The structure of the infinite models in integer programming

Amitabh Basu, MC, Marco Di Summa, Joseph Paat.

January 10, 2017

Not Again!

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18:00-18:30 Santanu Dey Facets of High-Dimensional Infinite Group Problem

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a lot of research since then...

MIPs in tableau form

$b \in \mathbb{R}^n \setminus \mathbb{Z}^n$.

$$x_B + \sum_{r \in R} r s(r) + \sum_{p \in P} p y(p) = b, \quad x_B \in \mathbb{Z}^n, s(r) \in \mathbb{R}_+, y(p) \in \mathbb{Z}_+$$

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$$\sum_{r \in R} r s(r) + \sum_{p \in P} p y(p) \in b + \mathbb{Z}^n, \;\; s(r) \in \mathbb{R}_+, y(p) \in \mathbb{Z}_+$$

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BFS: $s(r) = y(p) = 0$, $x_B = b$. Want $x_B \in \mathbb{Z}^n$

The mixed-integer model

Mixed-integer infinite group relaxation $b \in \mathbb{R}^n \setminus \mathbb{Z}^n$, $s : \mathbb{R}^n \to \mathbb{R}_+$ and $y : \mathbb{R}^n \to \mathbb{Z}_+$

$$M_b = \{s, y \in \mathbb{R}_+^{(\mathbb{R}^n)} \times \mathbb{R}_+^{(\mathbb{R}^n)} : \sum_{r \in \mathbb{R}^n} r s(r) + \sum_{p \in \mathbb{R}^n} p y(p) \in b + \mathbb{Z}^n\}.$$

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(ψ, π, α), $\psi, \pi : \mathbb{R}^n \to \mathbb{R}$, $\alpha \in \mathbb{R}$,

$$H_{\psi,\pi,\alpha} := \left\{(s, y) \in \mathbb{R}^{(\mathbb{R}^n)} \times \mathbb{R}^{(\mathbb{R}^n)} : \sum_{r \in \mathbb{R}^n} \psi(r)s(r) + \sum_{p \in \mathbb{R}^n} \pi(p)y(p) \geq \alpha\right\}$$

(ψ, π, α) is a valid tuple (functions) for M_b if $M_b \subseteq H_{\psi,\pi,\alpha}$.
equivalently: $\text{conv}(M_b)$. $\alpha \in \{-1, 0, 1\}$.

A face: The pure integer model

The pure integer infinite group relaxation $y : \mathbb{R}^n \to \mathbb{Z}_+$.

$$I_b = \{y : (0, y) \in M_b\} = \{y \in \mathbb{R}_+^{(\mathbb{R}^n)} : \sum_{p \in \mathbb{R}^n} p y(p) \in b + \mathbb{Z}^n\}.$$

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Gomory functions

$n = 1$, $\alpha = 1$, very simple to describe ($0 < b < 1$).

$$\psi(r) = \begin{cases} \frac{r}{b} & \text{if } r \geq 0 \\ \frac{-r}{1-b} & \text{if } r < 0 \end{cases} \qquad \pi(p) = \begin{cases} \frac{f(p)}{b} & \text{if } f(p) \leq b \\ \frac{1-f(p)}{1-b} & \text{if } f(p) > b \end{cases}$$

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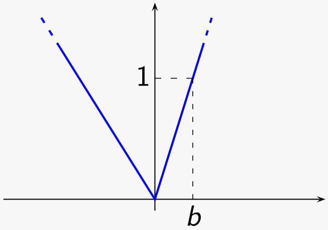

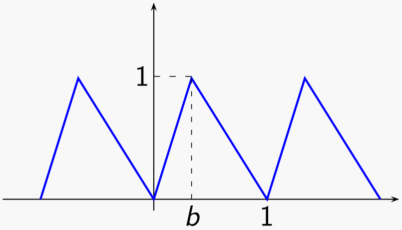

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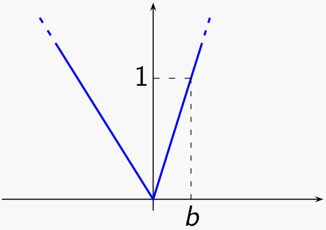

