ON DECOMPOSABILITY OF MULTILINEAR SETS

Alberto Del Pia, University of Wisconsin-Madison

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Joint work with Aida Khajavirad, Carnegie Mellon University

- Let $\mathcal{J}$ be a family of subsets of $\{1, \dots, n\}$ of cardinality at least two.
- We define the Multilinear set

$$\mathcal{S} := \Big\{(x, y) : x \in \{0, 1\}^n, \ y_I = \prod_{i \in I} x_i, \ \forall I \in \mathcal{J}\Big\}.$$

Example:

$$\begin{aligned} \mathcal{S} := \Big\{(x, y) : x \in \{0, 1\}^4, \ & y_{12} = x_1 x_2, \\ & y_{14} = x_1 x_4, \\ & y_{34} = x_3 x_4, \\ & y_{123} = x_1 x_2 x_3, \\ & y_{234} = x_2 x_3 x_4 \Big\}. \end{aligned}$$

- The Multilinear polytope is

$$\mathrm{MP} := \mathrm{conv}(\mathcal{S}).$$

If each $I \in \mathcal{J}$ has cardinality two, then

- $\mathcal{S}$ is a Quadratic set,
- MP is Padberg's Boolean quadric polytope.

Alberto: Multilinear sets subsume the complexity of a variety of problems:

- 0/1 multilinear optimization:

$$\begin{aligned} \text{maximize} \quad & 6x_1x_2 + 2x_1x_4 - x_3x_4 - 2x_1x_2x_3 + x_2x_3x_4 \\ \text{subject to} \quad & x \in \{0,1\}^4. \end{aligned}$$

- Also with mixed-0/1 variables.
- Also with box constraints on continuous variables.
- 0/1 polynomial optimization.
- 0/1 nonlinear optimization.

Aida: Multilinear sets allow us to construct sharp relaxations for many MINLP problems via factorable reformulation.

Previous work in many different communities:

- Boolean Quadric Polytope,
- Max-cut,
- Pseudo-Boolean optimization,
- Global optimization.

Some authors that worked on higher degrees:

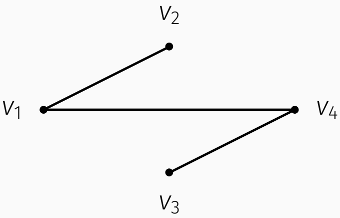
W.P. Adams, F.A. Al-Khayyal, X. Bao, E. Boros, C. Buchheim, S. Cafieri, Y. Crama, J.E. Falk, C.A. Floudas, P.L. Hammer, J. Lee, L. Liberti, J.T. Linderoth, J. Luedtke, C.A. Meyer, M. Namazifar, M. Padberg, J-P. Richard, A.D. Rikun, E. Rodríguez-Heck, H.S. Ryoo, S. Rudeanu, N.V. Sahinidis, H.D. Sherali, M. Tawarmalani, S. Ursic, M.J. Wilczak, C. Xiong.

REPRESENTATION

There is a bijection between Quadratic sets and graphs.

- One node v_i for each variable x_i.
- One edge for each variable y_I.
- Edge corresponding to y_I is $\{v_i : i \in I\}$.

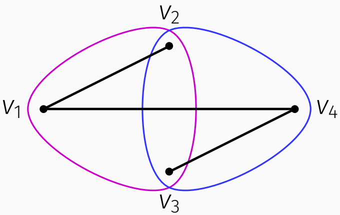
$$
\begin{aligned}
y_{12} &= x_1 x_2 \\
y_{14} &= x_1 x_4 \\
y_{34} &= x_3 x_4
\end{aligned}
\quad \longleftrightarrow
$$

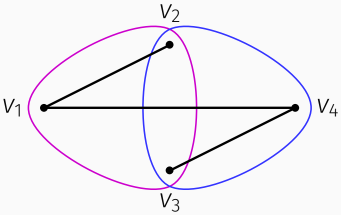


Similarly, there is a bijection between Multilinear sets and hypergraphs.

