Recent Developments in Disjunctive Programming

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Recent Developments in Disjunctive Programming

1. Background and basic results
2. Convexification and extended formulations
3. L&P cuts from split disjunctions
4. General (non-split, multiple-term) disjunctions
5. The convex hull in $\mathbb{R}^n$

(Parts 4-5 based on joint work with, respectively, Tamas Kis and Aleksandr Kazachkov)

1. Background and basic results

- Convexity – the dividing line between "tame" and "wild" problems
- Nonconvex sets can often be modeled via Integer Programming — which is $\mathcal{NP}$-complete
- Disjunctive programming: optimization over *disjunctive sets*, defined by inequalities joined by connectives $\wedge, \vee, \Rightarrow, \neg$ (nonconvex because of $\vee$)
- The largest known class of nonconvex sets convexifiable in polynomial time
- Many equivalent forms, two extreme:
  - CNF: intersection of elementary disjunctive sets

$$F = \bigwedge_{j \in T} S_j, \quad S_j = \{x : \bigvee_{i \in Q_j} (a^i x \geq a_{i0})\}, \quad j \in T$$

  - DNF: union of polyhedra

$$F = \bigcup_{i \in Q} P^i, \quad P^i = \{x : A^i x \geq b_i\}, \quad i \in Q$$

Any disjunctive set can be brought to either form by a sequence of simple steps.

Disjunctive sets in DNF:

Unions of polyhedra are a large class of nonconvex sets with a compact convex representation.

The union of q polyhedra in $\mathbb{R}^n$ (= the disjunction between q systems of linear inequalities in n variables) has a convex hull representation as a polyhedron in $\mathbb{R}^{(qn)}$ (= a linear system in $O(qn)$ variables).

Disjunctive sets in CNF:

If facial, any such set can be convexified *sequentially*, i.e. by imposing the elementary disjunctions one at a time, each time generating the convex hull of the current set:

0-1 MIP's are *facial*, (general) MIP's are *not*.

This is *the* basic property distinguishing 0-1 programs from general integer programs.

The convex hull of a disjunctive set

(E.B., Disjunctive Programming, CMU MSRR 348, 1974, *Discrete Applied Math*, 1998)

Basic observation

An inequality $\alpha x \geq \alpha_0$ is valid for the disjunction

$$\bigvee_{i \in Q} (A^i x \geq b^i)$$

if and only if it is valid for each system $A^i x \geq b^i$, $i \in Q$.

With this in mind, we have

Theorem 1.1 Farkas' Lemma for disjunctive sets

Let $F = \bigvee_{i \in Q} (A^i x \geq b^i)$, and $Q^* := \{i \in Q : \{x : A^i x \geq b^i\} \neq \emptyset\}$. The inequality $\alpha x \geq \beta$ is satisfied by all $x \in F$ if and only if there exist vectors $u^i \in \mathbb{R}^m$, $u^i \geq 0$, such that $\alpha = u^i A^i$, $\beta \leq u^i b^i$, $i \in Q^*$.

Thus, an elementary disjunction of the form $\bigvee_{i \in Q} (a^i x \geq b^i)$, where $a^i \in \mathbb{R}^n$, $b^i > 0$, $i \in Q$, and each term is feasible, implies the valid inequality

$$\sum_{j=1}^{n} \max_{i \in Q} \left\{ \frac{a^i}{b^i} \right\} x_j \geq 1.$$

which is a weakening of each $a^i x \geq b^i$.

Theorem 1.2 Let $F = \bigcup_{i \in Q} P^i$, $P^i = \{x \in \mathbb{R}^n : A^i x \geq b^i\}$, $i \in Q$, and let $Q^* = \{i \in Q : P^i \neq \emptyset\}$. Then

$$
\begin{aligned}
\operatorname{conv} F = \{x \in \mathbb{R}^n : x \; - \sum_{i \in Q^*} y^i \qquad\qquad &= \; 0 \\
A^i y^i \; - b^i y_0^i \; &\geq \; 0 \\
y_0^i \; &\geq \; 0 \qquad i \in Q^* \qquad\qquad (1.1) \\
\sum_{i \in Q^*} y_0^i \; &= \; 1 \\
\text{for some vectors } (y^i, y_0^i), i \in Q^*\}.
\end{aligned}
$$

Denoting $C^i := \{x : A^i x \geq 0\}$, $i \in Q$, Q^* can be replaced with Q if

$$\left(\bigcup_{i \in Q \setminus Q^*} C^i \right) \subseteq \left(\bigcup_{i \in Q^*} C^i \right)$$

To obtain the convex hull in $\mathbb{R}^n$, project the set $\mathcal{S}(Q)$ defined by (1.1) onto the x-space.

Projection cone:

$$W = \left\{ \begin{array}{rrrcll} (\alpha, \beta, \{u^i\}_{i \in Q}) : -\alpha & + & u^i A^i & = & 0 & \\ \beta & - & u^i b^i & \geq & 0 & i \in Q \\ & & u^i & \geq & 0 & \end{array} \right\}$$

Theorem 1.3 $\text{Proj}_x \mathcal{S}(Q) = \{x \in \mathbb{R}^n : \alpha x \geq \beta \text{ for all } (\alpha, \beta, \{u^i\}_{i \in Q}) \in \text{extr} W\}$

The inequalities $\alpha x \geq \beta$ can be generated as the (α, β)-components of extreme rays of W, i.e. by solving a system of $O(qn)$ variables.

The above representation of conv F was derived by using the H-polyhedral representation of F.

What about the V-polyhedral representation?

$$F = \bigcup_{i \in Q} P^i, \quad P^i = \text{conv}\, V^i + \text{cone}\, W^i, \quad i \in Q$$

Theorem 1.4 (M. Perregaard and E.B., IPCO 2001) The inequality $\gamma x \geq \delta$ is valid for F if and only if

$$\left.\begin{array}{l} \gamma p \geq \delta \text{ for all } p \in V^i \\ \gamma r \geq 0 \text{ for all } r \in W^i \end{array}\right\} i \in Q \tag{1.2}$$

Proof. $\gamma x \geq \delta$ is valid for F if and only if it is valid for all P_i, $i \in Q$. □

Thus conv F is defined by the inequalities $\gamma x \geq \delta$ corresponding to basic solutions (γ, δ) of the system (1.2).

The system (1.2) defines conv F in $\mathbb{R}^n$, but it has $\bigcup_{i \in Q}(|V^i| + |W^i|)$ inequalities.

