

CHARACTERIZING POLYTOPES IN THE 0/1-CUBE WITH BOUNDED CHVÁTAL-GOMORY RANK

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ABSTRACT. Let $S \subseteq \{0, 1\}^n$ and R be any polytope contained in $[0, 1]^n$ with $R \cap \{0, 1\}^n = S$. We prove that R has bounded Chvátal-Gomory rank (CG-rank) provided that S has bounded *pitch* and bounded *gap*, where the pitch is the minimum integer p such that all p -dimensional faces of the 0/1-cube have a nonempty intersection with S , and the gap is a measure of the size of the facet coefficients of $\text{conv}(S)$.

Let $H[\bar{S}]$ denote the subgraph of the n -cube induced by the vertices not in S . We prove that if $H[\bar{S}]$ does not contain a subdivision of a large complete graph, then both the pitch and the gap are bounded. By our main result, this implies that the CG-rank of R is bounded as a function of the treewidth of $H[\bar{S}]$. We also prove that if S has pitch 3, then the CG-rank of R is always bounded. Both results generalize a recent theorem of Cornuéjols and Lee [5], who proved that the CG-rank is always bounded if the treewidth of $H[\bar{S}]$ is at most 2.

1. INTRODUCTION

Given a polytope $R \subseteq \mathbb{R}^n$, its first *Chvátal-Gomory-closure* (CG-closure) is defined as $R' := \{x \in \mathbb{R}^n : c^\top x \geq \lceil \min_{y \in R} c^\top y \rceil \ \forall c \in \mathbb{Z}^n\}$, which can be shown to be again a (rational) polytope with $R' \cap \mathbb{Z}^n = R \cap \mathbb{Z}^n$ (see Dadush, Dey, and Vielma [7]). By setting $R^{(0)} := R$ and $R^{(t)} := (R^{(t-1)})'$ for every $t \in \mathbb{Z}_{\geq 1}$, one recursively defines the t -th CG-closure $R^{(t)}$ of R . It is well-known that there exists a number $t \in \mathbb{Z}_{\geq 0}$ such that $R^{(t)} = \text{conv}(R \cap \mathbb{Z}^n)$, and the smallest such number is called the *Chvátal-Gomory-rank* (CG-rank) of R . In this paper, we give new bounds on the CG-rank of a polytope R contained in $[0, 1]^n$ that only depend on properties of $S = R \cap \{0, 1\}^n$ and not on R itself. This is in the spirit of Conforti, Del Pia, Di Summa, Faenza, and Grappe [4] except that we only consider relaxations contained in $[0, 1]^n$.

One particular reason to study the CG-rank is to obtain bounds on lengths of *cutting-plane proofs* as introduced by Chvátal, Cook, and Hartmann [2, Sec. 6]. Letting k be the CG-rank of $R \subseteq \mathbb{R}^n$, the length of a cutting-plane proof is at most $(n^{k+1} - 1)/(n - 1)$. In fact, if k is a fixed constant and $R \subseteq \mathbb{R}^n$ has CG-rank k , then optimizing a linear function over $R \cap \mathbb{Z}^n$ is one of the few problems that is known to be in $\text{coNP} \cap \text{NP}$ but not known to be in P (see Cornuéjols & Li [6]). While the CG-rank of general polytopes in \mathbb{R}^n can be arbitrarily large compared to n (even for $n = 2$), Eisenbrand & Schulz [9] showed that the CG-rank of a polytope contained in $[0, 1]^n$ is always bounded by $\mathcal{O}(n^2 \log n)$. Unfortunately, there exist polytopes in $[0, 1]^n$ whose CG-rank grows quadratically in n (see Rothvoß & Sanità [10]).

This motivates the study for situations in which the CG-rank is at most a constant independent of n . This question has been recently addressed by Cornuéjols & Lee [5]. In their work, given a set $S \subseteq \{0, 1\}^n$, they consider the graph $H[\bar{S}]$ whose vertices are the points of $\bar{S} := \{0, 1\}^n \setminus S$ where two points are adjacent if they differ in exactly one coordinate. Their main result is that if the treewidth of $H[\bar{S}]$ (denoted $\text{tw}(H[\bar{S}])$) is at most 2, then the CG-rank of any polytope $R \subseteq [0, 1]^n$ with $R \cap \{0, 1\}^n = S$ is bounded (they prove a bound of 4, which is tight). One corollary of our work is that this holds for *all* values of treewidth: the CG-rank of every polytope $R \subseteq [0, 1]^n$ with $R \cap \{0, 1\}^n = S$ is bounded by a function that only depends on the treewidth of $H[\bar{S}]$.

In order to state our main result, we define the *pitch* of a subset $S \subseteq \{0, 1\}^n$ as the smallest $p \in \mathbb{Z}_{\geq 0}$ such that every p -dimensional face of $[0, 1]^n$ has a nonempty intersection with S . If S is empty, we define $p := n + 1$. We remark that this definition of pitch is consistent with the original definition due to Bienstock & Zuckerberg [1]. Furthermore, we define the *gap* of S as the smallest $\Delta \in \mathbb{Z}_{\geq 0}$ such that $\text{conv}(S)$ can be described as the set of solutions $x \in \mathbb{R}^n$

satisfying inequalities of the form

$$(1) \quad \sum_{i \in I} c_i x_i + \sum_{j \in J} c_j (1 - x_j) \geq \delta$$

where I, J are disjoint subsets of $[n]$, $\delta, c_1, \dots, c_n \in \mathbb{Z}_{\geq 0}$ with $\delta \leq \Delta$. We require that for each inequality in the description, the corresponding equation (obtained from (1) by replacing the inequality sign by an equality sign) defines a hyperplane spanned by 0/1-points. Notice that if S is empty, then we have $\Delta = 1$.

