopes contained in the unit hypercube

In this section, we prove NP-completeness of the following problem.

Problem 20. *Given a rational polytope P contained in the n-dimensional unit hypercube $[0,1]^n$, is the Chvátal closure of P empty?*

First, we will construct a polynomial reduction from the knapsack problem to Problem 20. Let $a_1, \cdots, a_{n-4}, b$ be positive integers that satisfy $1 \le a_1, \cdots, a_{n-4} < b$. Let a rational polytope $P_1 \subseteq [0,1]^n$ be defined as the convex hull of the following $n+6$ vectors $v^1, \cdots, v^{n+6} \in [0,1]^n$:

$$
\begin{array}{rcllllllllll}
v^1 & := & (& \frac{1}{2b}, & 0, & \cdots, & 0, & 0, & 0, & \frac{1}{2b}, & 0, & 0 \quad) \\
v^2 & := & (& 0, & \frac{1}{2b}, & \cdots, & 0, & 0, & 0, & \frac{1}{2b}, & 0, & 0 \quad) \\
 & & & & & & & \vdots & & & & \\
v^{n-4} & := & (& 0, & 0, & \cdots, & 0, & \frac{1}{2b}, & 0, & \frac{1}{2b}, & 0, & 0 \quad) \\
v^{n-3} & := & (& 0, & 0, & \cdots, & 0, & 0, & 0, & 1/2, & 1/2, & 1/2 \quad) \\
v^{n-2} & := & (& 1, & 1, & \cdots, & 1, & 1, & 1, & 1/2, & 1/2, & 1/2 \quad) \\
v^{n-1} & := & (& 1/2, & 1/2, & \cdots, & 1/2, & 1/2, & 1/2, & 1, & 1, & 1 \quad) \\
v^{n} & := & (& 1/4, & 1/4, & \cdots, & 1/4, & 1/4, & 1/4, & 1/4, & 1/4, & 1/4 \quad) \\
v^{n+1} & := & (& 1/2, & 1/2, & \cdots, & 1/2, & 1/2, & 1/2, & 1, & 1, & 1/2 \quad) \\
v^{n+2} & := & (& 1/2, & 1/2, & \cdots, & 1/2, & 1/2, & 1/2, & 0, & 0, & 1/2 \quad) \\
v^{n+3} & := & (& 1/2, & 1/2, & \cdots, & 1/2, & 1/2, & 1/2, & 1/2, & 1, & 1 \quad) \\
v^{n+4} & := & (& 1/2, & 1/2, & \cdots, & 1/2, & 1/2, & 1/2, & 1/2, & 0, & 0 \quad) \\
v^{n+5} & := & (& \frac{a_1}{2b}, & \frac{a_2}{2b}, & \cdots, & \frac{a_{n-5}}{2b}, & \frac{a_{n-4}}{2b}, & 0, & 0, & \frac{1}{2} - \frac{1}{4b}, & 0 \quad) \\
v^{n+6} & := & (& 1 - \frac{a_1}{2b}, & 1 - \frac{a_2}{2b}, & \cdots, & 1 - \frac{a_{n-5}}{2b}, & 1 - \frac{a_{n-4}}{2b}, & 1, & \frac{1}{2} + \frac{1}{4b}, & 0, & 0 \quad)
\end{array}
$$

Lemma 21. *A linear description of P_1 can be obtained in polynomial time, and P_1 contains no integer point.*

Proof. We can easily show that P_1 is a full-dimensional polytope, so the number of facets of P_1 is at most $\binom{n+6}{n} \le n^6$. Given n affinely independent points among $v^1, \cdots, v^{n+6}$, we can compute the hyperplane containing theses n points using the Gaussian elimination method. Since the encoding size of each v^i is polynomially bounded by $\log a_1, \cdots, \log a_{n-4}, \log b$, and n, the complexity of the hyperplane is also polynomially bounded by the input encoding size. Therefore, we can polynomially find each facet of P_1. Lastly, $P_1 := \text{conv}\{v^1, \cdots, v^{n+6}\} \subset [0,1]^n$ contains no integer point, because every point in $\{v^1, \cdots, v^{n+6}\}$ is fractional. $\square$

Let $Q_1 := \{x \in \mathbb{Z}_+^{n-4} : \sum_{i=1}^{n-4} a_i x_i = b\}$. The following two lemmas show that $P_1' = \emptyset$ if and only if $Q_1 \neq \emptyset$, thereby providing a polynomial reduction from the knapsack problem to Problem 20.

Lemma 22. *If $Q_1 \neq \emptyset$, then $P_1' = \emptyset$.*

Proof. Let $(w_1, \cdots, w_{n-4}) \in Q_1$. Then $\sum_{i=1}^{n-4} a_i w_i = b$ and $w_i \geq 0$ for $i = 1, \cdots, n-4$. Let $d := (w_1, \cdots, w_{n-4}, -\sum_{i=1}^{n-4} w_i, 1, -1, 1) \in \mathbb{Z}^n$. Notice that $w_k \leq a_k w_k \leq \sum_{i=1}^{n-4} a_i w_i = b$, so we get $\frac{w_k}{2b} \leq \frac{1}{2}$. Since $b > 1$, we know that $0 < \frac{1}{2b} \leq \frac{1}{4}$. Thus, $0 < dv^k = \frac{w_k}{2b} + \frac{1}{2b} < 1$ for $k = 1, \cdots, n-4$. It is easy to show that $dv^{n-3} = dv^{n-2} = dv^{n+1} = dv^{n+2} = dv^{n+3} = dv^{n+4} = \frac{1}{2}$, $dv^n = \frac{1}{4}$, and $dv^{n-1} = 1$. In addition, $dv^{n+5} = dv^{n+6} = \frac{1}{4b}$. That means $0 < dv^i < 1$ for $i \neq n-1$ and $dv^{n-1} = 1$.

We know that $dx > 0$ is valid for P_1, because $dv^k > 0$ for $k = 1, \cdots, n+6$ and P_1 is the convex hull of $v^1, \cdots, v^{n+6}$. This implies $dx \geq 1$ is valid for P_1'. In fact, $P_1 \cap \{x \in \mathbb{R}^n : dx \geq 1\} = \{v^{n-1}\}$. Notice that $x_{n-3} + x_{n-2} + x_{n-1} + x_n \leq \frac{7}{2}$ is also valid for P_1, because it is valid for every v^k for $k = 1, \cdots, n+6$. Then $x_{n-3} + x_{n-2} + x_{n-1} + x_n \leq 3$ is valid for P_1', and v^{n-1} violates this inequality. Therefore, $P_1 \cap \{x \in \mathbb{R}^n : dx \geq 1, x_{n-3} + x_{n-2} + x_{n-1} + x_n \leq 3\}$ is empty. Hence, $P_1' = \emptyset$. □

Lemma 23. *If $P_1' = \emptyset$, then $Q_1 \neq \emptyset$.*

Proof. Let $u := \frac{1}{2}v^{n-3} + \frac{1}{2}v^{n-2}$. Then $u = (\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Since $P_1' = \emptyset$, there is a Chvátal inequality that is violated by u. In other words, there exists a valid inequality $dx \leq d_0 + \alpha$ for P_1 such that $(d, d_0) \in \mathbb{Z}^{n+1}$, $0 < \alpha < 1$, and $du > d_0$. We claim that d and d_0 satisfy the following five properties:

1) $d_{n-3} = -\sum_{i=1}^{n-4} d_i$.

2) $d_0 = -1$.

3) $d_{n-2} = d_n = -1$ and $d_{n-1} = 1$.

4) $\sum_{i=1}^{n-4} a_i d_i = -b$.

5) $d_i \leq 0$ for $i = 1, \cdots, n-4$.