Two consequences of the compact convex hull representation

(1) The fact that a nonconvex set in $\mathbb{R}^n$ can be described by a convex polyhedron in a higher dimensional space gave rise to *extended formulations*.

Two types of benefits:

(a) tighter LP relaxations
(b) integrality of higher dimensional representation proves integrality of original polyhedron

(2) The description of the convex hull of a disjunctive set by the inequalities corresponding to basic solutions of a CGLP has led to the study of lift-and-project cuts for MIP's. First to be studied were cuts from *split disjunctions*.

2. Convexification and extended formulations

(E.B., *SIAM J on Algebraic Discrete Methods*, 1985)

- A general technique for tightening formulations of MIP's: replace the " big M" representation of some disjunctive subset by its convex hull representation
- CNF and DNF are both *intersections of unions of polyhedra*

$$F = \bigcap_{j \in T} S_j, \quad S_j = \bigcup_{i \in Q} P_i, \quad j \in T \tag{2.1}$$

- Call (2.1) a *regular form* (RF)

Theorem 2.1 Any disjunctive set F in RF(2.1) can be brought to DNF by $|T| - 1$ applications of the following basic step, which preserves regularity: For some $k, \ell \in T$, bring $S_k \cap S_\ell$ to DNF, i.e. replace it with

$$S_{k\ell} = \cup_{\substack{i \in Q_k \\ j \in Q_\ell}} (P_i \cap P_j)$$

Given a disjunctive set in regular form (2.1), define the *hull-relaxation of* $F = \bigcap_{j \in T} S_j$ as h-rel $F = \bigcap_{j \in T} \operatorname{conv} S_j$

Theorem 2.2 For $i = 0, 1, \ldots, t$, let

$$F_i = \bigcap_{j \in T_i} S_j$$

be a sequence of regular forms, such that

(i) F_0 is in CNF

(ii) F_t is in DNF

(iii) for $i = 1, \ldots, t$, F_i is obtained from F_{i-1} by a basic step.

Then

$$h\text{-rel}\, F_0 \supseteq h\text{-rel}\, F_1 \supseteq \cdots \supseteq h\text{-rel}\, F_t = \operatorname{conv} F_t$$

Clearly,

$$\operatorname{conv} S_{k\ell} \subseteq \operatorname{conv} S_k \cap \operatorname{conv} S_\ell.$$

When to replace a disjunctive set in RF by its convex hull?

–When it makes $\operatorname{conv} S_{k\ell}$ *tighter* than $(\operatorname{conv} S_k) \cap (\operatorname{conv} S_\ell)$.

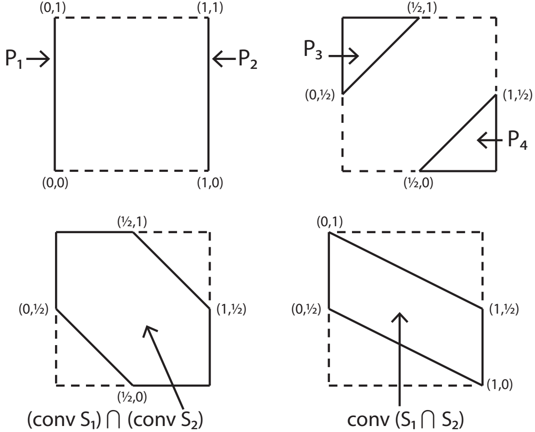
Theorem 2.3 Let $S_j = \underset{i \in Q_j}{\cup} P_i$, $j = 1, 2$. Then

$$\operatorname{conv}(S_1 \cap S_2) = (\operatorname{conv} S_1) \cap (\operatorname{conv} S_2)$$

if and only if every extreme point (direction) of $(\operatorname{conv} S_1) \cap (\operatorname{conv} S_2)$ is an extreme point (direction) of $P_i \cap P_k$ for some $(i, k) \in Q_1 \times Q_2$.

Example. Let P_i, $i = 1, \ldots, 4$ be as in the figure, and

$$F = S_1 \cap S_2, \quad S_1 = P_1 \cup P_2, \quad S_2 = P_3 \cup P_4$$



Then $\text{conv}\,(S_1 \cap S_2) \subsetneq (\text{conv}\, S_1) \cap (\text{conv}\, S_2)$ since vertices $(\frac{1}{2}, 0)$ and $(\frac{1}{2}, 1)$ of $\text{conv}\, S_1 \cap \text{conv}\, S_2$ are *not* vertices of either $P_1 \cap P_3$, $P_1 \cap P_4$, $P_2 \cap P_3$ or $P_2 \cap P_4$.

If a conjunct $S_j = \bigcup_{i \in Q_j} P_i$ of a RF $F_k = \bigcap_{j \in T} S_j$, is replaced with $\operatorname{conv} S_j$, we get

$$\left.\begin{array}{rcll} x - \sum_{i \in Q_j} y^i & = & 0 & \\ A^i y^i \; - \; b^i y_0^i & \geq & 0 & i \in Q_j \\ y_0^i & \geq & 0 & \\ \sum_{i \in Q_j} y_0^i & = & 1 & \end{array}\right\} j \in T \qquad (2.2)$$

In any solution nonbasic for the j-th subsystem, $y_0^i \notin \{0, 1\}$

Imposing $y_0^i \in \{0, 1\}$ for $i \in Q_j$, $j \in T$?

No: if the CNF is

$$F_0 = \bigwedge_{r \in T_0} S_r, \quad S_r = \bigvee_{s \in Q_r} (a_s x \geq b_s),$$

then replace the last equation of (2.2) by

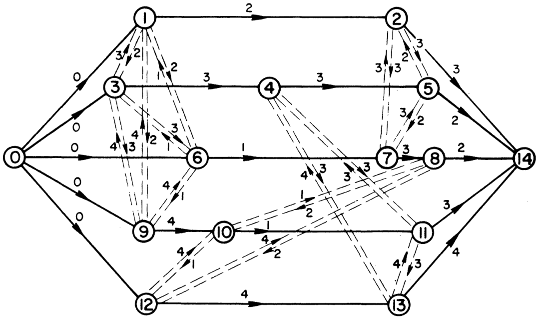
$$\begin{array}{rcllll} \sum_{i \in Q_j | s \in M_i} y_0^r - \delta^r & = & 0, & s \in Q_r & r \in T_0 & (M_i = \text{ row index set of } A^i) \\ \sum_{s \in Q_r} \delta_s^r & = & 1 & & r \in T_0 & \\ \delta_s^r \in \{0, 1\} & & & s \in Q_r, & r \in T_0 & \end{array}$$

i.e. we need the same number of 0-1 variables as in CNF.