The gap is well-defined since for every $S \subseteq \{0, 1\}^n$, $\text{conv}(S)$ has a description by inequalities in which every corresponding hyperplane is generated by 0/1-points. To see this, consider a full-dimensional 0/1-polytope $\text{conv}(T)$ where $S \subseteq T \subseteq \{0, 1\}^n$ such that $\text{conv}(S)$ is a face of $\text{conv}(T)$ (this exists since the set $\{0, 1\}^n$ is full-dimensional). Clearly, the bounding hyperplane of every facet of $\text{conv}(T)$ is generated by 0/1-points. Since $\text{conv}(S)$ is the intersection of the facets of $\text{conv}(T)$ which contain it, the claimed description directly follows.

Our main result is the following.

Theorem 1. *Let $S \subsetneq \{0, 1\}^n$ be a set with pitch p and gap Δ . Then the CG-rank of any polytope $R \subseteq [0, 1]^n$ with $R \cap \{0, 1\}^n = S$ is at most $p + \Delta - 1$.*

In order to generalize the result of Cornuéjols & Lee, we will show that p and Δ are both bounded in terms of $\text{tw}(H[\bar{S}])$. In fact, we will not even need the definition of treewidth because we actually prove a stronger result. We let K_t be a clique on t vertices. A *subdivision* of K_t is a graph obtained from K_t by replacing each edge of K_t by a path.

Corollary 2. *Let $S \subseteq \{0, 1\}^n$ and let t be the maximum integer such that $H[\bar{S}]$ contains a subdivision of K_{t+1} . Then the CG-rank of any polytope $R \subseteq [0, 1]^n$ with $R \cap \{0, 1\}^n = S$ is at most $t + 2t^{t/2}$.*

To see that Corollary 2 is a generalization of the result by Cornuéjols & Lee, the only thing the reader needs to know is that if a graph G has a subdivision of K_{t+1} , then $\text{tw}(G) \geq t$. This is an easy fact (see Diestel [8] for a gentle introduction to treewidth).

Paper structure. In Section 2 we discuss the meaning of the parameters p and Δ , and in particular their relation to the CG-rank. For instance, we give examples that show that the CG-rank of a polytope in $[0, 1]^n$ cannot be bounded in only one of the two parameters. Furthermore, we observe that optimizing a linear function over S can be done with $\mathcal{O}(n^p)$ oracle calls using an oracle that decides if a point $x \in \{0, 1\}^n$ belongs to S , see Proposition 4. This algorithm already appears in [5], but we show that it is valid under a weaker hypothesis.

Section 3 contains the proof of Theorem 1. In Section 4 we complement our main theorem by a result quantifying how well the t -th CG-closure approximates $\text{conv}(S)$ for constant t and constant p , this time without bounding Δ . In Section 5 we investigate the convex hulls of sets with pitch $p = 3$. In this case, we show that Δ is automatically bounded and give a complete linear description of $\text{conv}(S)$. We show that treewidth at most 2 implies pitch at most 3, but not vice versa, hence this result also strictly generalizes the main result of Cornuéjols & Lee [5].

2. DISCUSSION OF THE PARAMETERS

In this section, we discuss how the parameters *pitch* and *gap* of a set $S \subseteq \{0, 1\}^n$ influence the CG-rank of polytopes $R \subseteq [0, 1]^n$ with $R \cap \{0, 1\}^n = S$.

2.1. Small CG-rank implies small pitch. We first observe that, in order to get a constant bound on the CG-rank, one has to restrict to sets S with bounded pitch. Although this follows directly from known results, we include a proof for completeness. We point out that a little more work gives a lower bound of p , which is tight.

Proposition 3. *Let $S \subseteq \{0, 1\}^n$ have pitch p . Then there exists a polytope $R \subseteq [0, 1]^n$ with $R \cap \{0, 1\}^n = S$ whose CG-rank is at least $p - 1$.*

Proof. Following [5], we let R be the worst possible¹ relaxation of $\text{conv}(S)$:

$$(2) \quad R := \left\{ x \in [0, 1]^n \mid \forall a \in \bar{S} : \sum_{i:a_i=0} x_i + \sum_{i:a_i=1} (1 - x_i) \geq \frac{1}{2} \right\}.$$

By the definition of p , there exists a $(p - 1)$ -dimensional face F of $[0, 1]^n$ such that $F \cap S = \emptyset$. The CG-rank of R is at least that of its face $F \cap R$ (see for instance [3, Lem. 5.17]), which can be shown to be exactly $p - 1$ using [2, Lem. 7.2]. \square

It turns out that the structure of sets $S \subseteq \{0, 1\}^n$ with small pitch p can be efficiently exploited with respect to certain optimization tasks. For instance, the p -th level of the Bienstock-Zuckerberg hierarchy [1] gives a tight description of $\text{conv}(S)$, at least when applied to sets S of set-covering type. A much simpler observation is that linear programming over S is easy if p is constant.

2.2. Optimization algorithm for small pitch. Let $S \subseteq \{0, 1\}^n$ have pitch p . Assume that we have an oracle for deciding if a given point $x \in \{0, 1\}^n$ belongs to S . Here, we prove that optimizing a linear function over S can be done after performing at most $\mathcal{O}(n^p)$ oracle calls, and spending an extra polynomial time to select an optimum solution.