Then, $(-d_1, \cdots, -d_{n-4}) \in Q_1$, so $Q_1 \neq \emptyset$.

Since $d_0 < du \leq d_0 + \alpha < d_0 + 1$, we get $d_0 < \frac{1}{2}\sum_{i=1}^{n} d_i < d_0 + 1$. In addition, we know that $dv^k \leq d_0 + \alpha < d_0 + 1$ for $k = 1, \cdots, n+6$. The integrality of $\sum_{i=1}^{n} d_i$ implies that $\frac{1}{2}\sum_{i=1}^{n} d_i$ should be equal to $d_0 + \frac{1}{2}$, and thus we get $\sum_{i=1}^{n} d_i = 2d_0 + 1$ and $du = d_0 + \frac{1}{2}$. Consider dv^{n-3} and dv^{n-4}:

$$d_0 + 1 > dv^{n-3} = du - \frac{1}{2}\sum_{i=1}^{n-3} d_i = d_0 + \frac{1}{2} - \frac{1}{2}\sum_{i=1}^{n-3} d_i, \tag{1}$$

$$d_0 + 1 > dv^{n-2} = du + \frac{1}{2}\sum_{i=1}^{n-3} d_i = d_0 + \frac{1}{2} + \frac{1}{2}\sum_{i=1}^{n-3} d_i. \tag{2}$$

By (1) and (2), we get $-1 < \sum_{i=1}^{n-3} d_i < 1$. Since $\sum_{i=1}^{n-3} d_i$ is an integer, $\sum_{i=1}^{n-3} d_i = 0$ and the first property is satisfied. Then we know that $d_{n-2} + d_{n-1} + d_n = 2d_0 + 1$. Now, consider dv^{n-1} and dv^n:

$$d_0 + 1 > dv^{n-1} = du + \frac{1}{2}(d_{n-2} + d_{n-1} + d_n) = 2d_0 + 1, \tag{3}$$

$$d_0 + 1 > dv^n = \frac{1}{2}du = \frac{1}{2}d_0 + \frac{1}{4}. \tag{4}$$

By (3) and (4), we obtain $-\frac{3}{2} < d_0 < 0$ and thus $d_0 = -1$. So the second property holds and $d_{n-2} + d_{n-1} + d_n = -1$. Consider dv^{n+1} and dv^{n+2}:

$$d_0 + 1 > dv^{n+1} = du + \frac{1}{2}(d_{n-2} + d_{n-1}) = d_0 + \frac{1}{2} + \frac{1}{2}(d_{n-2} + d_{n-1}), \tag{5}$$

$$d_0 + 1 > dv^{n+2} = du - \frac{1}{2}(d_{n-2} + d_{n-1}) = d_0 + \frac{1}{2} - \frac{1}{2}(d_{n-2} + d_{n-1}). \tag{6}$$

By (5) and (6), we know that $-1 < d_{n-2} + d_{n-1} < 1$. So, $d_{n-2} + d_{n-1} = 0$. Similarly, we get $d_{n-1}+d_n = 0$ by considering dv^{n+3} and dv^{n+4}. Together with the observation $d_{n-2}+d_{n-1}+d_n = -1$, we get $d_{n-1} = 1$ and $d_{n-2} = d_n = -1$. Hence, the third property is satisfied. To prove the fourth property, we consider dv^{n+5} and dv^{n+6}:

$$dv^{n+5} = \frac{1}{2b}\sum_{i=1}^{n-4} a_i d_i + (\frac{1}{2} - \frac{1}{4b}) < d_0 + 1 = 0, \tag{7}$$

which implies that $\sum_{i=1}^{n-4} a_i d_i < -b + \frac{1}{2}$, so $\sum_{i=1}^{n-4} a_i d_i \le -b$ since the sum is an integer;

$$dv^{n+6} = \sum_{i=1}^{n-3} d_i - \frac{1}{2b}\sum_{i=1}^{n-4} a_i d_i - (\frac{1}{2} + \frac{1}{4b}) = -\frac{1}{2b}\sum_{i=1}^{n-4} a_i d_i - (\frac{1}{2} + \frac{1}{4b}) < d_0 + 1 = 0, \tag{8}$$

which implies that $\sum_{i=1}^{n-4} a_i d_i > -b - \frac{1}{2}$, so $\sum_{i=1}^{n-4} a_i d_i \ge -b$ since the sum is an integer. Therefore, $\sum_{i=1}^{n-4} a_i d_i = -b$. Lastly, consider dv^k for $k = 1, \cdots, n-4$:

$$dv^k = \frac{1}{2b}d_k - \frac{1}{2b} < d_0 + 1 = 0. \tag{9}$$

By (9), $d_k < 1$ and thus $d_k \le 0$. □

We are now ready to state the first main result of this section.

Theorem 24. *Deciding emptiness of the Chvátal closure of a rational polytope contained in the unit hypercube is NP-complete.*

Proof. First, deciding emptiness of the Chvátal closure of a rational polyhedron is in NP [9]. By Lemma 21, Lemma 22 and Lemma 23, we also know that there is a polynomial reduction from the knapsack problem to Problem 20. Thus, the decision problem is NP-complete. □

4.2 The case of simplices

A full-dimensional polytope $P \subset \mathbb{R}^n$ is called a *simplex* if it is the convex hull of $n+1$ affinely independent points. The integer programming feasibility problem over a rational simplex is NP-complete because of the following polynomial reduction of the knapsack problem to it [32]: consider positive integers $a_1, \cdots, a_n, b$. Let $v^i := \frac{b}{a_i}e^i$ where e^i denotes the ith unit vector for $i = 1, \cdots, n$. Let $v^{n+1} := \frac{b-\frac{1}{2}}{n}(\frac{1}{a_1}, \cdots, \frac{1}{a_n})$. Let $\text{conv}\{v^1, \cdots, v^{n+1}\}$ denote the convex hull of $v^1, \cdots, v^{n+1}$. Note that $av^{n+1} = b - \frac{1}{2}$ and $av^i = b$ for $i = 1, \cdots, n$. Then, $\text{conv}\{v^1, \cdots, v^{n+1}\} \cap \mathbb{Z}^n = \text{conv}\{v^1, \cdots, v^n\} = \{x \in \mathbb{Z}^n : \sum_{i=1}^n a_i x_i = b,\ x \ge 0\}$.

In this section, we prove NP-completeness of the following problem.

Problem 25. *Given a rational simplex $P \subset \mathbb{R}^n$, is the Chvátal closure of P empty?*

Let $a_1, \cdots, a_{n-1}, b$ be positive integers satisfying $1 \le a_1, \cdots, a_{n-1} < b$. Using the given $a_1, \cdots, a_{n-1}, b$, we define the following $n+1$ vectors in $\mathbb{R}^n$:

$$\begin{array}{rllllll} v^1 := & (& \frac{1}{2rB}, & 0, & \cdots, & 0, & \frac{1}{2r} - \frac{b}{2rBA} &) \\ v^2 := & (& 0, & \frac{1}{2rB}, & \cdots, & 0, & \frac{1}{2r} - \frac{b}{2rBA} &) \\ & & & & \vdots & & & \\ v^{n-1} := & (& 0, & 0, & \cdots, & \frac{1}{2rB}, & \frac{1}{2r} - \frac{b}{2rBA} &) \\ v^n := & (& ra_1, & ra_2, & \cdots, & ra_{n-1} & -rb + \frac{1}{2} &) \\ v^{n+1} := & (& -ra_1, & -ra_2, & \cdots, & -ra_{n-1} & rb + 1 &) \end{array}$$

where A and B denote $\sum_{i=1}^{n-1} a_i$ and the smallest integer greater than $\frac{b}{A}$, respectively and $r := 2016b + \frac{1}{2b}$. Let $P_2 := \text{conv}\{v^1, \cdots, v^{n+1}\} \subset \mathbb{R}^n$. It is straightforward to show that P_2 is an n-dimensional simplex.