Application: job shop scheduling with sequence-dependent setup times

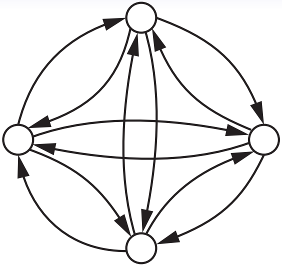
$$\begin{array}{lll} \min t_j & & \\ t_j - t_i \geq d_{ij} & & (i,j) \in A \\ t_j - t_i \geq d_{ij} \vee t_i - t_j \geq d_{ji} & & (i,j) \in E_k, \quad k \in M \end{array}$$

t_i = start time of job i
d_{ij} = duration of i + setup time for j
A = set of precedence arcs
M = set of machines
E_k = set of disjunctive pairs of arcs for machine k



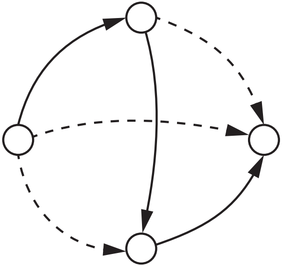
Machine-clique

- Selection: one of each pair
- Feasible: non-conflicting and acyclic (=tournament)
- Transitive closure of a directed Hamilton path



Finding an optimal jobshop schedule amounts to finding in each machine-clique an optimal directed Hamilton path subject to time window constraints

Tight formulation: path variables



Union of polytopes in different spaces

(E.B., A.Bockmayr, N. Pisaruk, L. Wolsey)

Convex hull description without additional variables.

Theorem 2.2. Let

$$Z = P \cup Q \quad Z := \{(x, y) : x \in P \text{ or } y \in Q\}$$

where P and Q are upper mononone polytopes, i.e.

$$P := \{x \in \{0,1\}^m : Ax \geq 1\}, \quad Q := \{y \in \{0,1\}^n : By \geq 1\},$$

with $A \geq 0$, $B \geq 0$, and let $M = \{1, \dots, m\}$, $N = \{1, \dots, n\}$. Then

$$\operatorname{conv} Z = \{(x, y) \in \{0,1\}^{m+n} : \sum_{j \in S} \frac{a_{ij}}{1 - \sum\limits_{h \in M \setminus S} a_{ih}} x_j + \sum_{\ell \in T} \frac{b_{k\ell}}{1 - \sum\limits_{h \in N \setminus T} b_{kh}} y_\ell \geq 1$$

for all $S \subseteq M$ s.t. $\sum\limits_{h \in M \setminus S} a_{ih} < 1$ and $T \subseteq N$ s.t. $\sum\limits_{h \in N \setminus T} a_{ih} < 1\}$.

Application: logical inference

$$\begin{aligned} \sum_{i\in M} x_i \geq k \quad &\Rightarrow \quad \sum_{j\in N} y_j \geq \ell \qquad M=\{1,\ldots,m\}, N=\{1,\ldots,n\} \\ \sum_{i\in M} x_i \leq k-1 \quad &\vee \quad \sum_{j\in N} y_j \geq \ell \\ \sum_{i\in M} \bar{x}_i \geq p \quad &\vee \quad \sum_{j\in N} y_j \geq \ell, \qquad \bar{x}_i = 1-x_i, \quad p = m-k+1 \end{aligned} \tag{2.3}$$

Denote $\bar{x}(S) = \sum_{i\in S} \bar{x}_i, \quad y(T) = \sum_{j\in T} y_j$

Theorem 2.3 The convex hull of points $(\bar{x}, y) \in \{0,1\}^{m+n}$ satisfying (2.3) is defined by

$$0 \leq \bar{x}_i \leq 1, \quad i \in M, \quad 0 \leq y_j \leq 1, \quad j \in N, \text{ and}$$

$$(|T| + \ell - n)\bar{x}(S) + (|S| + p - m)y(T) \geq (|S| + p - m)(|T| + \ell - n) \tag{2.4}$$

for all $S \subseteq M$, $m - p + 1 \leq |S| \leq m$ and all $T \subseteq N$, $n - \ell + 1 \leq |T| \leq n$.

- For $1 \leq p \leq m-1$, $1 \leq \ell \leq n-1$, every inequality of (2.4) is facet defining
- Given some $(x^*, y^*) \in \{0,1\}^{m+n}$, there is an algorithm for identifying the inequality of (2.4) most violated by (x^*, y^*) in $O(m+n)$ time

3. Disjunctive (lift-and-project) cuts from split disjunctions

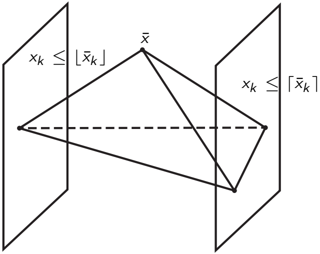
Cut from a split disjunction $x_k \le \lfloor \bar{x}_k \rfloor \vee x_k \ge \lceil \bar{x}_k \rceil$, where

$$x_k = a_{k0} - \sum_{j \in J} a_{kj} x_j, \quad (a_{k0} = \bar{x}_k)$$

is $\alpha x \ge 1$, with

$$\alpha_j = \max \left\{ \frac{a_{kj}}{a_{k0}}, \frac{-a_{kj}}{1 - a_{k0}} \right\}.$$

But $\alpha x \ge 1$ can also be viewed as an *intersection cut* from the convex set $S := \{x : \lfloor \bar{x}_k \rfloor \le x_k \le \lceil \bar{x}_k \rceil\}$



More generally:

If

- $\bar{x}$ is a basic solution to

$$P := \{x : Ax \geq b, x \geq 0\},$$

  and

- S is some convex set s.t.
  - $\bar{x} \in \operatorname{int} S$
  - $P_I \cap \operatorname{int} S = \emptyset$, where $P_I := P \cap \mathbb{Z}^n$ (S is a P_I-free convex set) then

- Intersect the n extreme rays $\bar{x} + r_j\lambda_j, \lambda_j \geq 0, j \in J$, of the LP cone $C(\bar{x})$ with $\operatorname{bd} S$

- Result: the hyperplane through the n intersection points $r \cap \operatorname{bd} S$ defines a valid cut—the intersection cut

Two definitions of intersection cuts:

- original (standard) def. (E.B., *Oper Res* 1971)

  S is a P_I-free convex set ($P_I \cap \operatorname{int} S = \emptyset$)

- more recent (restricted) def.