The algorithm is as follows. Given a cost vector $c \in \mathbb{R}^n$, we let $x^* \in \{0, 1\}^n$ be defined as $x_i^* := 0$ if $c_i \geq 0$ and $x_i^* := 1$ if $c_i < 0$. Note that this is an optimum solution of $\min\{c^\top x \mid x \in \{0, 1\}^n\}$. Next, among all the vertices of the cube $x \in \{0, 1\}^n$ that are at Hamming distance at most p from x^* , output any vertex x that belongs to S and has minimum cost.

Proposition 4. *For every $S \subseteq \{0, 1\}^n$ with pitch p and every $c \in \mathbb{R}^n$, the algorithm described above solves $\min\{c^\top x \mid x \in S\}$ in $\mathcal{O}(n^p)$ oracle calls.*

Proof. Clearly, the number of oracle calls performed by the algorithm is at most

$$\sum_{k=0}^p \binom{n}{k} \leq (n + 1)^p = \mathcal{O}(n^p).$$

There is always a feasible solution $x \in \{0, 1\}^n$ at Hamming distance at most p from x^* , since otherwise there would exist a p -dimensional face of the cube that is disjoint from S , which contradicts that the pitch of S is p . Therefore, the algorithm always outputs some feasible solution.

In order to finish proving the correctness of the algorithm, consider an optimum solution x^{opt} in S whose Hamming distance $d_H(x^{\text{opt}}, x^*)$ to x^* is minimum. Let $I := \{i \in [n] \mid x_i^{\text{opt}} \neq x_i^*\}$ be the set of indices of bits of x^* that are flipped in x^{opt} , so that we can express the optimum value as

$$\text{OPT} = c^\top x^{\text{opt}} = c^\top x^* + \sum_{i \in I} |c_i|.$$

Now consider the set F of vertices $x \in \{0, 1\}^n$ that are obtained by flipping the bits of x^* indexed by some set $J \subseteq I$. Thus, $F = \{x \in \{0, 1\}^n \mid \forall i \in [n] \setminus I : x_i = x_i^* = x_i^{\text{opt}}\}$. Clearly, F is the vertex set of some face of the cube. Every $x \in F$ has cost at most OPT since we have

$$c^\top x = c^\top x^* + \sum_{i \in J} |c_i| \leq c^\top x^* + \sum_{i \in I} |c_i| = c^\top x^{\text{opt}} = \text{OPT}.$$

By minimality of $d_H(x^{\text{opt}}, x^*)$, no $x \in F \setminus \{x^{\text{opt}}\}$ belongs to S . Thus, $d_H(x^{\text{opt}}, x^*) \leq p$ (otherwise, F would contain a p -dimensional face of $[0, 1]^n$ disjoint from S), and x^{opt} is one of the feasible solutions considered by the algorithm. The result follows. \square

¹In the sense that $R' \supseteq Q'$ for all polytopes $Q \subseteq [0, 1]^n$ such that $Q \cap \{0, 1\}^n = R \cap \{0, 1\}^n = S$.

2.3. Small CG-rank implies small gap. One might wonder whether sets $S \subseteq \{0, 1\}^n$ with small pitch are already simple enough to ensure that every relaxation for S contained in $[0, 1]^n$ has small CG-rank. However, it turns out that such sets S also need to have a description with bounded coefficients only, as illustrated by the next two results. Here, we denote by $\|A\|_\infty$ the maximum absolute value of an entry of A .

Lemma 5. *Let $P = \{x \in \mathbb{R}^n \mid Ax \geq b\}$, where $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$. Letting P' denote the first CG-closure of P , there is a description $P' = \{x \in \mathbb{R}^n \mid Bx \geq c\}$ with B and c integer such that $\|B\|_\infty \leq n\|A\|_\infty$.*

Proof. Every valid inequality for P' can be written as $\lambda^\top Ax \geq \lceil \lambda^\top b \rceil$ for some $\lambda \in \mathbb{R}_+^m$. By Carathéodory's theorem, we may assume that λ has at most n non-zero entries. Furthermore, it is well known that one can replace every entry of λ by its non-integral part to obtain an inequality that is valid for P' and at least as strong as the original one (see, e.g., [3, Lem. 5.13]). In other words, we may assume that $\lambda \in [0, 1]^m$ and λ has at most n non-zero entries. By the triangle inequality, this implies

$$\|\lambda^\top A\|_\infty = \left\| \sum_{i:\lambda_i \neq 0} \lambda_i A_i \right\|_\infty \leq \sum_{i:\lambda_i \neq 0} \underbrace{\lambda_i \|A_i\|_\infty}_{\leq \|A\|_\infty} \leq n\|A\|_\infty,$$

and the lemma follows. \square

Proposition 6. *Let $S \subsetneq \{0, 1\}^n$ be nonempty with gap Δ . Then there exists a polytope $R \subseteq [0, 1]^n$ with $R \cap \{0, 1\}^n = S$ such that the CG-rank of R is at least $\frac{\log \Delta}{\log n} - 1$.*

Proof. First, we claim that every integer matrix A for which there is an integer vector b with $\text{conv}(S) = \{x \in \mathbb{R}^n \mid Ax \geq b\}$ satisfies $\|A\|_\infty \geq \frac{\Delta}{n}$. Indeed, every inequality in such a description is of the form

$$\sum_{i \in I} c_i x_i - \sum_{j \in [n] \setminus I} c_j x_j \geq \beta$$

where $I \subseteq [n]$, $c \in \mathbb{Z}_{\geq 0}^n$, $\beta \in \mathbb{Z}$. Letting $\delta := \beta + \sum_{j \in [n] \setminus I} c_j$ we can rewrite this inequality as

$$\sum_{i \in I} c_i x_i + \sum_{j \in [n] \setminus I} c_j (1 - x_j) \geq \delta.$$

By the definition of Δ , for at least one of these inequalities we must have $\delta = \Delta$. Since $\text{conv}(S) \subseteq [0, 1]^n$ is nonempty, this implies $\|c\|_\infty \geq \frac{\Delta}{n}$.