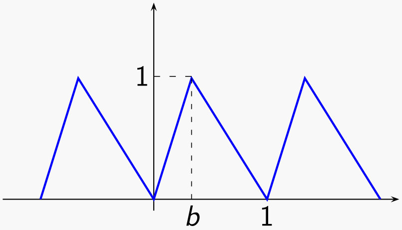

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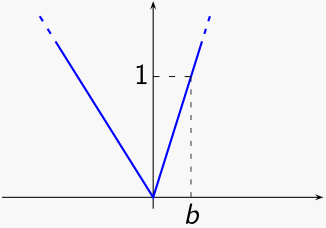

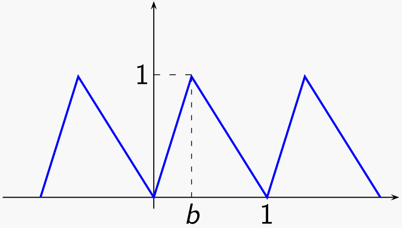

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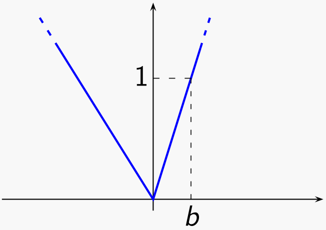

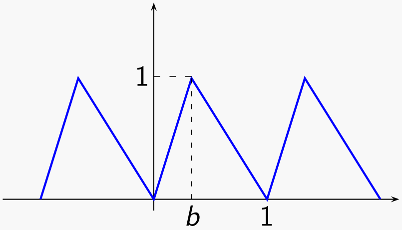

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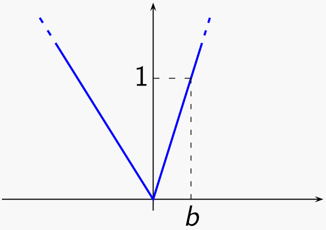

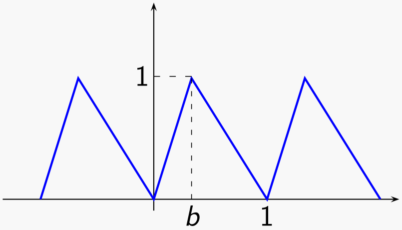

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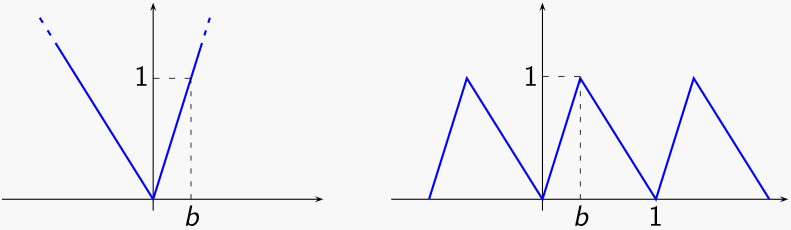

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Symmetry condition ϕ satisfies $\phi(r) + \phi(b - r) = 1$. (π)

Attractiveness of valid functions: Plug and play

$$-3.6s_1 + 1.7s_2 + .2y_1 - .2y_2 = .4 + \mathbb{Z}$$
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....Library of useful functions in IP solvers.

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Nontrivial: Not valid for $\mathbb{R}_+^{(\mathbb{R}^n)} \times \mathbb{R}_+^{(\mathbb{R}^n)}$.
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Extreme: Later.
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...but ignorance should not be an excuse.... We try to answer this.

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Theorem (Yildiz and Cornuéjols, related to Johnson.)