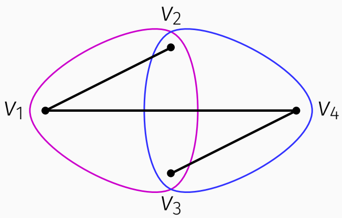
- One node v_i for each variable x_i.
- One edge for each variable y_I.
- Edge corresponding to y_I is $\{v_i : i \in I\}$.

$$\begin{aligned} y_{12} &= x_1x_2 \\ y_{14} &= x_1x_4 \\ y_{34} &= x_3x_4 \\ y_{123} &= x_1x_2x_3 \\ y_{234} &= x_2x_3x_4 \end{aligned} \quad \longleftrightarrow$$

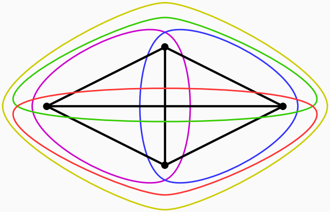




- Section hypergraph of G induced by U: $G' = (U, E')$, where E' consists of all the edges contained in U.



- Section hypergraph of G induced by U: $G' = (U, E')$, where E' consists of all the edges contained in U.
- A hypergraph $G = (V, E)$ is complete if all subsets of V of cardinality at least two are present in E.



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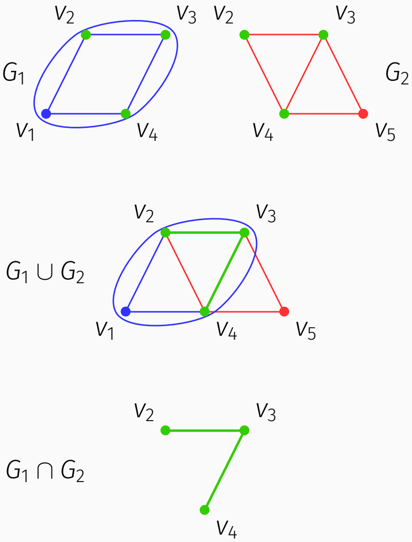
HYPERGRAPH DEFINITIONS



Given hypergraphs $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$,

- $\underline{G_1 \cup G_2}$ is the hypergraph $(V_1 \cup V_2, E_1 \cup E_2)$.
- $\underline{G_1 \cap G_2}$ is the hypergraph $(V_1 \cap V_2, E_1 \cap E_2)$.

DECOMPOSABILITY

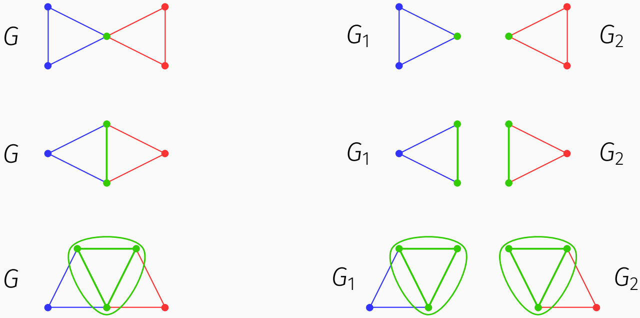
Let G_1 and G_2 be section hypergraphs of G such that $G_1 \cup G_2 = G$. We say that G is <u>decomposable</u> into G_1 and G_2 if

$$\mathrm{MP}_G = \mathrm{MP}_{G_1} \cap \mathrm{MP}_{G_2}.$$

- Why is decomposition important?
- We present a necessary and sufficient condition for decomposability of G based on the structure of $G_1 \cap G_2$.

Theorem

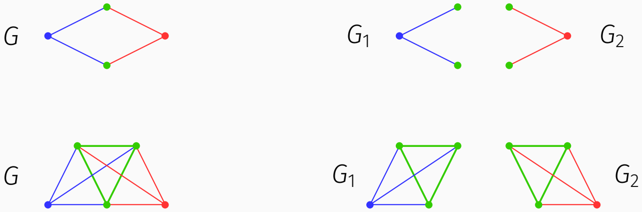
If $G_1 \cap G_2$ is a complete hypergraph, then G is decomposable into G_1 and G_2.