Lemma 26. *One can polynomially find a linear description of P_2.*

Proof. Since P_2 is a full-dimensional rational simplex in $\mathbb{R}^n$, it contains exactly $n+1$ facets. One can obtain each facet-defining inequality of P_2 by the Gaussian elimination method, and the complexity of each facet-defining inequality polynomially bounded by $\log a_1, \cdots, \log a_{n-1}$, and $\log b$. Hence, the lemma is proved. □

Let $Q_2 := \{x \in \mathbb{Z}_+^{n-1} : \sum_{i=1}^{n-1} a_i x_i = b\}$. Lemma 27 and Lemma 28 show that $P_2' = \emptyset$ if and only if $Q_2 \neq \emptyset$, which implies a polynomial reduction from the knapsack problem to Problem 25.

Lemma 27. *If $Q_2 \neq \emptyset$, then $P_2' = \emptyset$.*

Proof. Let $(w_1, \cdots, w_{n-1}) \in Q_2$. Then $\sum_{i=1}^{n-1} a_i w_i = b$ and $w_i \ge 0$ for $i = 1, \cdots, n-1$. Let $d := (w_1, \cdots, w_{n-1}, 1) \in \mathbb{Z}^n$. So $dv^k = \frac{w_k}{2rB} + \frac{1}{2r} - \frac{b}{2rBA}$ for $k = 1, \cdots, n-1$. Since $0 < B - \frac{b}{A} \le 1$ by the definition of B, we get that $0 < \frac{1}{2r} - \frac{b}{2rBA} \le \frac{1}{2rB}$. This implies $0 < dv^k \le \frac{w_k+1}{2rB}$. Because $w_k \le a_k w_k \le b$, we have $w_k + 1 \le 2b$. Hence, $0 < dv^k < 1$ for $k = 1, \cdots, n-1$. Besides, $dv^n = \frac{1}{2}$ and $dv^{n+1} = 1$. We now know that $dv^1, \cdots, dv^{n+1}$ are all positive, so $dx \ge 1$ is valid for P_2'. In addition, $x_n \le rb + 1 = 2016b^2 + \frac{3}{2}$ is valid for P_2, so $x_n \le rb + \frac{1}{2} = 2016b^2 + 1$ is valid for P_2'. Since $P_2 \cap \{x \in \mathbb{R}^n : dx \ge 1\} = \{v^{n+1}\}$ and the last component of v^{n+1} is greater than $rb + \frac{1}{2}$, we know $P_2 \cap \{x \in \mathbb{R}^n : dx \ge 1, \ x_n \le rb + \frac{1}{2}\} = \emptyset$. Therefore $P_2' = \emptyset$. □

Lemma 28. *If $P_2' = \emptyset$, then $Q_2 \neq \emptyset$.*

Proof. Let $u^1 := \frac{1}{A} \sum_{i=1}^{n-1} a_i v^i$. Then $u^1 = (\frac{a_1}{2rBA}, \cdots, \frac{a_{n-1}}{2rBA}, \frac{1}{2r} - \frac{b}{2rBA}) \in P_2$. Let $u^2 := \frac{1}{2}v^n + \frac{1}{2}v^{n+1} = (0, \cdots, 0, \frac{3}{4})$ and $u^3 := \frac{1}{3}u^1 + \frac{2}{3}u^2 = (\frac{a_1}{6rBA}, \cdots, \frac{a_{n-1}}{6rBA}, \frac{1}{6r} + \frac{1}{2} - \frac{b}{6rBA})$. Thus, both u^2 and u^3 are in P_2. If $P_2' = \emptyset$, at least one Chvátal inequality is violated by u^3. In other words, there exists an inequality $dx \le d_0 + \alpha$ valid for P_2 such that $(d, d_0) \in \mathbb{Z}^{n+1}$, $0 < \alpha < 1$, and $d_0 < du^3$. We claim that d and d_0 satisfy the following four properties:

1) $\sum_{i=1}^{n-1} a_i d_i = b d_n$.

2) $d_n = -1$.

3) $d_0 = -1$.

4) $d_i \le 0$ for $i = 1, \cdots, n-1$.

Then, $(-d_1, \cdots, -d_{n-1}) \in Q_2$, so $Q_2 \neq \emptyset$.

Let $\Delta := \sum_{i=1}^{n-1} a_i d_i - b d_n$. Then Δ is an integer. Note that $r\Delta + \frac{1}{2}d_n - 1 < \lfloor dv^n \rfloor \le d_0$. Then $r\Delta + \frac{1}{2}d_n - 1 < d_0 < du^3 = \frac{1}{6rBA}\Delta + (\frac{1}{2} + \frac{1}{6r})d_n$, and thus

$$6r\left(r - \tfrac{1}{6rBA}\right)\Delta - 6r \quad < \quad d_n. \tag{10}$$

Besides, $-r\Delta + d_n - 1 < \lfloor dv^{n+1} \rfloor \le d_0$. Since $-r\Delta + d_n - 1 < d_0 < du^3 = \frac{1}{6rBA}\Delta + (\frac{1}{2} + \frac{1}{6r})d_n$, we get

$$\left(\tfrac{1}{2} - \tfrac{1}{6r}\right) d_n \quad < \quad 1 + \left(r + \tfrac{1}{6rBA}\right)\Delta. \tag{11}$$

Suppose that $\Delta \neq 0$ for the sake of contradiction. There are four cases to consider; $\Delta > 0$ and $d_n \ge 0$; $\Delta > 0$ and $d_n < 0$; $\Delta < 0$ and $d_n \ge 0$; and $\Delta < 0$ and $d_n < 0$.

Case 1: $\Delta > 0$ and $d_n \ge 0$.

We know $(\frac{1}{2} - \frac{1}{6r}) > \frac{1}{3}$. By (11), we get $d_n < (3r + \frac{1}{2rBA})\Delta + 3$. Together with (10), we have

$$6\Delta r^2 - (3\Delta + 6)r - \tfrac{1}{BA}\Delta - \tfrac{1}{2rBA}\Delta \quad < \quad 3.$$

Note that $-6\Delta r \le -6r$ and $-2\Delta \le -\frac{1}{BA}\Delta - \frac{1}{2rBA}\Delta$. Hence, we obtain the following:

$$\Delta(6r^2 - 9r - 2) \quad < \quad 3,$$

which cannot be true, because $r \ge 2016$ and $\Delta \ge 1$. Therefore, Case 1 is not possible.

Case 2: $\Delta > 0$ and $d_n < 0$. By (10), $6\Delta r^2 - \frac{1}{BA}\Delta - 6r < d_n$. Since $\Delta \ge 1$, $-\Delta r^2 \le -\frac{1}{BA}\Delta$ and we get $5\Delta r^2 - 6r < d_n$. Because $\Delta \ge 1$ and $r \ge 2016$, $d_n > 0$, which contradicts the assumption $d_n < 0$.

Case 3: $\Delta < 0$ and $d_n \ge 0$. Since $\Delta \le -1$ and $\frac{1}{6rBA} > 0$, the right hand side of the inequality (11) is less than $1 - r$ which has negative value. Because $\frac{1}{2} > \frac{1}{6r}$, the inequality (11) implies that d_n be also negative, which contradicts the assumption $d_n \ge 0$.