Let $\psi : \mathbb{R}^n \to \mathbb{R}$, $\pi : \mathbb{R}^n \to \mathbb{R}$ *be any functions, and* $\alpha \in \{-1, 0, 1\}$. *Then* (ψ, π, α) *is a nontrivial minimal valid tuple for* M_b *if and only if:*

- π *is subadditive;*
- $\psi(r) = \sup_{\epsilon>0} \frac{\pi(\epsilon r)}{\epsilon} = \lim_{\epsilon\to 0^+} \frac{\pi(\epsilon r)}{\epsilon} = \limsup_{\epsilon\to 0^+} \frac{\pi(\epsilon r)}{\epsilon}$ *for every* $r \in \mathbb{R}^n$*; sublinear*
- π *is Lipschitz continuous with Lipschitz constant* $L := \max_{\|r\|=1} \psi(r)$*;*
- $\pi \geq 0$, $\pi(z) = 0$ *for every* $z \in \mathbb{Z}^n$*, and* $\alpha = 1$*;*
- π *satisfies the symmetry condition (and is periodic).*

(One of our) Goal(s)

$$
\begin{aligned}
Q_b &= \mathbb{R}_+^{(\mathbb{R}^n)} \times \mathbb{R}_+^{(\mathbb{R}^n)} \bigcap_{(\psi,\pi,\alpha) \text{ valid}} H_{\psi,\pi,\alpha} \\
&= \mathbb{R}_+^{(\mathbb{R}^n)} \times \mathbb{R}_+^{(\mathbb{R}^n)} \bigcap_{(\psi,\pi,\alpha) \text{ minimal, nontrivial}} H_{\psi,\pi,\alpha}
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Clearly $\mathrm{conv}(M_b) \subseteq Q_b$. But what is Q_b?

Norms, closed sets...

While in finite dimensions all norms are equivalent to the Euclidean norm, In infinite dimensions this is not so.....
Norm on $\mathbb{R}^{(\mathbb{R}^n)} \times \mathbb{R}^{(\mathbb{R}^n)}$ (BCCZ):

$$|(s,y)|_* := |s(0)| + \sum_{r \in \mathbb{R}^n} \|r\| |s(r)| + |y(0)| + \sum_{p \in \mathbb{R}^n} \|p\| |y(p)|$$

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Under the topology induced by $|(\cdot,\cdot)|_*$,

$$Q_b = \text{cl}(\text{conv}(M_b)) = \text{conv}(M_b) + \mathbb{R}_+^{(\mathbb{R}^n)} \times \mathbb{R}_+^{(\mathbb{R}^n)}.$$

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$\mathrm{conv}(M_b) \subsetneq \mathrm{conv}(M_b) + \mathbb{R}_+^{(\mathbb{R}^n)} \times \mathbb{R}^{(\mathbb{R}^n)}$

Liftable functions

$G_b = \{y \in \mathbb{R}^{(\mathbb{R}^n)} : (0, y) \in Q_b\}$
(Minimal, nontrivial) (π, α) valid for I_b liftable if $\exists\ \psi$ s.t. (ψ, π, α) (Minimal, nontrivial) valid for M_b.

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Canonical faces, Finite faces

Ultimately we want valid inequalities for Integer Programs (with rational data).
Given $R, P \subseteq \mathbb{R}^n$, let

$$V_{R,P} = \left\{(s, y) \in \mathbb{R}^{(\mathbb{R}^n)} \times \mathbb{R}^{(\mathbb{R}^n)} : s(r) = 0 \, \forall r \notin R,\ y(p) = 0 \, \forall p \notin P\right\}.$$

A canonical face of $\text{conv}(M_b)$ is $F = \text{conv}(M_b) \cap V_{R,P}$. When R, P finite, F is a finite canonical face.

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Same for $\mathrm{conv}(I_b)$. Finite canonical faces of $\mathrm{conv}(I_b)$ are the corner polyhedra.

Rational finite faces

What happens when $R,\ P \subset \mathbb{Q}^n$?

Theorem

Let $R, P \subset \mathbb{Q}^n$. *Then* $\mathrm{conv}(M_b) \cap V_{R,P} = \mathrm{cl}(\mathrm{conv}(M_b)) \cap V_{R,P}$.
Let $P \subset \mathbb{Q}^n$. *Then* $\mathrm{conv}(I_b) \cap V_P = \mathrm{cl}(\mathrm{conv}(I_b)) \cap V_P$.