Second, consider the polytope $R := \{x \in [0, 1]^n \mid \forall a \in \bar{S} : \sum_{i:a_i=0} x_i + \sum_{j:a_j=1} (1 - x_j) \geq 1\}$. Note that $R \subseteq [0, 1]^n$ and $R \cap \{0, 1\}^n = S$. Furthermore, R has a description of the form $R = \{x \in \mathbb{R}^n \mid Cx \geq d\}$ with C, d integer and $\|C\|_\infty = 1$. Thus, letting k be the CG-rank of R and in view of Lemma 5, we obtain $n^k \geq \frac{\Delta}{n}$, which yields the claim. \square

Let $S \subseteq \{0, 1\}^n$ with pitch p and gap Δ , and denote by k the largest CG-rank of a polytope $R \subseteq [0, 1]^n$ with $R \cap S = \{0, 1\}^n$. Summarizing the previous observations, we have seen that k can be bounded from below in terms of p (Proposition 3), and also in terms of Δ and n (Proposition 6). This explains the occurrence of both parameters in the statement of Theorem 1.

In what follows next, we would like to discuss that none of the two parameters p and Δ can be bounded by a function that only depends on the other. To see that p cannot be bounded by a function in Δ , observe that the set $S = \{x \in \{0, 1\}^n \mid x_p + x_{p+1} + \dots + x_n \geq 1\}$ has pitch p and gap 1.

2.4. Bounded Pitch Does Not Imply Bounded CG-rank. Next, we show that neither the parameter Δ nor the CG-rank can be bounded in terms of p alone.

Proposition 7. *For each $n \in \mathbb{N}$, there exists $S_n \subseteq \{0, 1\}^{2n+2}$ such that S_n has pitch at most 7 but gap at least 2^{n+1} .*

Proof. Fix $n \in \mathbb{N}$. We define a vector $c \in \mathbb{R}^{2n+2}$ by setting $c_1 = 2^n$, $c_2 = 2^{n-1}$, $c_i = c_{i-1}$ if $i \in [3, 2n+1]$ is odd, $c_i = (2^n - c_{i-1})/2$ if $i \in [3, 2n+1]$ is even, and $c_{2n+2} = 2^n - c_{2n+1}$. Now consider the inequality $\sum_{i=1}^{2n+2} c_i x_i \geq 2^{n+1}$, and let S_n be the set of vectors in $\{0, 1\}^{2n+2}$ for which this inequality is satisfied.

By definition, $\sum_{i=1}^{2n+2} c_i x_i \geq 2^{n+1}$ is a valid inequality for $\text{conv}(S_n)$. We claim that it is actually a facet of $\text{conv}(S_n)$. This follows by observing that $c_1 + c_2 + c_3 = 2^{n+1}$, $c_1 + c_{2i-1} + c_{2i} + c_{2i+1} = c_1 + c_{2i-2} + c_{2i} + c_{2i+1} = 2^{n+1}$ for all $i \in [2, n]$, $c_1 + c_{2n} + c_{2n+2} = c_1 + c_{2n+1} + c_{2n+2} = 2^{n+1}$ and $c_2 + c_3 + c_{2n+1} + c_{2n+2} = 2^{n+1}$.

Note that the greatest common divisor of the entries of c is 1 since c_1 is a power of 2 and c_{2n} is odd. Since all c_i are non-negative, this implies that the gap of S_n is at least 2^{n+1} .

Finally, we show that S_n has pitch at most 7. That is, we must show that the 7 smallest entries of c sum to at least 2^{n+1} . This is easily checked by hand if $n < 8$, so we may assume $n \geq 8$. By solving a linear recurrence of degree 1, we find that $c_{2i} = c_{2i+1} = 2^n \cdot (1 - (-1/2)^i)/3$ for $i \in [1, n]$. It follows that the 7 smallest entries of c are $c_4, c_5, c_8, c_9, c_{12}, c_{13}$, and c_{16} . The sum of these entries is

$$2^n \cdot \left(\frac{1}{4} + \frac{1}{4} + \frac{5}{16} + \frac{5}{16} + \frac{21}{64} + \frac{21}{64} + \frac{85}{256} \right) = 2^n \cdot \frac{541}{256} > 2^n \cdot 2 = 2^{n+1},$$

as required. \square

By Proposition 6 and Proposition 7 we directly obtain.

Corollary 8. *For each n , there exists a polytope $R \subseteq [0, 1]^{2n+2}$ such that $S = R \cap \{0, 1\}^{2n+2}$ has pitch at most 7, but the CG-rank of R is $\Omega(\frac{n}{\log n})$.*

2.5. Bounded Treewidth Implies Bounded Pitch and Gap. Finally, we demonstrate that Theorem 1 can indeed be seen as a generalization of the results of Cornuéjols & Lee [5]. To this end, it suffices to show that p and Δ can be bounded in terms of the treewidth of $H[\bar{S}]$. Recall that the largest t such that $H[\bar{S}]$ contains a subdivision of K_{t+1} is at most $\text{tw}(H[\bar{S}])$.

