Lattice closure of polyhedra

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Cutting planes for mixed-integer sets

- *Consider the mixed-integer set:*

$$P^{IP} \; = \; \{x \in \mathcal{R}^n \; : \; Ax \geq b\} \cap (\mathcal{Z}^{n_1} \times \mathcal{R}^{n_2})$$

where A *and* b *are rational, and* $n_1 + n_2 = n$ *.*

- *An inequality* $ax \geq b$ *is valid for* P^{IP} *if*

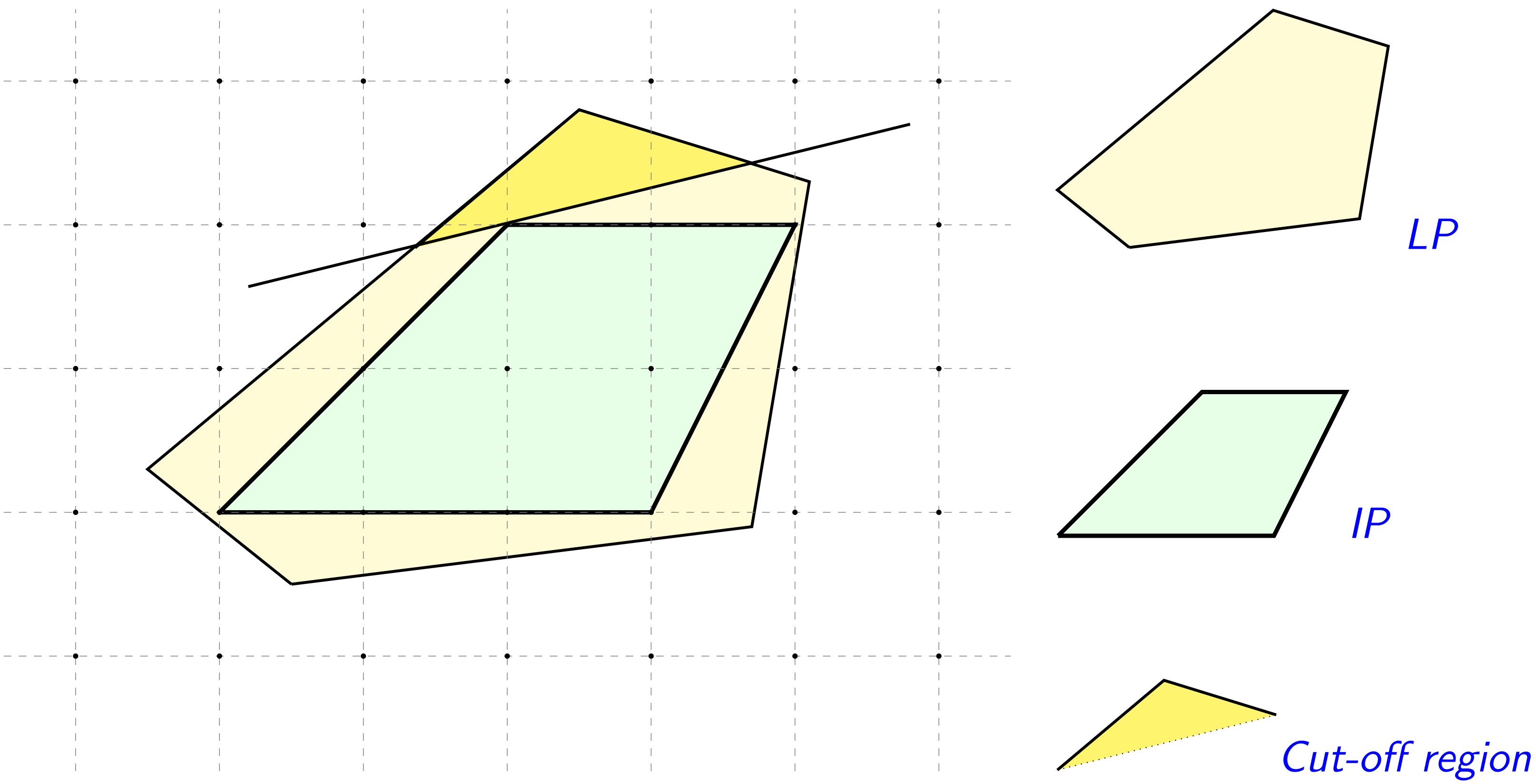
$$P^{IP} \; \subseteq \; \{x \in \mathcal{R}^n \; : \; ax \geq b\}$$

and a cutting plane if

$$P^{LP} \; = \; \{x \in \mathcal{R}^n \; : \; Ax \geq b\} \; \not\subseteq \; \{x \in \mathcal{R}^n \; : \; ax \geq b\}.$$