Corollary

The restrictions of the minimal, nontrivial valid tuples give all the (nontrivial) facets of rational mixed-integer polyhedra.
The restrictions of the minimal, nontrivial liftable functions give all the (nontrivial) facets of rational corner polyhedra,

Extreme functions and facets

$\pi \geq 0$ extreme if $(\pi, 1)$ valid and $\pi_1 = \pi_2 = \pi$ for every $\pi_1 \geq 0$, $\pi_2 \geq 0$ such that $(\pi_1, 1)$ $(\pi_2, 1)$ valid and $\pi = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2$.

$\pi \geq 0$ facet if $(\pi, 1)$ valid and $\pi_1 = \pi$ for every $\pi_1 \geq 0$, such that $H_{\pi_1,1} \cap I_b \subset H_{\pi_1,1} \cap I_b$.

π facet $\rightarrow$ π extreme. The converse is not known.

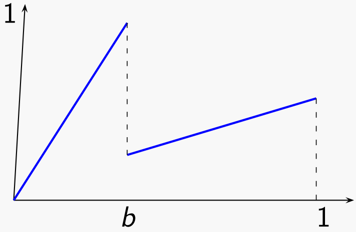
Köppe and Zhou: Coincide for the case of continuous piecewise linear functions On the notions of facets, weak facets, and extreme functions of the Gomory-Johnson infinite group problem $n = 1$,

software to test extremality of piecewise linear functions.

Discontinuous extreme functions ($n = 1$)

Dey, Richard, Li and Miller: The following function is extreme: $n = 1$, $0 < b < \frac{1}{2}$,

$$\pi : \pi(r) = \begin{cases} \frac{r}{b} & 0 \leq r \leq b \\ \frac{r}{1+b} & b < r < 1 \end{cases}$$



More discontinuous functions ($n = 1$)

Letchford, Lodi "Strong fractional functions" Minimal, dominate fractional functions.
Dash, Günlük "Extended two-step MIR" (mixed integer rounding) functions. limit of sequences of two-step MIR functions, dominate LetcLo.
Hildebrand , "two-sided discontinuous at the origin with 1 or 2 slopes", extreme
Köppe, Zhou: Extreme functions that are continuous but not Lipschitz continuous.

see Köppe, Zhou Equivariant perturbation in Gomory and Johnson's infinite group problem. vi. the curious case of two-sided discontinuous functions.

Not all extreme functions are needed

Summarizing our results:

$$\mathrm{cl}(\mathrm{conv}(I_b)) = \mathrm{conv}(I_b) + \mathbb{R}_+^{(\mathbb{R}^n)} =$$
$$= \mathbb{R}_+^{(\mathbb{R}^n)} \cap \bigcap \{H_{\pi,\alpha} : (\pi, \alpha) \text{ minimal nontrivial liftable tuple}\}.$$

When $P \subseteq \mathbb{Q}^n$, we have that $\mathrm{cl}(\mathrm{conv}(I_b)) \cap V_P = \mathrm{conv}(I_b) \cap V_P$

(π, α) minimal nontrivial liftable tuple:$\rightarrow$ $\pi \geq 0$, $\alpha = 1$, π Lipschitz continuous.

ONLY THESE ARE NEEDED

Cauchy equation and Hamel bases

The Cauchy functional equation in $\mathbb{R}^n$:

$$\theta(u) + \theta(v) = \theta(u + v) \text{ for all } u, v \in \mathbb{R}^n.$$

(subadditivity) $\theta(x) = c^T x$ is obviously a solution to the equation.

A Hamel basis B in a basis of $\mathbb{R}^n$ over the field $\mathbb{Q}$. i.e. a subset of $\mathbb{R}^n$ s.t. $\forall x \in \mathbb{R}^n$, there exists a unique finite subset $\{\beta_1, \ldots, \beta_t\} \subseteq B$ and $\lambda_1, \ldots, \lambda_t \in \mathbb{Q}$ such that $x = \sum_{i=1}^t \lambda_i \beta_i$. (axiom of choice).

Cauchy equation and Hamel bases II

For $\beta \in B$, let $c(\beta) \in \mathbb{R}$ be a real number. Define θ as:

$$\theta(x) = \sum_{i=1}^{t} \lambda_i c(\beta_i).$$

...θ solves the Cauchy equation.