Case 4: $\Delta < 0$ and $d_n < 0$. Notice that $\frac{1}{2rBA}\Delta + \frac{1}{2r}d_n - 1 < \lfloor du^1 \rfloor \le d_0$. Then $\frac{1}{2rBA}\Delta + \frac{1}{2r}d_n - 1 < d_0 < du^3 = \frac{1}{6rBA}\Delta + (\frac{1}{2} + \frac{1}{6r})d_n$. Then $\frac{1}{3rBA}\Delta - 1 < (\frac{1}{2} - \frac{1}{3r})d_n < \frac{1}{3}d_n$, and thus $\frac{1}{rBA}\Delta - 3 < d_n$. By (11) and the assumption $d_n < 0$, we also know that $d_n < 1 + (r + \frac{1}{6rBA})\Delta$. Then, we obtain

$$-4 \quad < \quad \left(r - \tfrac{5}{6rBA}\right)\Delta.$$

Since $\Delta \le -1$, the right hand side of the above inequality is at most $-r + \frac{5}{6rBA}$, which is less than $1 - r$. Then we get $-4 < 1 - r$, but this cannot be true because $r \ge 2016$.

All the four cases are not possible, so we get contradiction to the supposition that $\Delta \neq 0$. Therefore, $\Delta = 0$, *i.e.*, (d, d_0) satisfies the first property. In this case, $du^1 = \frac{1}{2r}d_n$, $du^3 = (\frac{1}{2} + \frac{1}{6r})d_n$, $dv^n = \frac{1}{2}d_n$, and $dv^{n+1} = d_n$. If $d_n = 0$, then d_0 satisfies $d_0 < du^3 = 0 < d_0 + 1$ which is not true since d_0 is an integer. Hence, $d_n \neq 0$. If $d_n \ge 1$, then the following relation holds.

$$\lfloor du^3 \rfloor \quad = \quad \lfloor (\tfrac{1}{2} + \tfrac{1}{6r})d_n \rfloor \quad < \quad d_n \quad = \quad \lfloor d_n \rfloor \quad = \quad \lfloor dv^{n+1} \rfloor \quad \le \quad \lfloor d_0 + \alpha \rfloor \quad = \quad d_0.$$

However, we assumed that $d_0 < du^3$, and this implies $d_0 \le \lfloor du^3 \rfloor$. We got contradiction, so $d_n \le -1$. Note that $\frac{1}{2r}d_n - 1 < \lfloor du^1 \rfloor \le d_0$. Since $d_0 < du^3 = (\frac{1}{2} + \frac{1}{6r})d_n$, we get $-1 < (\frac{1}{2} - \frac{1}{3r})d_n$

and thus $-2 \leq d_n$. If $d_n = -2$, $\lfloor dv^n \rfloor = -1$ and $\lfloor du^3 \rfloor = -2$. Then $\lfloor du^3 \rfloor < \lfloor dv^n \rfloor \leq d_0$, but this contradicts the observation $d_0 \leq \lfloor du^3 \rfloor$. Therefore, $d_n = -1$, *i.e.*, (d, d_0) satisfies the second property.

Since $d_n = -1$, $du^3 = -\frac{1}{2} - \frac{1}{6r}$. This means $-1 < du^3 < 0$, so it also implies $d_0 = -1$ which is the third property. To prove the fourth property, let us consider dv^k for $k = 1, \cdots, n-1$. $dv^k = \frac{d_k}{2rB} - (\frac{1}{2r} - \frac{b}{2rBA}) < d_0 + 1 = 0$. Then, $d_k < B - \frac{b}{A}$. Since B is the smallest integer greater than $\frac{b}{A}$, $B \leq \frac{b}{A} + 1$. Therefore, $d_k < 1$ and thus $d_k \leq 0$ for $k = 1, \cdots, n-1$. □

Theorem 29. *Deciding emptiness of the Chvátal closure of a rational simplex is NP-complete.*

Proof. The decision problem is obviously in NP. The NP-completeness is implied by Lemma 26, Lemma 27 and Lemma 28. □

4.3 Optimization and separation over Chvátal closure

Eisenbrand [16] showed that the following separation problem is NP-hard, answering an early question of Schrijver [31].

Separation problem over the Chvátal closure. Given a rational polyhedron $P \subset \mathbb{R}^n$ and a point $\bar{x} \in \mathbb{Q}^n$, either show that $\bar{x} \in P'$ or find a valid Chvátal inequality $dx \leq d_0$ for P' such that $d\bar{x} > d_0$.

According to a general result given by Grötschel, Lovász and Schrijver [20], the above separation problem and the following optimization problems are equivalent up to a polynomial time overhead.

Optimization problem over the Chvátal closure. Given a rational polyhedron $P \subset \mathbb{R}^n$ and an objective $c \in \mathbb{Q}^n$, find a vector $x^* \in P'$ satisfying $cx^* = \max\{cx : x \in P'\}$, or show $P' = \emptyset$, or find a ray z of the recession cone of P' for which cz is positive.

Theorem 24 and Theorem 29 trivially imply the NP-hardness of the optimization problem over the Chvátal closure of a rational polytope in the unit hypercube or over the Chvátal closure of a rational simplex, so we have the following conclusion.

Corollary 30. *The optimization and separation problems over the Chvátal closure of a rational polytope in the unit hypercube or over the Chvátal closure of a rational simplex are NP-hard.*

5 Deciding whether Chvátal closure equals integer hull

A result of Cornuéjols and Li [9] is that it is NP-hard to decide whether the Chvátal rank of a rational polyhedron is 1. In this section, we improve their result by showing that the decision problem remains NP-hard even when the input polyhedron is contained in the unit hypercube or is a rational simplex.

Theorem 31. *Let $P \subseteq [0,1]^n$ be a rational polytope. It is NP-hard to decide whether $P' = P_I$.*

Proof. Recall that P_1 contains no integer point by Lemma 21. That means $(P_1)_I = \emptyset$. By Lemma 22 and Lemma 23, $P_1' = (P_1)_I$ if and only if there exists a solution to $\{x \in \mathbb{Z}_+^{n-4} : \sum_{i=1}^{n-4} a_i x_i = b\}$. Therefore, there exists a polynomial reduction from the knapsack problem to the problem of deciding whether a given rational polytope $P \subseteq [0,1]^n$ satisfies $P' = P_I$, so the latter problem is also NP-hard. □

Now, we turn our attention to simplices.

Theorem 32. *Let $P \subset \mathbb{R}^n$ be a rational simplex. It is NP-hard to decide whether $P' = P_I$.*

Proof. Recall that the polyhedron P_2 constructed in Section 4.2 is a rational simplex, and Lemma 27 and Lemma 28 show that P_2' is empty if and only if there exists a solution to $\{x \in \mathbb{Z}_+^n : \sum_{i=1}^{n-1} a_i x_i = b\}$, thereby providing a polynomial reduction from the knapsack problem to the problem of deciding whether $P' = \emptyset$ for a given rational simplex P.

To prove Theorem 32, it suffices to show that P_2 contains no integer point, which means that $P_2' = \emptyset$ is equivalent to $P_2' = (P_2)_I$. Suppose that P_2 contains an integer point for the sake of contradiction. Then, there exists an integer $d \in [-rb + \frac{1}{2}, rb + 1]$ such that $P_2(d) := \{x \in P_2 : x_n = d\}$ contains an integer point.