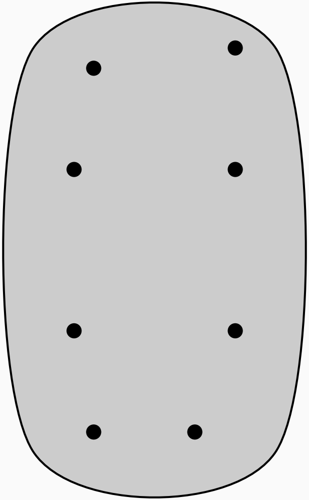


A NECESSARY CONDITION FOR DECOMPOSABILITY

Theorem

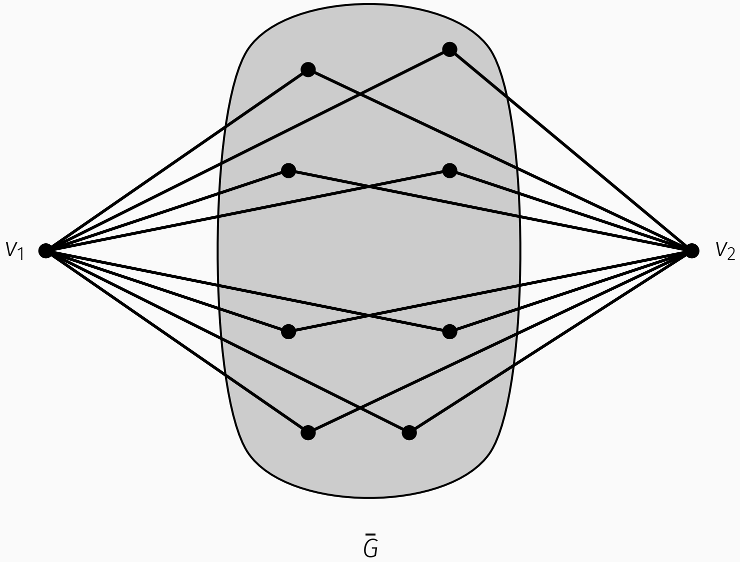
Let $\bar{G}$ be a hypergraph that is not complete. Then there exist hypergraphs G_1, G_2, with $\bar{G} = G_1 \cap G_2$, such that $G_1 \cup G_2$ is not decomposable into G_1 and G_2.