  S is a lattice-free convex set ($\mathbb{Z}^n \cap \operatorname{int} S = \emptyset$)

(Difference: a P_I-free set is not necessarily lattice-free.)

Theorem (M. Conforti, G. Cornuéjols and G. Zambelli, *OR Letters* 2010)
Any intersection cut from an LP cone $C(\bar{x})$ and a lattice-free polyhedron S is valid for the corner polyhedron $\operatorname{conv}(C(\bar{x}) \cap \mathbb{Z}^n)$. All the facets of the corner polyhedron are defined by intersection cuts from lattice-free polyhedra.

Lift-and-project cuts for 0-1 MIP's

(E.B., S. Ceria and G. Cornuéjols, *Math Prog* 1993)

- $\min\{cx : Ax \geq b,\ x \geq 0,\ x_j \in \{0,1\}, j \in N_1 \subset N\}$ (MIP)

  $\min\{cx : \tilde{A}x \geq \tilde{b}\}$ Optimal LP solution: $\bar{x}$ (LP)

- L&P cut $\alpha x \geq \beta$ from $\begin{pmatrix} \tilde{A}x \geq \tilde{b} \\ -x_k \geq 0 \end{pmatrix} \vee \begin{pmatrix} \tilde{A}x \geq \tilde{b} \\ x_k \geq 1 \end{pmatrix}$

$$
\begin{array}{lllllllll}
\min & \alpha\bar{x} & -\beta & & & & & & \\
& \alpha & & -u\tilde{A} & & +u_0 e_k & & = & 0 \\
& \alpha & & & -v\tilde{A} & & -v_0 e_k & = & 0 \\
& & -\beta & +u\tilde{b} & & & & = & 0 \\
& & -\beta & & +v\tilde{b} & & +v_0 & = & 0 \\
& & & & & & ue + ve + u_0 + v_0 & = & 1 \\
& & & & & & u, v, u_0, v_0 & \geq & 0
\end{array}
\qquad (\text{CGLP})_k
$$

The last equation is a normalization constraint.

Theorems (3.1-3.6)

1. Any basic feasible solution $(\alpha, \beta, u, u_0, v, v_0)$ to $(\text{CGLP})_k$ with $u_0, v_0 > 0$ yields a valid cut $\alpha x \geq \beta$; and together these cuts define $\text{conv}\{x \in \mathbb{R}^n : \tilde{A}x \geq \tilde{b}, x_k \in \{0,1\}\}$
2. Solutions with $u_0 = 0$ or $v_0 = 0$ yield inequalities of P
3. The cut $\alpha x \geq \beta$ is facet defining for
$$\text{conv}\{x : \tilde{A}x \geq \tilde{b}, x_k \in \{0,1\}\}$$
iff (α, β) is an extreme ray of W_0
4. If the normalization constraint of $(\text{CGLP})_k$ is replaced with $u_0 + v_0 = 1$, then the cut $\alpha x \geq \beta$ defined by the optimal solution to $(\text{CGLP})_k$ is the simple disjunctive cut (MIG cut) from $x_k \leq \lfloor \bar{x}_k \rfloor \vee x_k \geq \lceil \bar{x}_k \rceil$, where $x_k = \bar{a}_{k0} - \sum_{j \in J} \bar{a}_{kj} x_j$
5. If some of the variables x_j, $j \in J$, are integer-constrained, the cut $\alpha x \geq \beta$ can be strengthened by modularization of its coefficients
6. If the cuts generated under 1 are added to $Ax \geq b$ and the procedure is iterated for $k = 1, \ldots, n$, the cuts generated define $\text{conv}\{x \in \mathbb{R}^n : \tilde{A}x \geq \tilde{b}, x \in \{0,1\}^n\}$

The correspondence between lift-and-project cuts and mixed integer Gomory cuts

(E.B. and M. Perregaard, *Mathematical Programming* 2003)

Theorem A. Let $\alpha x \geq \beta$ be the lift-and-project cut associated with the basic feasible solution $(\alpha, \beta, u, v, u_0, v_0)$ to $(\mathrm{CGLP})_k$, where $u_0, v_0 > 0$ and

$M_1 := \{i : u_i \text{ is basic }\}, \qquad M_2 := \{i : v_i \text{ is basic }\}.$

Then

$\alpha x \geq \beta$ is equivalent to $\pi s_J \geq \pi_0$,

the cut from the disjunction $x_k \leq 0 \ \vee \ x_k \geq 1$ applied to $x_k = a_{k0} - \sum_{j \in J} a_{kj} s_j$ where

$J = M_1 \cup M_2.$

Theorem B. Let $\pi s_J \geq \pi_0$ be the cut from the disjunction $x_k \leq 0 \ \vee \ x_k \geq 1$ applied to $x_k = a_{k0} - \sum_{j \in J} a_{kj} s_j$, and let (M_1, M_2) be any partition of J such that

$j \in M_1$ if $a_{kj} < 0$, $\qquad j \in M_2$ if $a_{kj} > 0$. Then

$\pi s_J \geq \pi_0$ is equivalent to $\alpha x \geq \beta$,

the lift-and-project cut associated with any basic solution $(\alpha, \beta, u, v, u_0, v_0)$ to $(\mathrm{CGLP})_k$ such that $u_0, v_0 > 0$ and

u_i is basic if $i \in M_1$, $\qquad v_i$ is basic in M_2.

The above correspondence implies that every lift-and-project cut from a basic feasible solution to $(\mathrm{CGLP})_k$ corresponds to a Gomory mixed integer cut from $x_k \leq 0 \ \vee \ x_k \geq 1$, with x_k expressed in terms of some feasible or infeasible basis of the LP.

Upshot I:

- There is a many-to-one correspondence between basic feasible solutions to $(\mathrm{CGLP})_k$ such that u_i and v_i are basic for $i \in M_1$ and $i \in M_2$, respectively, and the basic (feasible or infeasible) solution to (LP) corresponding to the nonbasic set indexed by $J = M_1 \cup M_2$
- Hence: $(\mathrm{CGLP})_k$ can be solved implicitly, by mimicking its pivots on the (LP) tableau
  - the reduced costs of $(\mathrm{CGLP})_k$ can be computed from the LP tableau
  - a pivot in the LP tableau corresponds to several pivots in the CGLP tableau

Upshot II:

The procedure for solving the CGLP by pivoting in the LP simplex tableau can be viewed as an algorithm for finding the best mixed integer Gomory cut from the disjunction $x_k \leq 0 \vee x_k \geq 1$; more precisely, for finding the best J to yield such a cut. Namely:

- Start with mixed the MIG cut from the x_k row of the optimal LP simplex tableau.
- Evaluate it as a lift-and-project cut (by computing the reduced costs).
- If it is not optimal (as a lift-and-project cut) then identify a pivot (in the LP tableau) which can improve it.
- After the pivot (which creates an LP tableau that is in general neither optimal nor feasible), the MIG cut from the x_k row is guaranteed to be stronger.