Lemma 9. *Let $S \subseteq \{0, 1\}^n$, and let p and Δ respectively denote the pitch and the gap of S . If t is maximum such that $H[\bar{S}]$ contains a subdivision of K_{t+1} , then $p \leq t + 1$ and $\Delta \leq 2t^{t/2}$.*

Proof. Note that the d -dimensional cube contains a subdivision of K_{d+1} , where the branch vertices are the vectors with support at most 1, and the subdivision vertices are the vectors with support 2. Now, since $H[\bar{S}]$ contains a subgraph isomorphic to the $(p-1)$ -dimensional cube, it contains a subdivision of K_p and we have $t \geq p-1$.

To show $\Delta \leq 2t^{t/2}$, observe that it suffices to prove the following. For any hyperplane $H := \{x \in \mathbb{R}^n \mid \sum_{i \in I} c_i x_i + \sum_{j \in [n] \setminus I} c_j (1 - x_j) = 1\}$ that is spanned by 0/1-points such that $\sum_{i \in I} c_i x_i + \sum_{j \in [n] \setminus I} c_j (1 - x_j) \geq 1$ is valid for S and $c_1, \dots, c_n \in \mathbb{Q}_{>0}$, there exists some integer number $K \in [1, 2t^{t/2}]$ such that every c_i is an integer multiple of $1/K$.

By switching the coordinates indexed by $[n] \setminus I$, we may assume that $I = [n]$. Define $I_{<1/2} := \{i \in [n] \mid c_i < 1/2\}$, and $I_{=1/2}, I_{>1/2}$ similarly. We have that $|I_{<1/2}| \leq t$ since otherwise $H[\bar{S}]$ contains a subdivision of a clique of size $t+2$ whose branch vertices are the characteristic vectors of the empty set \emptyset and the singletons $\{i\}$ for $i \in I_{<1/2}$ and whose subdivision vertices are the characteristic vectors of the pairs $\{i, j\}$ for $i, j \in I_{<1/2}$.

Let $x \in \{0, 1\}^n \cap H$ and denote by T its support. Then one of the following holds: (i) $|T \cap I_{=1/2}| = |T \cap I_{>1/2}| = 0$, (ii) $|T \cap I_{=1/2}| = 1$ and $|T \cap I_{>1/2}| = 0$, (iii) $|T \cap I_{=1/2}| = 0$ and $|T \cap I_{>1/2}| = 1$, or (iv) $|T \cap I_{=1/2}| = 2$ and $|T \cap I_{>1/2}| = 0$. Thus, the vector c is the unique solution of a system of linear equations of the following form

$$\begin{pmatrix} A \\ B & * \\ C & & D \\ & & & \mathbb{I} \end{pmatrix} c = b,$$

where the coefficient matrix has integer entries, A, B, C are 0/1-matrices with columns indexed by $I_{<1/2}$, \mathbb{I} is an identity matrix with columns indexed by $I_{=1/2}$, D is a 0/1-matrix with columns

indexed by $I_{>1/2}$ and exactly one 1 per row, and b is a column vector with entries in $\{1/2, 1\}$. The last rows of the above system are meant to be the trivial equations $c_i = 1/2$, which are obviously valid for all $i \in I_{=1/2}$. Since every row in D contains exactly one 1, we can perform elementary row operations to obtain an equivalent system of the form

$$\begin{pmatrix} E & * \\ * & \mathbb{I} \\ & \mathbb{I} \end{pmatrix} c = b',$$

where the coefficient matrix has integer entries, E is a matrix with entries in $\{-1, 0, 1\}$ and columns indexed by $I_{<1/2}$, and b' a column vector with entries in $\{0, 1/2, 1\}$. By removing some rows in the topmost block, we may assume that the coefficient matrix is a regular $n \times n$ -matrix whose determinant is $\pm \det(E)$. Thus, by Cramer's rule, every c_i is an integer multiple of $\frac{1}{2|\det(E)|}$. Since E is a matrix with entries in $\{-1, 0, 1\}$ and $|I_{<1/2}| \leq t$ columns and rows, by the Hadamard bound we obtain

$$K = 2|\det(E)| \leq 2t^{\frac{1}{2}t},$$

as claimed. \square

3. PROOF OF MAIN THEOREM

Lemma 10. *Let $R \subseteq [0, 1]^n$ be a polytope and $I, J \subseteq [n]$ with $I \cap J = \emptyset$ such that*

$$(3) \quad \sum_{i \in I} x_i + \sum_{j \in J} (1 - x_j) \geq 1$$

holds for every $x \in R \cap \{0, 1\}^n$. Then (3) is also valid for $R^{(n+1-(|I|+|J|))}$.