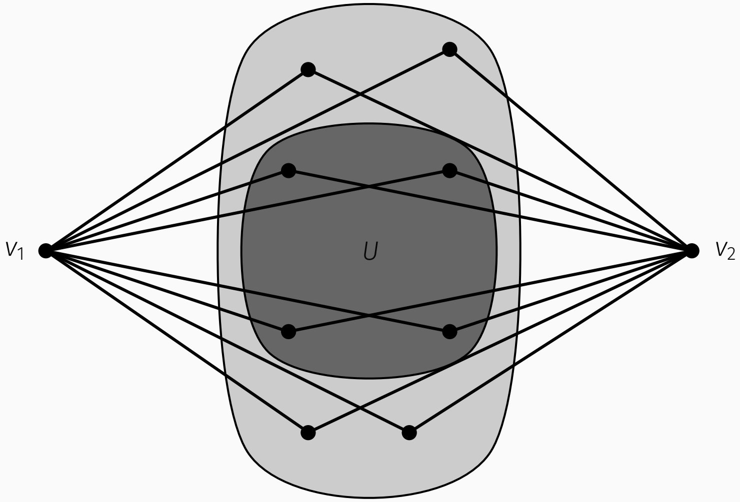


$\bar{G}$

PROOF SKETCH

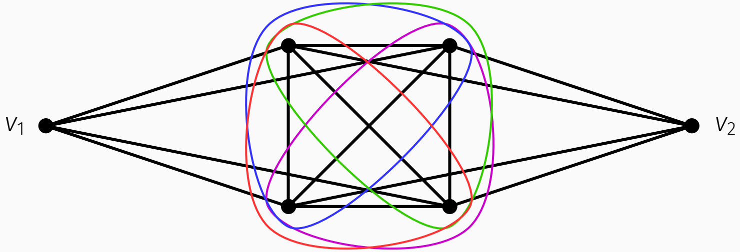


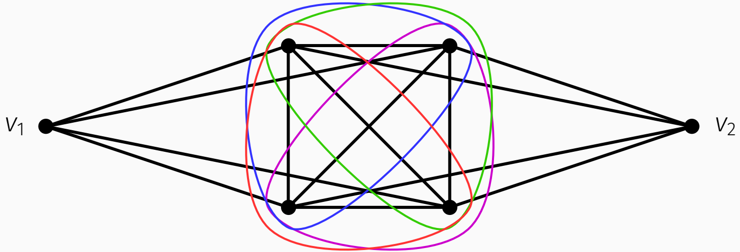
PROOF SKETCH



$\bar{G}$

PROOF SKETCH



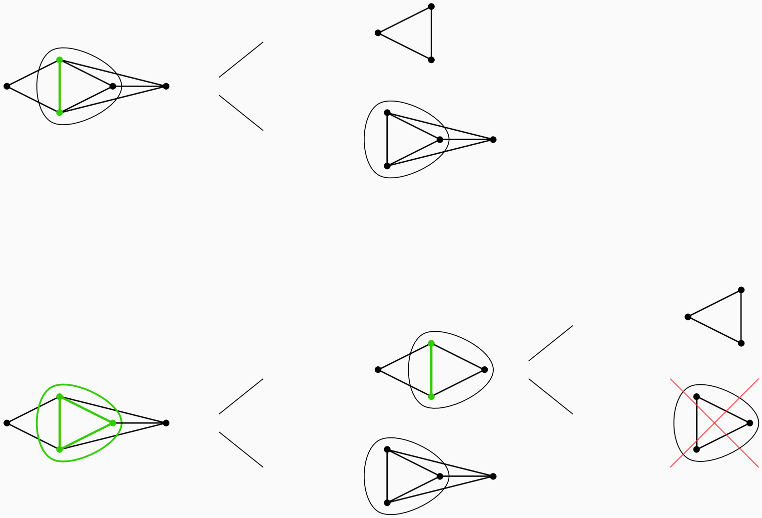


$$r(r-1)x_1+(r-1)x_2+r\sum_{i\in U}x_i-(r-1)\sum_{i\in U}y_{1i}-\sum_{i\in U}y_{2i}-\sum_{\substack{e\subset\{3,\dots,r+3\}\\|e|=r}}y_e\leq r^2-1$$

is facet-defining for the Multilinear polytope of this hypergraph. $\square$

DECOMPOSITION ALGORITHM

DECOMPOSITION



A decomposition of G is a family of hypergraphs G_k, with the following properties:

(i) $\mathrm{MP}_G = \bigcap_k \mathrm{MP}_{G_k}$,

(ii) No G_k is further decomposable,

(iii) No hypergraph G_k is a section hypergraph of another hypergraph $G_{k'}$.

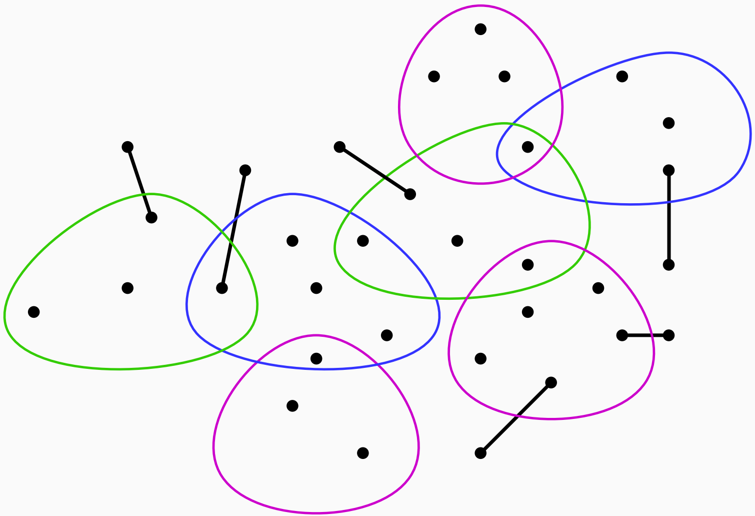
Theorem

Consider a hypergraph G with n nodes and m edges. The decomposition of G is unique and can be found with $O(rm(n+m))$ operations, where r is the maximum cardinality of an edge.