From this perspective, lift-and-project theory provides the means by which, given a variable x_k, we can find among all possible bases containing x_k, the one in which the MIG cut from x_k is strongest.

Implementations

Commercial codes:

XPRESS	–	Perregaard
MOPS	–	Wesselmann
CPLEX	–	Tramontani

Publicly available implementation (P. Bonami):

COIN-OR's Cut Generation Library (Cgl):

`http://projects.coin-or.org/Cgl/Wiki/CglL&P`

Computational Testing

E.B. and P. Bonami, *Math Prog Computation* 2009)

- 3 different normalizations
- 2 ways of selecting the pivot column
- strengthening through disjunctive modularization

Test-bed: all 65 instances of MIPLIB3 library

- MIG cuts vs. 3 variants of L&P cuts
- 2 comparisons:
  (a) integrality gap after 10 rounds of 50 cuts at root node
  (b) CPU time and tree size (# nodes) required for solving by branch and bound after 10 rounds at root node

(a) For the 33 harder instances

Integrality gap closed at the root node (%)	L&P			
	MIG	Variant 1	Variant 2	Variant 3
	21	30	33	30

(b) For the 24 hardest instances

	L&P							
	MIG		Variant 1		Variant 2		Variant 3	
Complete branch-and-bound run	CPU(s)	Nodes	CPU(s)	Nodes	CPU(s)	Nodes	CPU(s)	Nodes
Geo Mean	73	23,449	42	10,444	61	16,590	42	10,021

Generalized intersection cuts

E.B. and F. Margot, *Math Programming* 2013

- Polyhedral relaxation C of P
- P_I-free convex set S ($\bar{x} \in \text{int}\, S$, $\text{int}\, S \cap P_I = \emptyset$)
- Intersection points $p^j = r^j \cap \text{bd}\, S$, $j \in \mathcal{P}$, of rays (extended edges) of C with $\text{bd}\, S$

Theorem 3.7. Let p^j, $j \in \mathcal{P}$, be a *proper* collection of intersection points. Let $\bar{\alpha}$ be any solution to either of the systems

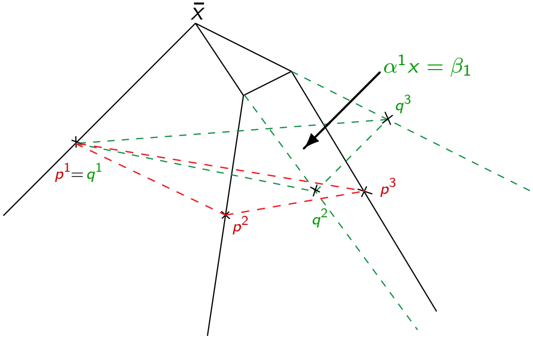
$$\alpha p^j \geq \beta, \quad j \in \mathcal{P}$$

for $\beta \in \{1, -1, 0\}$ such that $\alpha \bar{x} < \beta$.

Then $\bar{\alpha} x \geq \beta$ is a valid generalized intersection cut (GIC) for P_I.

- Motivation: to get rid of *recursive* cut generation that leads to numerical difficulties
- New paradigm: (a) create a proper collection Q,
  (b) use it to generate deep cuts without recursion

Generalized intersection cuts



q^1, q^2, q^3 – intersection points of extreme rays of C_1 with bd S

p^1, p^2, p^3 – intersection points of extreme rays of C with bd S

4. General disjunctive cuts

(E.B. and T. Kis, *Math Programming* 2016)

General disjunction:

$$\bigvee_{t\in T}\begin{pmatrix} \tilde{A}x & \geq & \tilde{b} \\ D^t x & \geq & d_0^t \end{pmatrix} \qquad (4.1)$$

Convex hull representation still compact (grows linearly with $|Q|$)

$$\begin{aligned} &\min \alpha\bar{x} - \beta \\ &\alpha \qquad - u^t\tilde{A} - v^t D^t = 0 \\ &\quad -\beta + u^t\tilde{b} + v^t d^t = 0 \quad t \in T \\ &\sum_{t\in T} u^t e^m + \sum_{t\in T} v^t e^p = 1 \\ &\qquad u^t, v^t \geq 0, \quad t \in T \end{aligned} \qquad (CGLP)_T$$

If the set $D^t x \geq d_0^t$ is replaced with a single inequality then (4.1) is called simple.

L&P cuts and GIC's

Let

$$S := \{x \in \mathbb{R}^N : d^t x \le d_{t0},\ t \in T\}$$

be a maximal P_I-free polyhedron.

Let

$$F := \left\{ x \in \mathbb{R}^N : \bigvee_{t \in T} \left(\begin{array}{ccc} \tilde{A}x & \ge & \tilde{b} \\ d^t x & \ge & d_{t0} \end{array} \right) \right\}$$

and

$$\text{conv}\, F := \{x \in \mathbb{R}^N : \alpha x \ge \beta \text{ for all } (\alpha, \beta) \text{ satisfying } (*)\}$$

where

$$\begin{array}{rcrcrcl} \alpha & - & u^t \tilde{A} & - & u_0^t d^t & = & 0 \\ -\beta & + & u^t \tilde{b} & + & u_0^t d_{t0} & = & 0 \\ \multicolumn{3}{r}{\sum_{t \in T}(u^t e} & + & u_0^t) & = & 1 \\ & & & & u^t, u_0^t & \ge & 0 \end{array} \quad t \in T \qquad (*)$$

Theorem 4.1. The family of GIC's from S is equivalent to the family of L&P cuts $\alpha x \ge \beta$ from basic feasible solutions to $(*)$ such that $u_0^t > 0$ for all $t \in T$.