- P^{LP} *strengthened with cutting planes gives a better approximation of* conv $\left(P^{IP}\right)$

Cutting Planes for MILP



- *The region cut-off by the valid inequality is strictly lattice-free (i.e. no integer points).*
- *Conversely, excluding lattice-free regions from LP gives valid inequalities.*
- *Proving that a set is lattice-free is hard in general.*
- *Easier to work with sets* **known** *to be lattice-free for cut generation.*

Split sets and split cuts

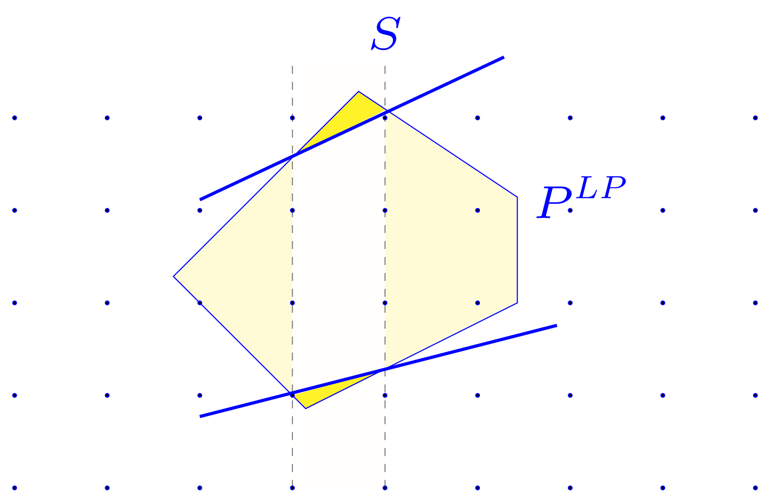
Consider the **split set** *for* $M = (\mathcal{Z}^{n_1} \times \mathcal{R}^{n_2})$:

$$S(\pi, \gamma) = \{x \in \mathcal{R}^n \ : \ \gamma + 1 > \pi x > \gamma\}$$

where $\pi \in \mathcal{Z}^n$, $\gamma \in \mathcal{Z}$, *and* $\pi_j \neq 0$ *only if* $j \leq n_1$.

Clearly

$$P^{LP} \supseteq \mathit{conv}\left(P^{LP} \setminus S(\pi, \gamma)\right) \supseteq P^{IP} = P^{LP} \cap M.$$



Split closure

- *Elementary split closure of* P^{LP} *is polyhedral:*

$$SC(P^{LP}) = \bigcap_{\pi \in \Pi} \bigcap_{\gamma \in \mathcal{Z}} \mathit{conv}\left(P^{LP} \setminus S(\pi, \gamma)\right)$$

where $\Pi = \mathcal{Z}^{n_1} \times \{0\}^{n_2}$ *(integer vectors with last* n_2 *elements zero.)*

- *There is an alternate definition of this closure*

$$SC(P^{LP}) = \bigcap_{\pi \in \Pi} \mathit{conv}\left(P^{LP} \cap \{x \,:\, \pi x \in \mathcal{Z}\}\right)$$

Note: $S(\pi, \gamma) \cap \{x \,:\, \pi x \in \mathcal{Z}\} = \emptyset$

- *Repeat:* $SC^2(P) = SC(SC(P))$, $SC^3(P),\ SC^4(P), \ldots$

- *There is a set in* $\mathcal{Z}^2 \times \mathcal{R}_+$ that has facets with unbounded split rank.