Proof. Consider the set

$$F := \{x \in R \mid x_i = 0 \forall i \in I, x_j = 1 \forall j \in J\},$$

which is a face of R of dimension $k \leq n - (|I| + |J|)$. Since (3) is valid for $R \cap \{0, 1\}^n$, we have $F \cap \mathbb{Z}^n = \emptyset$. Since $F \subseteq [0, 1]^n$, this implies $F^{(k)} = \emptyset$ (by [9, Lem. 2.2]). This implies $R^{(k)} \cap F = \emptyset$ (see [3, Lem. 5.17]) and hence there exists an $\varepsilon > 0$ such that

$$\sum_{i \in I} x_i + \sum_{j \in J} (1 - x_j) \geq \varepsilon$$

is valid for $R^{(k)}$. This means that (3) holds for $R^{(k+1)}$, as claimed. \square

Proof of Theorem 1. By the definition of Δ , we can find a description of $\text{conv}(R \cap \{0, 1\}^n)$ by means of linear inequalities where every inequality is of the form

$$(4) \quad \sum_{i \in I} c_i x_i + \sum_{j \in J} c_j (1 - x_j) \geq \delta$$

for some $I, J \subseteq [n]$ with $I \cap J \neq \emptyset$, where $\delta \in \mathbb{Z}_{\geq 0}$, $c_i \in \mathbb{Z}_{\geq 1}$ for all $i \in I \cup J$, and $\delta \leq \Delta$. Note that every such inequality with $\delta = 0$ is already valid for R . For inequalities with $\delta \geq 1$, we may assume that $c_i \leq \delta$ holds for every $i \in I \cup J$.

By induction on $\delta \geq 1$ we will show that (4) holds for every $x \in R^{(p+\delta-1)}$, which then yields the claim. If $\delta = 1$, then we have $c_i = 1$ for all $i \in I \cup J$. By Lemma 10, we know that Inequality (4) is valid for $R^{(t)}$, where $t = n + 1 - (|I| + |J|)$. It remains to show that $t \leq p + \delta - 1 = p$. To this end, consider the set

$$F = \{x \in [0, 1]^n \mid x_i = 0 \forall i \in I, x_j = 1 \forall j \in J\},$$

which is a face of the cube, and note that no point of F satisfies (4). Thus, we indeed obtain $p \geq \dim(F) + 1 = n + 1 - (|I| + |J|) = t$.

Now let $\delta \geq 2$. We may assume that $|I| + |J| \geq 1$, otherwise we can divide (4) by δ and proceed by induction. For every $i_0 \in I$ consider the inequality

$$\sum_{i \in I \setminus \{i_0\}} c_i x_i + (c_{i_0} - 1)x_{i_0} + \sum_{j \in J} c_j (1 - x_j) \geq \delta - x_{i_0} \geq \delta - 1,$$

which is valid for $R \cap \{0, 1\}^n$. Similarly, for every $j_0 \in J$ the inequality

$$\sum_{i \in I} c_i x_i + \sum_{j \in J \setminus \{j_0\}} c_j (1 - x_j) + (c_{j_0} - 1)(1 - x_{j_0}) \geq \delta - (1 - x_{j_0}) \geq \delta - 1$$

is also valid for $R \cap \{0, 1\}^n$. Thus, by the induction hypothesis, both such inequalities are valid for $R^{(p+\delta-2)}$. Summing these $k := |I| + |J|$ many inequalities up and dividing them by $k \geq 1$, we obtain that

$$\sum_{i \in I} \left(c_i - \frac{1}{k}\right) x_i + \sum_{j \in J} \left(c_j - \frac{1}{k}\right) (1 - x_j) \geq \delta - 1$$

is valid for $R^{(p+\delta-2)}$. Choose $\varepsilon > 0$ such that $(1 + \varepsilon)(c_i - \frac{1}{k}) \leq c_i$ holds for all $i \in I \cup J$. Scaling the above inequality by $(1 + \varepsilon)$, we thus obtain that

$$\begin{aligned} \sum_{i \in I} c_i x_i + \sum_{j \in J} c_j (1 - x_j) &\geq (1 + \varepsilon) \left(\sum_{i \in I} \left(c_i - \frac{1}{k}\right) x_i + \sum_{j \in J} \left(c_j - \frac{1}{k}\right) (1 - x_j) \right) \\ &\geq (1 + \varepsilon)(\delta - 1), \end{aligned}$$

holds for every $x \in R^{(p+\delta-2)}$, and hence

$$\sum_{i \in I} c_i x_i + \sum_{j \in J} c_j (1 - x_j) \geq \lceil (1 + \varepsilon)(\delta - 1) \rceil \geq \delta$$

is valid for $R^{(p+\delta-1)}$, as claimed. \square

4. APPROXIMATING THE INTEGER HULL WHEN THE PITCH IS BOUNDED

We have shown in Section 2.4 that if we only assume that p is constant, it might take $\Omega(n/\log n)$ rounds of CG-cuts to converge to the integer hull: we have to control Δ also in order to guarantee bounded CG-rank. Here we prove that bounding p alone is in fact enough to obtain good *approximations* of the integer hull after a bounded number of rounds. This is in contrast with the results of Singh & Talwar [11], who show that for many problems performing a constant number of rounds of CG-cuts does not significantly decrease the integrality gap.