Let

$$F := \left\{ x \in \mathbb{R}^N : \vee_{t \in T} \begin{pmatrix} \tilde{A}x & \geq & \tilde{b} \\ D^t x & \geq & d_{t0} \end{pmatrix} \right\}$$

and

$$\text{conv}\, F := \{x \in \mathbb{R}^N : \alpha x \geq \beta \text{ for all } \alpha, \beta \text{ satisfying } (**)\},$$

where

$$\begin{array}{rcrcrcll} \alpha & - & u^t \tilde{A} & - & v^t D^t & = & 0 & \\ -\beta & + & u^t \tilde{b} & + & v^t d_{t0} & = & 0 & t \in T \\ & & \sum_{t \in T}(u^t e & + & v^t e) & = & 1 & \\ & & & & u^t, v^t & \geq & 0, & t \in T \end{array} \qquad (**)$$

Theorem 4.2. The L&P cut $\bar{\alpha} x \geq \bar{\beta}$ from a basic feasible solution $(\bar{\alpha}, \bar{\beta}, \{\bar{u}^t, \bar{v}^t\}_{t \in T})$ to (**) such that $\bar{v}^t e > 0$ for all $t \in T$ is equivalent to a GIC from the maximal P_I-free polyhedron

$$S := \{x \in \mathbb{R}^N : (\bar{v}^t D^t) x \leq \bar{v}^t d_{t0}, t \in T\}.$$

L&P cut from

GIC from

$$\bigvee_{t \in T} \begin{pmatrix} \tilde{A}x & \geq & \tilde{b} \\ d^t x & \geq & d_{t0} \end{pmatrix} \iff S := \{x \in \mathbb{R}^n : d^t x \leq d_{t0}, t \in T\}$$

(P_I- free polyhedron)

$$\bigvee_{t \in T} \begin{pmatrix} \tilde{A}x & \geq & \tilde{b} \\ D^t x & \geq & d_0^t \end{pmatrix} \iff S := \{x \in \mathbb{R}^n : v^t D^t x \leq v^t d_{t0}, t \in T\}$$

(family of P_I-free polyhedra)

L&P cuts and SIC's

Theorem 4.3. Let $\bar{x}$ be an optimal LP solution with nonbasic set J, and let

$$S := \{x \in \mathbb{R}^n : d^t x \leq d_{t0}, t \in T\}$$

be a maximal P_I-free polyhedron with $\bar{x} \in \operatorname{int} S$.

The SIC $\pi x \geq 1$ from $\bar{x}$ and S is equivalent to the L&P cut from a basic feasible solution to

$$\begin{array}{rcccccl} \alpha & - & u^t\tilde{A} & - & u_0^t d^t & = & 0 \\ -\beta & + & u^t\tilde{b} & + & u_0^t d_{t0} & = & 0 \\ & \sum_{t\in T}(& u^t e & + & u_0^t) & = & 1 \\ & & & & u^t, u_0^t & \geq & 0 \end{array} \quad t \in T \qquad (*)$$

in which, for each $t \in T$, all but one of the variables u_j^t with $j \in J$ are basic, and all the variables u_j^t with $j \notin J$ are nonbasic, except for the u_0^t, which are all basic and positive.

Theorem 4. Let $w := (\alpha, \beta, \{u^t, u_0^t\}_{t\in T})$ be a basic feasible solution to (*) such that $u_0^t > 0$, $t \in T$. If there exists a $n \times n$ nonsingular submatrix $\tilde{A}_J$ of $\tilde{A}$ such that $u_j^t = 0$ for all $t \in T$ and $j \notin J$, then the L&P cut $\alpha x \geq \beta$ is equivalent to the SIC $\pi x \geq 1$ from S.

Let $F = \bigvee_{t \in T} \begin{pmatrix} \tilde{A}x \geq \tilde{b} \\ D^t x \geq d_0^t \end{pmatrix}$, and consider the $(\text{CGLP})_T$ with the constraint set

$$\begin{array}{rcll} \alpha - u^t \tilde{A} - v^t D^t & = & 0 & \multirow{2}{*}{$t \in T$} \\ -\beta + u^t \tilde{b} + v^t d_0^t & = & 0 & \\ \sum_{t \in T} (u^t e + v^t e) & = & 1 & \\ u^t, v^t & \geq & 0 & \end{array} \qquad (**)$$

Theorem 4.5. Let $\bar{w} = (\bar{\alpha}, \bar{\beta}, \{\bar{u}^t, \bar{v}^t\}_{t \in T})$ be a basic feasible solution to (**) such that $\bar{v}^t e > 0$ for all $t \in T$. If there exists a $n \times n$ nonsingular submatrix $\tilde{A}_J$ of $\tilde{A}$ such that $\bar{u}_j^t = 0$ for all $t \in T$ and $j \notin J$, then the L&P cut $\bar{\alpha}x \geq \bar{\beta}$ is equivalent to the SIC $\pi x_J \geq 1$ from the LP solution with nonbasic set indexed by J, and the P_I-free polyhedron

$$S(\bar{v}) := \{x \in \mathbb{R}^n : (\bar{v}^t D^t)x \leq \bar{v}^t d_0^t, t \in T\}.$$

Theorem 4.6. Let $\bar{w} := (\bar{\alpha}, \bar{\beta}, \{\bar{u}^t, \bar{v}^t\}_{t\in T})$ be a basic feasible solution to

$$\begin{array}{rcrcrcl} \alpha & - & u^t\tilde{A} & - & v^tD^t & = & 0 \\ -\beta & + & u^t\tilde{b} & + & v^td_{t0} & = & 0 \\ \multicolumn{5}{r}{\sum_{t\in T}(u^te + v^te)} & = & 1 \\ & & & & u^t, v^t & \geq & 0 \end{array} \quad t \in T \qquad (**)$$

such that $\bar{v}^te > 0$ for all $t \in T$. If there is no $n \times n$ nonsingular $\tilde{A}_J$ such that $u_j^t = 0$ for all $t \in T$ and $j \notin J$, and there is no basic feasible solution $\tilde{w}$ to $(**)$ with $(\tilde{\alpha}, \tilde{\beta}) = \theta(\bar{\alpha}, \bar{\beta})$ for some $\theta > 0$ that satisfies this condition, then there exists no SIC from any member of the family of polyhedra

$$S(v) = \{x \in \mathbb{R}^n : (v^tD^t)x \leq v^td_{t0}, t \in T\},$$

where $v \geq 0$, $v \neq 0$, equivalent to $\bar{\alpha}x \geq \bar{\beta}$.