Theorem

Let B a Hamel basis of $\mathbb{R}^n$. Then every solution to the Cauchy equation is of this form.

The affine hull of conv(I_b)

Theorem (Basu, Hildebrand Köppe)

The affine hull of conv(I_b) *is described by the equations*

$$\sum_{p \in \mathbb{R}^n} \theta(p) y(p) = \theta(b)$$

for all solutions $\theta : \mathbb{R}^n \to \mathbb{R}$ *of the Cauchy equation such that* $\theta(p) = 0$ *for every* $p \in \mathbb{Q}^n$.

Extreme functions (without $\pi \geq 0$) do not exist.....

$\mathrm{aff}(\mathrm{cl}(\mathrm{conv}(I_b))) = \mathbb{R}^{(\mathbb{R}^n)}$
$\mathrm{aff}(\mathrm{conv}(M_b)) = \mathrm{aff}(\mathrm{cl}(\mathrm{conv}(M_b))) = \mathbb{R}^{(\mathbb{R}^n)} \times \mathbb{R}^{(\mathbb{R}^n)}$.

Every valid function is nonnegative.

Theorem

For every valid tuple (π, α) for I_b, there exists a unique solution of the Cauchy equation $\theta : \mathbb{R}^n \to \mathbb{R}$ such that $\theta(p) = 0$ for every $p \in \mathbb{Q}^n$ and the valid tuple $(\pi', \alpha') = (\pi + \theta, \alpha + \theta(b))$ satisfies $\pi' \geq 0$.

$\rightarrow$ Nonnegative valid functions form a compact, convex set. Its extreme points are the extreme functions and suffice to describe this set. (.....but not all of them are necessary)

Finite faces and recession cones

$\sqrt{2}y_1 - .2y_2 + (1 - \sqrt{2})y_3 \in .4 + \mathbb{Z}$
$y_1, y_2, y_3 \in \mathbb{Z}_+$

$y_1 = y_3$, $(1, 0, 1)$

Finite faces and recession cones

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$y_1, y_2, y_3 \in \mathbb{Z}_+$

$y_1 = y_3$, $(1, 0, 1)$

L the linear space parallel to $\mathrm{aff}(\mathrm{conv}(I_b))$

Theorem

For every $P \subseteq \mathbb{R}^n$ finite:

- *the face $C^P = \mathrm{conv}(I_b) \cap V_P$ is a rational polyhedron in $\mathbb{R}^P$;*
- *every extreme ray of C^P is spanned by some $r \in \mathbb{Z}_+^P$ such that $\sum_{p \in P} pr(p) \in \mathbb{Z}^n$;*
- $\mathrm{rec}(C^P) = (L \cap V_P) \cap \mathbb{R}_+^P$.

Finite faces and recession cones

Theorem

There are finite canonical faces of $\operatorname{conv}(M_b)$ *that are not closed.*
All the finite canonical faces of $\operatorname{conv}(I_b)$ *are rational polyhedra.*

More work?

$n = 1$: EVERY EXTREME FUNCTION $\pi \geq 0$ IS "NICE".

Gomory Johnson: Every extreme function is piecewise linear. NO
Basu, Conforti, Cornuejols, Zambelli.

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Dey and Richard Aussois 2008 Construct extreme functions that are piecewise linear and have > 4 slopes. YES

Hildebrand (2013) 6
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Is every "bad" function (discontinuous, non piecewise linear, ∞ slopes) the pointwise limit of a sequence of "good" functions?

Maybe "nice" functions suffice....

Is every facet of $\text{conv}(I_b) \cap V_P$, P finite (rational) the restriction of a piecewise linear function?

Maybe "nice" functions suffice....

Is every facet of $\mathrm{conv}(I_b) \cap V_P$, P finite (rational) the restriction of a piecewise linear function?

$\mathrm{conv}(I_b) = \mathrm{cl}(\mathrm{conv}(I_b)) \cap \mathrm{aff}(\mathrm{conv}(I_b))$?

THANK YOU FOR YOUR ATTENTION