First of all, we assume that $d > 0$. The hyperplane defined by $x_n = d$ intersects the line segments between v^{n+1} and v^i for $i = 1, \cdots, n$. In fact, $P_2(d)$ is the convex hull of those n intersection points. Let u_0^i denote the intersection point of $\{x \in \mathbb{R}^n : x_n = d\}$ and the line segment between v^{n+1} and v^i for $i = 1, \cdots, n$. Then, for $i = 1, 2, \cdots, n-1$,

$$\begin{aligned} u_0^i &:= \frac{-rd + \frac{1}{2} - \frac{b}{2BA}}{rb + 1 - \frac{1}{2r} + \frac{b}{2rBA}}(a_1, \cdots, a_{n-1}, 0) + \left(\frac{1}{2rB} - \frac{d - \frac{1}{2r} + \frac{b}{2rBA}}{2rB(rb + 1 - \frac{1}{2r} + \frac{b}{2rBA})}\right) e^i + de^n, \quad \text{and} \\ u_0^n &:= \frac{-rd + \frac{3}{4}r}{rb + \frac{1}{4}}(a_1, \cdots, a_{n-1}, 0) + de^n. \end{aligned}$$

Let a and u^i, where $i = 1, \cdots, n$, denote the projection of $(a_1, \cdots, a_{n-1}, 0)$ and u_0^i to the space defined by the first $n-1$ components, respectively. Then, $a = (a_1, \cdots, a_{n-1}) \in \mathbb{Z}^{n-1}$ and

$$\begin{aligned} u^i &:= \frac{-rd + \frac{1}{2} - \frac{b}{2BA}}{rb + 1 - \frac{1}{2r} + \frac{b}{2rBA}} a + \left(\frac{1}{2rB} - \frac{d - \frac{1}{2r} + \frac{b}{2rBA}}{2rB(rb + 1 - \frac{1}{2r} + \frac{b}{2rBA})}\right) e^i \quad \text{for } i = 1, 2, \cdots, n-1, \\ u^n &:= \frac{-rd + \frac{3}{4}r}{rb + \frac{1}{4}} a. \end{aligned}$$

Since $P_2(d)$ contains an integer point with the last component equal to d, the convex hull of $u^1, \cdots, u^n$, which is a simplex in $\mathbb{R}^{n-1}$ and is denoted by S, contains an integer point in $\mathbb{Z}^{n-1}$. Consider the line L' through 0 and a in $\mathbb{R}^{n-1}$. By a simple calculation, we see that the intersection of L' and S is the line segment between u^n and $\frac{1}{A}\sum_{i=1}^{n-1} a_i u^i$. Let u denote $\frac{1}{A}\sum_{i=1}^{n-1} a_i u^i$. Then

$$u = \left(\frac{-rd + \frac{1}{2} - \frac{b}{2BA}}{rb + 1 - \frac{1}{2r} + \frac{b}{2rBA}} + \frac{1}{2rBA} - \frac{d - \frac{1}{2r} + \frac{b}{2rBA}}{2rBA(rb + 1 - \frac{1}{2r} + \frac{b}{2rBA})}\right) a.$$

Now we show that the line segment between u^n and u does not contain an integer point. Since $a_1, \cdots, a_{n-1}$ are relatively prime integers, there is no integer point between ℓa and $(\ell+1)a$ for any $\ell \in \mathbb{Z}$. Let $C_1 := \frac{-rd + \frac{3}{4}r}{rb + \frac{1}{4}}$ and $C_2 := \left(\frac{-rd + \frac{1}{2} - \frac{b}{2BA}}{rb + 1 - \frac{1}{2r} + \frac{b}{2rBA}} + \frac{1}{2rBA} - \frac{d - \frac{1}{2r} + \frac{b}{2rBA}}{2rBA(rb + 1 - \frac{1}{2r} + \frac{b}{2rBA})}\right)$. Then $u^n = C_1 a$ and $u = C_2 a$. We will show that there exists an integer ℓ such that $C_1, C_2 \in (\ell, \ell+1)$. By doing so, we can prove that there is no integer point in the line segment between u^n and u.

Note that d can be expressed as $kb + h$ for some $0 \leq k \leq 2016b$ and $0 \leq h < b$. Then we can rewrite both u^n and u as follows:

$$u^n = \left(-k + \frac{-rh + \frac{1}{4}k + \frac{3}{4}r}{rb + \frac{1}{4}}\right) a = \left(-k - 1 + \frac{r(b-h) + \frac{1}{4} + \frac{1}{4}k + \frac{3}{4}r}{rb + \frac{1}{4}}\right) a,$$

$$\begin{aligned} u &= \left(-k + \frac{-rh+(r-k)(\frac{1}{2r}-\frac{b}{2rBA})+k}{rb+1-\frac{1}{2r}+\frac{b}{2rBA}} + \frac{1}{2rBA} - \frac{d-\frac{1}{2r}+\frac{b}{2rBA}}{2rBA(rb+1-\frac{1}{2r}+\frac{b}{2rBA})}\right) a \\ &= \left(-k-1 + \frac{r(b-h)+(r-k-1)(\frac{1}{2r}-\frac{b}{2rBA})+(k+1)}{rb+1-\frac{1}{2r}+\frac{b}{2rBA}} + \frac{1}{2rBA} - \frac{d-\frac{1}{2r}+\frac{b}{2rBA}}{2rBA(rb+1-\frac{1}{2r}+\frac{b}{2rBA})}\right) a. \end{aligned}$$

In the following, we consider three possible cases: $h = 0$; $h = 1$ and $k = 2016b = r - \frac{1}{2b}$; and $h \geq 1$ and $k \leq 2016b - 1 = r - 1 - \frac{1}{2b}$.

Case 1: $h = 0$. In this case, the integer part of C_1 is $-k$, while its fractional part is $\frac{\frac{1}{4}k+\frac{3}{4}r}{rb+\frac{1}{4}}$ since it is certainly positive and less than 1. Notice that $\frac{1}{2r} - \frac{b}{2rBA} = \frac{B-\frac{b}{A}}{2rB} \leq \frac{1}{2rB}$, because B is the smallest integer greater than $\frac{b}{A}$. Then $(r-k)(\frac{1}{2r}-\frac{b}{2rBA})+k \leq \frac{1}{2B}+r$. In addition, $0 < 1-\frac{d-\frac{1}{2r}+\frac{b}{2rBA}}{rb+1-\frac{1}{2r}+\frac{b}{2rBA}} < 1$, because $0 < d < rb+1$. Therefore, $\frac{-rh+(r-k)(\frac{1}{2r}-\frac{b}{2rBA})+k}{rb+1-\frac{1}{2r}+\frac{b}{2rBA}} + \frac{1}{2rBA} - \frac{d-\frac{1}{2r}+\frac{b}{2rBA}}{2rBA(rb+1-\frac{1}{2r}+\frac{b}{2rBA})}$ is positive and at most $\frac{1}{2rB} + \frac{1}{2rBA}$ which is less than 1. That means the integer part of C_2 is $-k$ and its fractional part is positive. In this case, $C_1, C_2 \in (-k, -k+1)$.

Case 2: $h = 1$ and $k = 2016b = r - \frac{1}{2b}$. Since $k < r$, $0 < r(b-1) + \frac{1}{4} + \frac{1}{4}k + \frac{3}{4}r < rb + \frac{1}{4}$. Thus, we get $0 < \frac{r(b-1)+\frac{1}{4}+\frac{1}{4}k+\frac{3}{4}r}{rb+\frac{1}{4}} < 1$. Then $C_1 \in (-k-1, -k)$. Note that

$$\begin{aligned} r(b-h) + (r-k-1)(\tfrac{1}{2r} - \tfrac{b}{2rBA}) + (k+1) &= r(b-1) + (\tfrac{1}{2b} - 1)(\tfrac{1}{2r} - \tfrac{b}{2rBA}) + 2016b + 1 \\ &= rb + 1 - \tfrac{1}{2r} + \tfrac{b}{2rBA} + \tfrac{1}{2b}(-1 + \tfrac{1}{2r} - \tfrac{b}{2rBA}). \end{aligned}$$

In addition,

$$\frac{1}{2rBA} - \frac{d-\frac{1}{2r}+\frac{b}{2rBA}}{2rBA(rb+1-\frac{1}{2r}+\frac{b}{2rBA})} = \frac{1}{4rBA(rb+1-\frac{1}{2r}+\frac{b}{2rBA})}.$$

In this case,

$$\frac{r(b-h)+(r-k-1)(\frac{1}{2r}-\frac{b}{2rBA})+(k+1)}{rb+1-\frac{1}{2r}+\frac{b}{2rBA}} + \frac{1}{2rBA} - \frac{d-\frac{1}{2r}+\frac{b}{2rBA}}{2rBA(rb+1-\frac{1}{2r}+\frac{b}{2rBA})} = 1 + \frac{\frac{1}{2b}(-1+\frac{1}{2r}-\frac{b}{2rBA})+\frac{1}{4rBA}}{rb+1-\frac{1}{2r}+\frac{b}{2rBA}}.$$

is less than 1, because $\frac{1}{2r} - \frac{b}{2rBA} + \frac{1}{4rBA} \leq \frac{1}{2rB} + \frac{1}{4rBA} < \frac{1}{2b}$. Therefore, we get that $C_2 \in (-k-1, -k)$.