Furthermore, if $\bar{\alpha}\bar{x} - \bar{\beta}$ uniquely minimizes $\alpha\bar{x} - \beta$ over $(**)$, then $\bar{\alpha}\bar{x} - \bar{\beta} < \bar{\alpha}\bar{x} - \bar{\beta}$ for any L&P cut $\tilde{\alpha}x \geq \tilde{\beta}$ equivalent to an intersection cut from $S(v)$.

A feasible basis for $(\text{CGLP})_T$ and the associated solution is called *regular* if the cut that it defines is equivalent to an intersection cut, i.e. if it satisfies the condition of Theorem 4.5, *irregular* otherwise.

A cut defined by an irregular solution w is called irregular, unless there exists a regular solution w' with the same (α, β)-component as that of w; in which case it is called regular.

There are two types of irregular basic solutions. Let B be a feasible basis matrix of $(\text{CGLP})_T$, and let $\tilde{A}_K$ be the submatrix of $\tilde{A}$ whose rows are contained in the columns B_j of B such that u_j^t is basic for some $t \in T$. Then the two types are:

Type 1. $\tilde{A}_K$ contains a $n \times n$ nonsingular submatrix $\tilde{A}_J$, but $K \setminus J$ is nonempty.

Type 2. $\tilde{A}_K$ contains no $n \times n$ nonsingular submatrix

L&P cuts and corner polyhedra

- Intersection cuts from $\bar{x}$ and a lattice-free polyhedron are valid for the corner polyhedron associated with $\bar{x}$; but
- Irregular L&P cuts obtained by minimizing $\alpha\bar{x} - \beta$ may cut off parts of the corner polyhedron $\text{conv}\,(C(\bar{x}) \cap \mathbb{Z}^h)$.

Frequency of irregular L&P cuts

Let $\tilde{A}$ be $m \times n$, $n < m \leq 3n$, and let the disjunction have $k = |T|$ terms.

- An arbitrary basis contains km candidate variables u_j^t for $(k-1)n$ places in a basis
- A regular basis contains variables u_j^t with exactly n different subscripts j
- All bases containing variables u_j^t with fewer or more than n different subscripts j are irregular

The ratio $\frac{\text{irregular}}{\text{regular}}$ increases with $k = |T|$.

Early computational results indicate the irregular cuts tend to vastly outnumber regular ones.

In a recent computational experiment (E.B. and T. Serra) with L&P cuts from disjunctions of the form

$$x_1 \leq 0 \vee x_2 \leq 0 \vee \cdots \vee x_k \leq 0 \vee (x_j \geq 1, j = 1, \ldots, k)$$

(orthant cuts), all the cuts for $k = 2, 3, 4$ were generated for each of the instances of Steiner triples with $n = 9, 15, 27$ and 45.

For $n = 9, 15$, all cuts were irregular of type 1.

For $n = 27, 45, \approx 90\%$ of the cuts were irregular of type 1, while $\approx 10\%$ were irregular of type 2.

Numerical Example

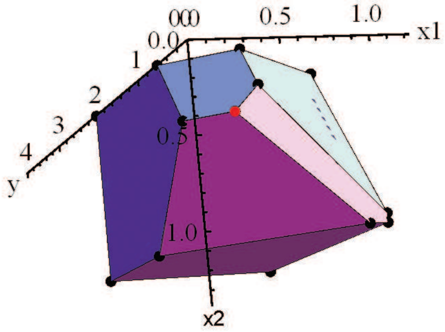
Consider the MIP

$$\begin{aligned}
&\min y \\
&\text{such that} \\
&y - 1.1x_1 + x_2 \geq -0.15 \\
&y + x_1 - 1.1x_2 \geq -0.2 \\
&y + x_1 + x_2 \geq 0.6 \\
&x_1, x_2 \in \{0,1\}, y \geq 0
\end{aligned}$$

The optimal solution of the LP relaxation is $x_1^* = 23/105, x_2^* = 8/21, y^* = 0$, the optimal simplex tableau is

s_1	x_1	x_2	y	s_2	s_3	RHS
1			$21/10$	1	$1/10$	$29/100$
	1		1	$-10/21$	$-11/21$	$23/105$
		1	0	$10/21$	$-10/21$	$8/21$

The LP feasible set



$$-x_1 \geq 0 \ \vee \ -x_2 \geq 0 \ \vee \ x_1 + x_2 \geq 2$$

After eliminating α, β, the CGLP is

$$\min \frac{29}{100} u_1^1 + \frac{23}{105} u_5^1 + \frac{82}{105} u_6^1 + \frac{8}{21} u_7^1 + \frac{13}{21} u_8^1 - \frac{23}{105} u_0^1$$

$$\begin{pmatrix} \tilde{A}^T \\ \tilde{b}^T \\ \tilde{A}^T \\ \tilde{b}^T \\ e^T \end{pmatrix} u^1 + \begin{pmatrix} -\tilde{A}^T \\ -\tilde{b}^T \\ 0 \\ 0 \\ e^T \end{pmatrix} u^2 + \begin{pmatrix} 0 \\ 0 \\ -\tilde{A}^T \\ -\tilde{b}^T \\ e^T \end{pmatrix} u^3 + \begin{pmatrix} -e_{x_1} \\ 0 \\ -e_{x_1} \\ 0 \\ 1 \end{pmatrix} u_0^1 +$$

$$\begin{pmatrix} e_{x_2} \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} u_0^2 + \begin{pmatrix} 0 \\ 0 \\ -e_{x_1,x_2} \\ -2 \\ 1 \end{pmatrix} u_0^3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

with

$$\tilde{A} = \begin{pmatrix} 1 & -1.1 & 1 \\ 1 & 1 & -1.1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \tilde{b} = \begin{pmatrix} -0.15 \\ -0.20 \\ 0.60 \\ 0 \\ 0 \\ -1 \\ 0 \\ -1 \end{pmatrix}$$

The optimal CGLP solution has basic variables

$$\{u_0^1, u_0^2, u_0^3; u_2^1, u_3^1, u_1^2, u_3^2, u_2^3, u_4^3\}$$

The solution is *irregular* of type 1, as $|\{1, 2, 3, 4\}| = 4 > 3 = n$

The resulting cut

$$3.5222y + 0.4049x_1 + 0.5997x_2 \geq 1 \tag{1}$$

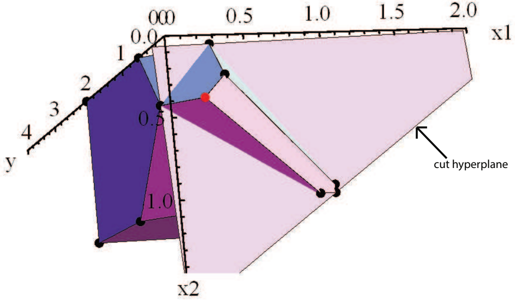
cuts off the LP optimum by more than any SIC.