Corollary 11. *Let $S \subseteq \{0, 1\}^n$ have pitch p and let $\varepsilon \in (0, 1)$ be such that $p\varepsilon^{-1} \in \mathbb{Z}_{\geq 0}$. For every $t \geq p\varepsilon^{-1} - 1$ and for every inequality $\sum_{i \in I} c_i x_i + \sum_{j \in J} c_j (1 - x_j) \geq \delta$ that is valid for $\text{conv}(S)$ with $\delta \geq c_1, \dots, c_n \geq 0$, the inequality $\sum_{i \in I} c_i x_i + \sum_{j \in J} c_j (1 - x_j) \geq (1 - \varepsilon)\delta$ is valid for $R^{(t)}$, where $R \subseteq [0, 1]^n$ is any polytope such that $R \cap \{0, 1\}^n = S$.*

Proof. After flipping some coordinates, we may assume that $J = \emptyset$. After scaling, we may further assume that $\delta = 1$. Let $K := p\varepsilon^{-1}$. Consider the valid inequality $\sum_{i \in I} \tilde{c}_i x_i \geq \tilde{\delta}$ where $\tilde{c}_i := \frac{1}{K} \lfloor K c_i \rfloor \in \{0, 1/K, 2/K, \dots, 1\}$ and $\tilde{\delta} := \min\{\sum_{i \in I} \tilde{c}_i x_i \mid x \in S\}$. We claim that $\tilde{\delta} \leq 1 - \varepsilon$. Indeed, let $x \in S$ be arbitrary and let $y \in S$ be such that $0 \leq y \leq x$ and y has support on at most p coordinates. Then

$$\begin{aligned} \sum_{i \in I} \tilde{c}_i x_i &\geq \sum_{i \in I} \tilde{c}_i y_i \\ &= \sum_{i \in I} \frac{1}{K} \lfloor K c_i \rfloor y_i \\ &\geq \sum_{i \in I} \frac{1}{K} (K c_i - 1) y_i \\ &\geq \sum_{i \in I} c_i y_i - \frac{p}{K} \\ &\geq 1 - \frac{p}{K} = 1 - \varepsilon \end{aligned}$$

so that $\delta \leq 1 - \varepsilon$. Now consider the valid inequality $\sum_{i \in I} K \tilde{c}_i x_i \geq K(1 - \varepsilon) = K - p$ with nonnegative integer coefficients. From the proof of Theorem 1, we see that this inequality is valid for the t -th CG-closure of R since $t = K - 1 = (K - p) + p - 1$. \square

5. THE PITCH-3 CASE

Theorem 12. *Let $S \subseteq \{0, 1\}^n$ have pitch $p \leq 3$. Then $P = \text{conv}(S)$ can be defined by $0 \leq x_i \leq 1$ for $i \in [n]$ together with inequalities that can be brought in the following form after flipping some coordinates, where for each inequality the subsets of indices are a partition of $[n]$ (we allow empty sets in the partition):*

$$\begin{aligned}
 (5) \quad & \sum_{i \in I_0} 0x_i + \sum_{i \in I_1} 1x_i \geq 1, & |I_0| \leq 2 \\
 (6) \quad & \sum_{i \in I_0} 0x_i + \sum_{i \in I_1} 1x_i + \sum_{i \in I_2} 2x_i \geq 2, & |I_0| \leq 1 \\
 (7) \quad & \sum_{i \in I_1} 1x_i + \sum_{i \in I_2} 2x_i + \sum_{i \in I_3} 3x_i \geq 3, & |I_1| \geq 3 \\
 (8) \quad & \sum_{i \in I_1} 1x_i + \sum_{i \in I_2} 2x_i + \sum_{i \in I_3} 3x_i + \sum_{i \in I_4} 4x_i \geq 4, & |I_1| = 2, |I_2| \geq 1 \\
 (9) \quad & \sum_{i \in I_2} 2x_i + \sum_{i \in I_3} 3x_i + \sum_{i \in I_4} 4x_i + \sum_{i \in I_6} 6x_i \geq 6, & |I_2| \geq 3
 \end{aligned}$$

In particular, S has gap $\Delta \leq 6$.

Proof. We may assume that $n \geq 3$, otherwise the theorem holds trivially. Thus, S is nonempty. Pick any nonredundant inequality description of $\text{conv}(S)$ such that the corresponding hyperplanes are spanned by 0/1-points. Let $c^\top x \geq \delta$ be any inequality in this description which is not of the form $x_i \geq 0$ or $1 - x_i \geq 0$. By flipping coordinates and scaling we may assume that $c_i \in \mathbb{Q}_{\geq 0}$ and $\delta = 1$. We choose a non-redundant system that uniquely defines c consisting of equations of the form $c_i^* = 0, c_i^* - c_j^* = 0$, and $\sum_{i \in I \subseteq [n]} c_i^* = 1$ such that equations of lower support are always included before equations of higher support. In particular, this implies that equations of the form $c_i^* = 0$ or $c_i^* = 1$ are always included if $c_i = 0$ or $c_i = 1$.

Sort the entries of c as $c_1 \leq c_2 \leq \dots \leq c_n$. Clearly, since S has pitch at most 3, $c_1 + c_2 + c_3 \geq 1$. Hence, if any equation has support greater than 3, then it is already implied by $c_1^* + c_2^* + c_3^* = 1$ together with equations of the form $c_i^* = 0$. Thus, no equation with support greater than 3 appears. If $c_1 + c_2 + c_3 = 1$, then any equation whose support has size 3 is already implied by the equation $c_1^* + c_2^* + c_3^* = 1$ together with equations of the form $c_\ell^* - c_m^* = 0$ for $\ell \in [3]$. If $c_1 + c_2 + c_3 > 1$, then no equation of the form $c_i^* + c_j^* + c_k^* = 1$ will appear. Thus, at most one equation of support 3 appears.