Case 3: $h \geq 1$ and $k \leq 2016b - 1 = r - 1 - \frac{1}{2b}$. As in the previous case, we can show that $C_1 \in (-k-1, -k)$. Notice that

$$\begin{aligned} r(b-h) + (r-k-1)(\tfrac{1}{2r} - \tfrac{b}{2rBA}) + (k+1) &\leq rb - \tfrac{1}{2b} + \tfrac{1}{2b}(\tfrac{1}{2r} - \tfrac{b}{2rBA}) \\ &= rb + 1 - \tfrac{1}{2r} + \tfrac{b}{2rBA} - (1 + \tfrac{1}{2b})(1 - \tfrac{1}{2r} + \tfrac{b}{2rBA}). \end{aligned}$$

We also have the following:

$$\frac{1}{2rBA} - \frac{d-\frac{1}{2r}+\frac{b}{2rBA}}{2rBA(rb+1-\frac{1}{2r}+\frac{b}{2rBA})} \leq \frac{1}{2rBA} \leq \frac{1}{2rb} \leq \frac{1}{rb+1-\frac{1}{2r}-\frac{b}{2rBA}}.$$

Since $1 - (1 + \frac{1}{2b})(1 - \frac{1}{2r} + \frac{b}{2rBA}) < 0$, we get

$$\frac{r(b-h)+(r-k-1)(\frac{1}{2r}-\frac{b}{2rBA})+(k+1)}{rb+1-\frac{1}{2r}+\frac{b}{2rBA}} + \frac{1}{2rBA} - \frac{d-\frac{1}{2r}+\frac{b}{2rBA}}{2rBA(rb+1-\frac{1}{2r}+\frac{b}{2rBA})} < 1.$$

It is obvious that $r(b-h) + (r-k-1)(\frac{1}{2r} - \frac{b}{2rBA}) + (k+1) > 0$, so $C_2 \in (-k-1, -k)$.

We have proved that there is no integer point in the line segment connecting u^n and u. Then any integer point in $\text{conv}\{u^1, \cdots, u^n\}$ is not on the line segment. Let $\tilde{v} \in \text{conv}\{u^1, \cdots, u^n\} \cap \mathbb{Z}^{n-1}$. Observe that there exists $\delta \in \mathbb{R}$ such that $\|\tilde{v} - \delta a\|_\infty \leq \frac{1}{rB}$, because $\|u^i - u\|_\infty \leq \frac{1}{rB}$ for $i = 1, \cdots, n-1$, $\|u^n - u^n\|_\infty = 0$, and $\tilde{v} \in \text{conv}\{u^1, \cdots, u^n\}$. In addition, $\tilde{v}$ is not on the line connecting 0 and a in $\mathbb{R}^{n-1}$. That means there exists an index $j \in \{1, \cdots, n-2\}$ such that $(\tilde{v}_j, \tilde{v}_{n-1})$ is not on the line through 0 and (a_j, a_{n-1}) in $\mathbb{R}^2$ [9]. The following can be proved with a simple geometric analysis in $\mathbb{R}^2$:

Claim 1. *Let $u := (u_1, u_2) \neq (0,0)$ and $v := (v_1, v_2)$ be two points in $\mathbb{Z}^2$. Then, for any $\delta \in \mathbb{R}^n$, $\|v - \delta u\|_\infty \geq \frac{|u_1 v_2 - u_2 v_1|}{|u_1| + |u_2|}$.*

By Claim 1,

$$\|(\tilde{v}_j, \tilde{v}_{n-1}) - \delta(a_j, a_{n-1})\|_\infty \geq \frac{|a_j \tilde{v}_{n-1} - a_{n-1}\tilde{v}_j|}{|a_j| + |a_{n-1}|}.$$

for any $\delta \in \mathbb{R}$. Because $(\tilde{v}_j, \tilde{v}_{n-1})$ is not on the line through 0 and (a_j, a_{n-1}), $|a_j \tilde{v}_{n-1} - a_{n-1}\tilde{v}_j| \geq 1$. Since $0 < a_j < a_{n-1} \leq b$,

$$\|(\tilde{v}_j, \tilde{v}_{n-1}) - \delta(a_j, a_{n-1})\|_\infty \quad > \quad \frac{1}{2b}. \tag{12}$$

Since $\|\tilde{v} - \delta a\|_\infty$ is at least as large as $\|(\tilde{v}_j, \tilde{v}_{n-1}) - \delta(a_j, a_{n-1})\|_\infty$, we now know that $\|\tilde{v} - \delta a\|_\infty$ is greater than $\frac{1}{2b}$ by (12) for all $\delta \in \mathbb{R}$. However, this contradicts the observation that $\|\tilde{v} - \delta a\|_\infty \leq \frac{1}{rB}$ for some $\delta \in \mathbb{R}$. Therefore, we conclude that $P_2(d)$ does not contain an integer point when $d > 0$.

Now, we assume that $d \leq 0$. Then the hyperplane $\{x \in \mathbb{R}^n : \ x_n = d\}$ intersects the line segment between v^n and v^i for $i = 1, \cdots, n-1, n+1$. We denote each intersection point by w_0^i accordingly. Then we can calculate, for $1 \leq i \leq n-1$,

$$\begin{aligned} w_0^i &:= \frac{rd - \frac{1}{2} + \frac{b}{2BA}}{-rb + \frac{1}{2} - \frac{1}{2r} + \frac{b}{2rBA}}(a_1, \cdots, a_{n-1}, 0) \ + \left(\frac{1}{2rB} - \frac{d - \frac{1}{2r} + \frac{b}{2rBA}}{2rB(-rb + \frac{1}{2} - \frac{1}{2r} + \frac{b}{2rBA})}\right) e^i \ + de^n \quad \text{, and} \\ w_0^{n+1} &:= \frac{-rd + \frac{3}{4}r}{rb + \frac{1}{4}}(a_1, \cdots, a_{n-1}, 0) \qquad + de^n. \end{aligned}$$

We know that $P_2(d)$ is the convex hull of the above n intersection points. Let w^i, where $i = 1, \cdots, n-1, n+1$, denote the projection of w_0^i to the space defined by the first $n-1$ components, respectively. Then,

$$\begin{aligned} w^i &:= \frac{rd - \frac{1}{2} + \frac{b}{2BA}}{-rb + \frac{1}{2} - \frac{1}{2r} + \frac{b}{2rBA}} a \ + \left(\frac{1}{2rB} - \frac{d - \frac{1}{2r} + \frac{b}{2rBA}}{2rB(-rb + \frac{1}{2} - \frac{1}{2r} + \frac{b}{2rBA})}\right) e^i \quad \text{for } 1 \leq i \leq n-1, \\ w^{n+1} &:= \frac{-rd + \frac{3}{4}r}{rb + \frac{1}{4}} a. \end{aligned}$$