At each iteration, this algorithm chooses one hypergraph $\tilde{G}$ in the current family and tests a set p for "decomposition" of $\tilde{G}$ of minimal cardinality that is not tested in $\tilde{G}$ and in any ancestor of $\tilde{G}$.

ACYCLIC HYPERGRAPHS

BERGE-ACYCLIC HYPERGRAPHS

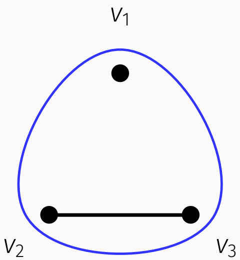


A hypergraph is called Berge-acyclic if it contains no Berge-cycles.

Definition

A Berge-cycle in G of length t for some $t \geq 2$, is a sequence $C = v_1, e_1, v_2, e_2, \ldots, v_t, e_t, v_1$ with the following properties:

- $v_1, v_2, \ldots, v_t$ are distinct nodes of G,
- $e_1, e_2, \ldots, e_t$ are distinct edges of G,
- for all i, the node v_i belongs to e_{i-1} and e_i.



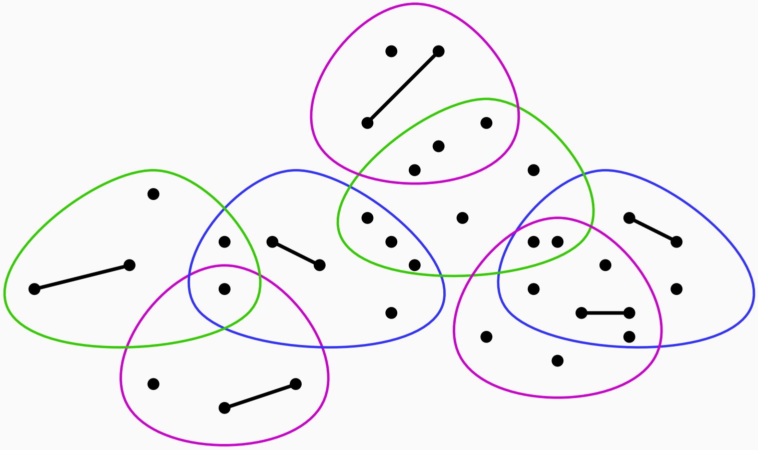
Berge-cycle: $v_2, \{v_2, v_3\}, v_3, \{v_1, v_2, v_3\}$.

Theorem

A hypergraph G is Berge-acyclic if and only if the polytope MP_G is described by the standard linear relaxation: [1]

$$
\begin{aligned}
x_v &\leq 1 && \forall v \in V, \\
y_e &\geq 0 && \forall e \in E, \\
y_e &\geq \sum_{v \in e} x_v - |e| + 1 && \forall e \in E, \\
y_e &\leq x_v && \forall v \in e,\ \forall e \in E.
\end{aligned}
$$

[1] This result has been obtained independently by Buchheim, Crama, and Rodríguez-Heck [2016].

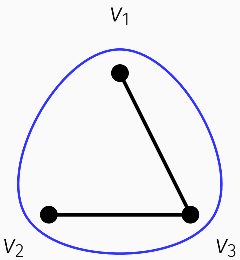


A hypergraph is called γ-acyclic if it contains no γ-cycles.

Definition

A γ-cycle in G of length t for some $t \geq 3$, is a sequence $C = v_1, e_1, v_2, e_2, \dots, v_t, e_t, v_1$ with the following properties:

- $v_1, v_2, \dots, v_t$ are distinct nodes of G,
- $e_1, e_2, \dots, e_t$ are distinct edges of G,
- for all $i \in \{1, \dots, t-1\}$, the node v_i belongs to e_{i-1}, e_i and no other e_j,
- v_t belongs to e_{t-1} and e_t (and possibly other e_js).



γ-cycle:
$v_1, \{v_1, v_2, v_3\}, v_2, \{v_2, v_3\}, v_3, \{v_1, v_3\}$.

Theorem

Given a γ-acyclic hypergraph G, we can optimize a linear function over MP_G in polynomial time.

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