Define a graph $G = ([n], E)$, where $E := \{ij : c_i^* + c_j^* = 1 \text{ or } c_i^* - c_j^* = 0\}$. Let $\Sigma := \{ij : c_i^* + c_j^* = 1\}$ and define a cycle of G to be *unbalanced* if it contains an odd number of edges of Σ . Since the system is non-redundant, each component of G contains at most one cycle, which will have to be unbalanced. For each $\gamma \in [0, 1]$, let $J_\gamma := \{i \in [n] : c_i = \gamma\}$. Note that $|J_0| \leq 2$, as $c_1 + c_2 + c_3 \geq 1$. Let $J'_{\frac{1}{2}}$ be the set of vertices of G contained in a component with an unbalanced cycle. Clearly, $J'_{\frac{1}{2}} \subseteq J_{\frac{1}{2}}$.

Let T_1, \dots, T_ℓ be the components of G which contain at least one edge and no cycles. Note that if $\ell \geq 2$, then the set of solutions of the system (minus the single equation of the form $c_i^* + c_j^* + c_k^* = 1$) has dimension at least 2. Thus, the solution set of the full system has dimension at least 1, which contradicts the uniqueness of c . Therefore, $\ell \leq 1$. We may partition the vertices of T_1 as $J'_\alpha \cup J'_\beta$ where $c_i = \alpha$ for all $i \in J'_\alpha$ and $c_i = 1 - \alpha := \beta$ for all $i \in J'_\beta$. Note that if $\alpha = 0$, then $J'_\alpha \subseteq J_0$ and $J'_\beta \subseteq J_1$, and if $\alpha = \frac{1}{2}$, then $J'_\alpha \cup J'_\beta \subseteq J_{\frac{1}{2}}$.

It follows that $[n] := J_0 \cup J_\alpha \cup J_{\frac{1}{2}} \cup J_\beta \cup J_1$, for some $0 < \alpha < \frac{1}{2}$ and $\beta := 1 - \alpha$ (some of these sets are possibly empty). There are now various cases to consider depending on where the indices of the single equation $c_i^* + c_j^* + c_k^* = 1$ belong.

First suppose that there does not exist an equation of the form $c_i^* + c_j^* + c_k^* = 1$. In this case, by the uniqueness of c , we must have $J_\alpha = J_\beta = \emptyset$. If $|J_0| = 2$, then $J_{\frac{1}{2}} = \emptyset$ and $J_1 \neq \emptyset$, so we get (5) with $(I_0, I_1) = (J_0, J_1)$. If $|J_0| \leq 1$, we get (6) with $(I_0, I_1, I_2) = (J_0, J_{\frac{1}{2}}, J_1)$.

We may hence assume there does exist an equation of the form $c_i^* + c_j^* + c_k^* = 1$ (with $i < j < k$). We may further assume that $\{i, j, k\} \cap J_0 = \emptyset$, because otherwise, the equation $c_i^* + c_j^* + c_k^* = 1$ is implied by the lower support equations $c_i^* = 0$ and $c_j^* + c_k^* = 1$. Similarly, $\{i, j, k\} \cap J_1 = \emptyset$.

Suppose $\{i, j, k\} \subseteq J_\alpha$. This implies that $\alpha = \frac{1}{3}$ and, since $c_1 + c_2 + c_3 \geq 1$, $J_0 = \emptyset$ and $|J_{\frac{1}{3}}| \geq 3$. If $J_{\frac{1}{2}} = \emptyset$ then we get (7) with $(I_1, I_2, I_3) = (J_{\frac{1}{3}}, J_{\frac{2}{3}}, J_1)$. If $J_{\frac{1}{2}} \neq \emptyset$, then we get (9) with $(I_2, I_3, I_4, I_6) = (J_{\frac{1}{3}}, J_{\frac{1}{2}}, J_{\frac{2}{3}}, J_1)$.

Suppose $\{i, j\} \subseteq J_\alpha$ and $k \in J_{\frac{1}{2}}$. This implies $\alpha = \frac{1}{4}$, and since $c_1 + c_2 + c_3 \geq 1$, we have $J_0 = \emptyset$, $|J_\alpha| = 2$, and $J_{\frac{1}{2}} \geq 1$. So, we get (8) with $(I_1, I_2, I_3, I_4) = (J_{\frac{1}{4}}, J_{\frac{1}{2}}, J_{\frac{3}{4}}, J_1)$.

Suppose $\{i, j\} \subseteq J_\alpha$ and $k \in J_\beta$. This implies $2\alpha + (1 - \alpha) = 1$, and so $\alpha = 0$. This contradicts $\alpha > 0$.

Finally if $|\{i, j, k\} \cap J_\alpha| \leq 1$, then $c_i + c_j + c_k > 1$, which is a contradiction. \square

Applying Theorem 1, we obtain the following result.

Corollary 13. *Let $S \subseteq \{0, 1\}^n$ be a set with pitch at most 3. Then the CG-rank of every polytope $R \subseteq [0, 1]^n$ with $R \cap \{0, 1\}^n = S$ is at most 8.*

Note that when $\text{tw}(H[\bar{S}]) \leq 2$, none of the inequalities (7), (8), or (9) can appear in the linear description of $\text{conv}(S)$ because for each of them there is a set of indices $I \subseteq [n]$ of size 3 such that the characteristic vector of every proper subset of I is in \bar{S} . This implies that $H[\bar{S}]$ contains a subdivision of K_4 . Hence, we recover the same upperbound of 4 on the CG-rank when $\text{tw}(H[\bar{S}]) \leq 2$ established by Cornuéjols & Lee [5]. On the other hand, the pitch 3 case includes graphs of unbounded treewidth. For example, if we let $S \subseteq \{0, 1\}^n$ be the set of vectors of support at least 3, then S has pitch 3 and $H[\bar{S}]$ contains a subdivision of K_{n+1} .

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