Since $P_2(d)$ contains an integer point, its projection, which is identical to the convex hull of $w^1, \cdots, w^{n-1}, w^{n+1}$, contains an integer point in $\mathbb{R}^{n-1}$. Let w denote $\frac{1}{A}\sum_{i=1}^{n-1} a_i w^i$. Then w can be written as

$$w \quad = \quad \left(\frac{rd - \frac{1}{2} + \frac{b}{2BA}}{-rb + \frac{1}{2} - \frac{1}{2r} + \frac{b}{2rBA}} + \frac{1}{2rBA} - \frac{d - \frac{1}{2r} + \frac{b}{2rBA}}{2rBA(-rb + \frac{1}{2} - \frac{1}{2r} + \frac{b}{2rBA})}\right) a.$$

In fact, the line though 0 and a in $\mathbb{R}^{n-1}$ intersects with $\text{conv}\{w^1, \cdots, w^{n-1}, w^{n+1}\}$ in the line segment between w^{n+1} and w. As in the previous case, we will show that the line segment between w^{n+1} and w does not contain an integer point. Let $D_1 := \frac{-rd + \frac{3}{4}r}{rb + \frac{1}{4}}$ and $D_2 := \frac{rd - \frac{1}{2} + \frac{b}{2BA}}{-rb + \frac{1}{2} - \frac{1}{2r} + \frac{b}{2rBA}} + \frac{1}{2rBA}$ $- \frac{d - \frac{1}{2r} + \frac{b}{2rBA}}{2rBA(-rb + \frac{1}{2} - \frac{1}{2r} + \frac{b}{2rBA})}$. We will prove that both $D_1, D_2 \in (\ell, \ell + 1)$ for some integer ℓ. Since d can be written as $kb+h$ for some $-2016b \leq k \leq -1$ and $0 \leq h < b$, we can rewrite both w^{n+1} and w as follows:

$$w^{n+1} = \left(-k + \frac{-rh+\frac{1}{4}k+\frac{3}{4}r}{rb+\frac{1}{4}}\right) a = \left(-k-1+\frac{r(b-h)+\frac{1}{4}+\frac{1}{4}k+\frac{3}{4}r}{rb+\frac{1}{4}}\right) a,$$

$$\begin{aligned} w &= \left(-k + \frac{rh+\frac{1}{2}k-(r+k)(\frac{1}{2r}-\frac{b}{2rBA})}{-rb+\frac{1}{2}-\frac{1}{2r}+\frac{b}{2rBA}} + \frac{1}{2rBA} - \frac{d-\frac{1}{2r}+\frac{b}{2rBA}}{2rBA(rb+1-\frac{1}{2r}+\frac{b}{2rBA})}\right) a \\ &= \left(-k-1+\frac{r(b-h)-\frac{1}{2}-(r+k-1)(\frac{1}{2r}-\frac{b}{2rBA})+\frac{1}{2}k}{rb-\frac{1}{2}+\frac{1}{2r}-\frac{b}{2rBA}} + \frac{1}{2rBA} - \frac{d-\frac{1}{2r}+\frac{b}{2rBA}}{2rBA(rb+1-\frac{1}{2r}+\frac{b}{2rBA})}\right) a. \end{aligned}$$

There are two cases to consider: $h=0$; $h>0$. When $h=0$, it is straightforward to show that $D_1, D_2 \in (-k, -k+1)$. When $h>0$, it is also easy to show that $D_1, D_2 \in (-k-1, -k)$. Therefore, the line segment between w^{n+1} and w does not contain any integer point. The rest of the proof is similar to that given in the previous case when d is positive. □

6 Deciding whether adding a certain number of Chvátal cuts can yield integer hull

Let k be any given positive integer. In this section, we prove that it is NP-hard to decide whether adding at most k Chvátal inequalities to the description of a rational polytope suffices to describe its integer hull. In particular, we show that this is true for the rational polytopes in the unit hypercube and for the rational simplices.

6.1 The case of polytopes contained in the unit hypercube

Theorem 33. *Let $P \subset [0,1]^n$ be a rational polytope and k be a positive integer. Deciding if we can obtain P_I by adding at most k Chvátal inequalities to the linear description of P is NP-hard.*

To prove Theorem 33, we use Lemma 34 for the case $k=1$ and Lemma 35 for $k \geq 2$.

Lemma 34. *Let $P \subseteq [0,1]^n$ be a rational polytope. Deciding whether there exists a Chvátal inequality of P such that adding it to the linear description of P gives P_I is NP-complete.*

Proof. We consider the partition problem:

Partition Problem (see [17]). Given positive integers $a_1, \cdots, a_n$, is there a subset K of the set of indices $[n]$ such that $\sum_{i\in K} a_i = \sum_{j\in[n]\setminus K} a_j$?

The partition problem is NP-complete. Mahajan and Ralphs [27] constructed a rational polytope in the unit hypercube using the input from an instance of the partition problem. We borrow their construction for the proof. Let $a_1, \cdots, a_{n-2}$ be the input for an instance of the partition problem with index set $[n-2]$. Let $\tilde{a}_k$ denote $\frac{1}{\sum_{j=1}^{n-2} a_j} a_k$ for $k = 1, \cdots, n-2$. We use the following $n+2$ vectors in $[0,1]^n$:

$$
\begin{array}{rlllllll}
v^1 := & (& \frac{1}{2}+\frac{1}{2(n-1)}, & \frac{1}{2(n-1)}, & \cdots, & \frac{1}{2(n-1)}, & 0, & 0 &)\\
v^2 := & (& \frac{1}{2(n-1)}, & \frac{1}{2}+\frac{1}{2(n-1)}, & \cdots, & \frac{1}{2(n-1)}, & 0, & 0 &)\\
 & & & & \vdots & & & & \\
v^{n-2} := & (& \frac{1}{2(n-1)}, & \frac{1}{2(n-1)}, & \cdots, & \frac{1}{2}+\frac{1}{2(n-1)}, & 0, & 0 &)\\
v^{n-1} := & (& \tilde{a}_1, & \tilde{a}_2, & \cdots, & \tilde{a}_{n-2}, & 1, & 1 &)\\
v^{n} := & (& \tilde{a}_1, & \tilde{a}_2, & \cdots, & \tilde{a}_{n-2}, & \frac{1}{2}-\frac{1}{2\sum_{j=1}^{n-2}a_j}, & 0 &)\\
v^{n+1} := & (& \tilde{a}_1, & \tilde{a}_2, & \cdots, & \tilde{a}_{n-2}, & 0, & \frac{1}{2}-\frac{1}{2\sum_{j=1}^{n-2}a_j} &)\\
v^{n+2} := & (& 0, & 0, & \cdots, & 0, & \frac{1}{2}, & 0 &)
\end{array}
$$

Let $P_3 := \text{conv}\{v^1, \cdots, v^{n+2}\}$ be the convex hull of the above $n+2$ vectors. Mahajan and Ralphs [27] proved that there is $(\pi, \pi_0) \in \mathbb{Z}^n$ such that $P_3 \subseteq \{x \in \mathbb{R}^n : \pi_0 < \pi x < \pi_0 + 1\}$ if and only if there exists a subset K of $[n-2]$ such that $\sum_{i\in K} a_i = \sum_{j\in[n-2]\setminus K} a_j$, thereby showing the decision problem for the existence of such (π, π_0) is NP-complete.