It also cuts off integer points of

every corner polyhedron

associated with any basis of the LP relaxation.

The lift-and-project cut (irregular)



5. Cuts from the V-polyhedral representation of $\text{conv} \bigcup_{i \in Q} P_i$

(E.B. and A. Kazachkov, in preparation)

$$F = \bigcup_{h \in Q} P^h, \quad P^h = \text{conv}\, V^h + \text{cone}\, W^h, \quad h \in Q$$

where V^h and W^h are the vertices and extreme rays of P^h. From Theorem 1.4, the set of inequalities $\gamma x \geq \delta$ defining $\text{conv}\, F$ is given by the solution set to the system

$$\begin{aligned} \gamma p \geq \delta, & \qquad \forall p \in \bigcup_{h \in Q} V^h \\ \gamma r \geq 0, & \qquad \forall r \in \bigcup_{h \in Q} W^h \end{aligned} \tag{5.1}$$

To generate valid cuts for $\text{conv}\, F$ that chop off $\bar{x}$ we need only a small subset of the inequalities (5.1).

V-polyhedral Disjunctive Cuts

$$F = \bigcup_{h \in Q} P^h, \quad P^h = \operatorname{conv} V^h + \operatorname{cone} W^h, \quad h \in Q$$

For $h \in Q$, let $x^h = \arg\min\{cx : x \in P^h\}$, let $C(x^h) = \{x^h + r^j \lambda_j, \lambda_j \geq 0, j \in J\}$ be the LP cone associated with x^h, and let $\tilde{P}^h$ be any relaxation of P^h contained in $C(x^h)$, i.e. such that $P^h \subseteq \tilde{P}^h \subseteq C(x^h)$. Finally, let $\tilde{V}^h$ and $\tilde{W}^h$ be the set of vertices and extreme rays, respectively, of $\tilde{P}^h$.

Theorem 5.1. Any solution to the system

$$\begin{aligned} \gamma p &\geq \delta \qquad p \in \bigcup_{h \in Q} \tilde{V}^h \\ \gamma h &\geq 0 \qquad r \in \bigcup_{h \in Q} \tilde{W}^h \end{aligned} \tag{5.2}$$

defines a valid cut $\gamma x \geq \delta$ for $\operatorname{conv} F$. Together, these cuts define $\operatorname{conv} \bigcup_{h \in Q} \tilde{P}^h$.

The system (5.2) is much smaller than (5.1). In case we choose $\tilde{P}^h = C(x^h)$, $h \in Q$, it has only $q = |Q|$ nonhomogeneous and $q(n-q)$ homogeneous inequalities.

V-polyhedral cuts versus L&P cuts

Let

$$\tilde{P}^h = \{x \in \mathbb{R}^n : \tilde{D}^h x \geq \tilde{d}_0^h\}, h \in Q$$

be the *H-polyhedral* representation of

$$\tilde{P}^h = \text{conv}\, \tilde{V}^h + \text{cone}\, \tilde{W}^h, h \in Q$$

Consider the L&P cut generating system

$$\begin{array}{rcll} \alpha - u^h \tilde{D}^h & = & 0 & \quad h \in Q \\ \beta + u^h \tilde{d}^h & \geq & 0 & \\ \sum_{h \in Q} u^h e & = & 1 & \\ & & u^h \geq 0, h \in Q & \end{array} \tag{5.3}$$

Theorem 5.2 The L&P cuts $\alpha x \geq \beta$ corresponding to basic solutions to (5.3) are equivalent to the V-polyhedral cuts $\gamma x \geq \delta$ corresponding to basic solutions to (5.2).

Proof. Both (5.2) and (5.3) describe conv $\bigcup_{h \in Q} \tilde{P}^h$. □

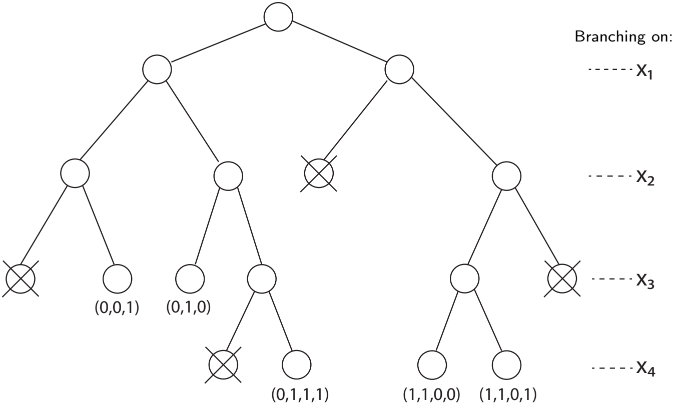
Harnessing branch-and-bound information for cut generation

- The set of active nodes of a partial branch-and-bound tree offer a convenient disjunction to derive V-polyhedral cuts from.
- At each active node h, x^h and $C(x^h)$ are readily available.
- Cuts $\gamma x \geq \delta$ can be derived from

$$\begin{aligned} \min \bar{x}^T \gamma - \delta & \\ \gamma x^h \geq \delta \qquad & h \in Q \\ \gamma r^j \geq 0 \qquad & r^j \in \text{ext}\, C(x^h), h \in Q \end{aligned}$$

- Cuts are valid for the whole tree

Cuts from a partial branch and bound tree



5-term disjunction, $|Q| = 5$

$D^1 x \leq d_0^1$	$D^2 x \leq d_0^2$	$D^3 x \leq d_0^3$	$D^4 x \leq d_0^4$	$D^5 x \leq d_0^5$
$x_1 \leq 0$	$x_1 \leq 0$	$x_1 \leq 0$	$x_1 \geq 1$	$x_1 \geq 1$
$x_2 \leq 0$	$x_2 \geq 1$	$x_2 \geq 1$	$x_2 \geq 1$	$x_2 \geq 1$
$x_3 \geq 1$	$x_3 \leq 0$	$x_3 \geq 1$	$x_3 \leq 0$	$x_3 \leq 0$
		$x_4 \geq 1$	$x_4 \leq 0$	$x_4 \geq 1$

Resulting cut can be added to the LP relaxation for each active node.

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