To complete the proof, we show that P_3 contains no integer point. Let $d := (1, \cdots, 1, 1, -1)$. Then $dv^i = 1 - \frac{1}{2(n-1)}$ for $i = 1, \cdots, n-2$. Besides, we can easily calculate $dv^{n-1} = 1$, $dv^n = \frac{3}{2} - \frac{1}{2\sum_{j=1}^{n-2} a_j}$, $dv^{n+1} = \frac{1}{2} + \frac{1}{2\sum_{j=1}^{n-2} a_j}$, and $dv^{n+2} = \frac{1}{2}$. Without loss of generality, we assume $\sum_{j=1}^{n-2} a_j \geq 1$. So $0 < dx < 2$ is valid for all $x \in P_3$, and thus $dx = 1$ is valid for P_3'.

Since $0 < a_1 < \sum_{j=1}^{n-2} a_j$ by the assumption, $0 < \tilde{a}_1 < 1$. This implies that the first component of each v^i be less than 1, so $x_1 \leq 0$ is valid for P_3'. Notice that $P_3 \cap \{x \in [0,1]^n : x_1 \leq 0\} = \{v^{n+2}\}$. Besides, $dv^{n+2} = \frac{1}{2} \neq 1$. Since $P_3' \subseteq P_3 \cap \{x \in [0,1]^n : dx = 1, \ x_1 \leq 0\}$ and $P_3 \cap \{x \in [0,1]^n : dx = 1, \ x_1 \leq 0\} = \emptyset$, we see $P_3' = \emptyset$. Hence, $(P_3)_I = \emptyset$. □

Lemma 35. *Let $P \subseteq [0,1]^n$ be a rational polytope and k be a positive integer. Deciding if we can obtain P_I by adding at most k Chvátal inequalities of P to the linear description of P is NP-hard.*

Proof. Let $a_1, \cdots, a_{n-4}, b$ be the input for an instance of the knapsack problem of dimension $n-4$. Recall that, in Section 4.1, $(w_1, \cdots, w_{n-4}) \in Q_1 = \{x \in \mathbb{Z}_+^{n-4} : \sum_{j=1}^{n-4} a_j x_j = b\}$ if and only if P_1' is empty. When P_1' is empty, we observe that two Chvátal inequalities $dx \geq 1$ where $d = (w_1, \cdots, w_{n-4}, -\sum_{i=1}^{n-4} w_i, 1, -1, 1)$ and $x_{n-3} + x_{n-2} + x_{n-1} + x_n \leq 3$ remove every point in P_1. Then, we can claim that the knapsack instance Q_1 has a solution if and only if adding at most k Chvátal inequalities is sufficient to describe the integer hull of P_1 which is empty for any $k \geq 2$. Therefore, the decision problem is NP-complete. □

Theorem 33 follows from Lemma 34 and Lemma 35. Note from the proof of Lemma 35 that the lemma also holds in the case that k is not a constant.

6.2 The case of simplices

Theorem 36. *Let $P \subset \mathbb{R}^n$ be a rational simplex and k be a positive integer. Deciding if we can obtain P_I by adding at most k Chvátal inequalities to the linear description of P is NP-hard.*

To prove Theorem 36, we use Lemma 37 for the case $k = 1$ and Lemma 38 for $k \geq 2$.

Lemma 37. *Let $P \subseteq [0,1]^n$ be a rational polytope. Deciding whether there exists a Chvátal inequality of P such that adding it to the linear description of P yields P_I is NP-complete.*

Proof. Mahajan and Ralphs [27] constructed a simplex in $\mathbb{R}^n$ using the input from an instance of the partition problem. Again, we borrow their construction for the proof. Let $a_1, \cdots, a_{n-1}$ be the input for an instance of the partition problem with index set $[n-1]$. Let $n+1$ vectors $v^1, \cdots, v^{n+1}$ be defined as follows:

$$\begin{array}{rllllll} v^1 := & (& \frac{1}{2}+\frac{1}{2n}, & \frac{1}{2n}, & \cdots, & \frac{1}{2n}, & \frac{1}{2n} &) \\ v^2 := & (& \frac{1}{2n}, & \frac{1}{2}+\frac{1}{2n}, & \cdots, & \frac{1}{2n}, & \frac{1}{2n} &) \\ & & & & \vdots & & & \\ v^{n-1} := & (& \frac{1}{2n}, & \frac{1}{2n}, & \cdots, & \frac{1}{2}+\frac{1}{2n}, & \frac{1}{2n} &) \\ v^n := & (& \frac{1}{2n}, & \frac{1}{2n}, & \cdots, & \frac{1}{2n}, & \frac{1}{2}+\frac{1}{2n} &) \\ v^{n+1} := & (& a_1, & a_2, & \cdots, & a_{n-1}, & \frac{1}{2}-\frac{1}{2}\sum_{i=1}^{n-1} a_i &) \end{array}$$

Let $P_4 := \text{conv}\{v^1, \cdots, v^{n+1}\}$ be the convex hull of these $n+1$ vectors. Mahajan and Ralphs [27] proved that there is $(\pi, \pi_0) \in \mathbb{Z}^{n+1}$ such that $P_4 \subseteq \{x \in \mathbb{R}^n : \ \pi_0 < \pi x < \pi_0 + 1\}$ if and only if there exists a subset K of $[n-1]$ such that $\sum_{i \in K} a_i = s$.

As in the proof of Lemma 34, it suffices to show that the integer hull of P_4 is empty. Suppose that $P_4 \cap \mathbb{Z}^n \neq \emptyset$ for the sake of contradiction. Let $d := (1, \cdots, 1, 2)$. Notice that $dv^{n+1} = 1$, $dv^n = \frac{3}{2} + \frac{1}{2n}$, and $dv^i = 1 + \frac{1}{2n}$ for $i = 1, \cdots, n-1$. Then $1 < dv^i < 2$ for $i = 1, \cdots, n$. Because $1 \leq dx < 2$ for all $x \in P_4$, $dz = 1$ for all $z \in P_4 \cap \mathbb{Z}^n$. Let $z^* \in P_4 \cap \mathbb{Z}^n$. Since $z^* \in P_4$, z^* is a convex combination of $v^1, v^2, \cdots, v^{n+1}$ with nonnegative multipliers $\alpha_1, \alpha_2, \cdots, \alpha_{n+1}$, respectively. Note that $dz = 1$ implies $\alpha_1 = \cdots = \alpha_n = 0$ and thus $z = v^{n+1}$. However, v^{n+1} is not integral, which contradicts the integrality of z^*. Hence, $(P_4)_I = \emptyset$. □

Lemma 38. *Let $P \subseteq \mathbb{R}^n$ be a rational simplex and k be a number greater than 1. Deciding if we can obtain P_I by adding at most k Chvátal inequalities of P to the linear description of P is NP-hard.*

Proof. Let $a_1, \cdots, a_{n-1}, b$ be the input from an instance of the knapsack problem of dimension $n-1$. Recall that, in Section 4.2, $(w_1, \cdots, w_{n-1}) \in Q_2 = \{x \in \mathbb{Z}^{n-1}_+ : \ \sum_{j=1}^{n-1} a_j x_j = b\}$ if and only if P_2' is empty. Also, when P_2' is empty, adding two Chvátal inequalities $dx \geq 1$, where $d = (w_1, \cdots, w_{n-1}, 1)$, and $x_n \leq rb + \frac{1}{2}$ to the description of P_2 results in an empty polytope. Then, we can claim that the knapsack instance Q_2 is feasible if and only if adding at most k Chvátal inequalities is sufficient to describe the integer hull of P_2 for any $k \geq 2$. Therefore, the NP-completeness is proved. □

Theorem 36 follows from Lemma 37 and Lemma 38. We also know from the proof of Lemma 38 that the lemma is true even when k is not a constant.

Acknowledgments: This work was supported in part by NSF grant CMMI 1560828 and ONR grant 00014-15-12082.

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