. [Cook/Kannan/Schrijver '90]

Some further results on split cuts

Let $\mathcal{S}^*$ *be the collection of all split sets for* $\mathcal{Z}^{n_1} \times \mathcal{R}^{n_2}$ *.*

- *For any* $\mathcal{S} \subset \mathcal{S}^*$

$$SC(P^{LP}, \mathcal{S}) = \bigcap_{S \in \mathcal{S}} \textit{conv}\left(P^{LP} \setminus S\right)$$

is also polyhedral.

[Andersen/Cornuéjols/Li '05]

- *Given* $\mathcal{S} \subseteq \mathcal{S}^*$ *, there exists a finite* $\mathcal{S}_F \subseteq \mathcal{S}$ *such that for any* $S \in \mathcal{S}$

$$\textit{conv}\left(P^{LP} \setminus S'\right) \subseteq \textit{conv}\left(P^{LP} \setminus S\right)$$

for some $S' \in \mathcal{S}_F$ *.*

[Averkov '12]

Two generalizations of split cuts

The split closure:

$$Cl(P^{LP}, \mathcal{S}^*) \;=\; \bigcap_{S \in \mathcal{S}^*} \textit{conv}\left(P^{LP} \setminus S\right) \;=\; \bigcap_{\pi \in \Pi} \textit{conv}\left(P^{LP} \cap \{x \;:\; \pi x \in \mathcal{Z}\}\right)$$

where $\mathcal{S}^*$ *is the collection of all split sets for* $(M = \mathcal{Z}^{n_1} \times \mathcal{R}^{n_2})$ *.*

- **t-branch split closure:**

$$Cl(P^{LP}, \mathcal{T}^*) = \bigcap_{T \in \mathcal{T}^*} \textit{conv}\left(P^{LP} \setminus T\right)$$

where $\mathcal{T}^*$ *is the collection of all* T *such that* $T = \cup_{i=1}^{t} S_i$ *where* $S_i \in \mathcal{S}^*$ *.*

- **k-lattice closure:**

$$Cl(P^{LP}, \mathcal{H}^*) = \bigcap_{(\pi_1,\ldots,\pi_k) \in \mathcal{H}*} \textit{conv}\left(P^{LP} \cap \{x \;:\; \pi_i x \in \mathcal{Z} \;\; \textit{for } i = 1, \ldots, k\}\right)$$

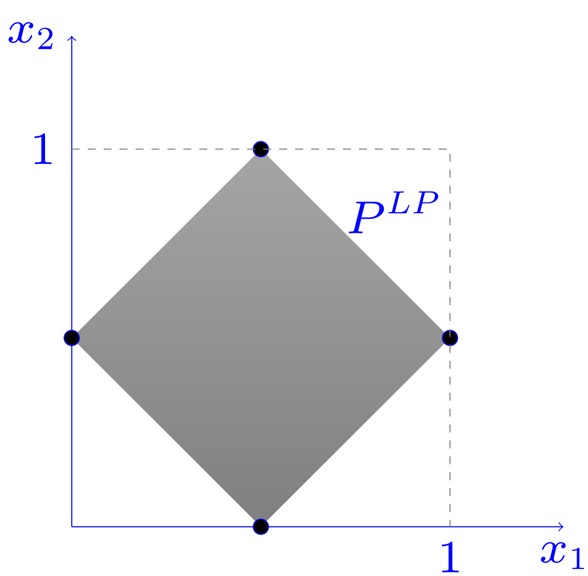
where $\mathcal{H}^*$ *is the collection of all* $(\pi_1, \ldots, \pi_k) \in \Pi^k$ *.*

a remark

- *In general*

$$\textit{conv}\left(P^{LP} \setminus (S_1 \cup S_2)\right) \subseteq \textit{conv}\left(P^{LP} \setminus S_1\right) \bigcap \textit{conv}\left(P^{LP} \setminus S_2\right)$$

and the inclusion can be strict when $P^{LP} \cap S_1 \cap S_2 \neq \emptyset$.



Let $S_k = \{1 > x_k > 0\}$ *for* $k = 1, 2$

Then

$$\textit{conv}\left(P^{LP} \setminus (S_1 \cup S_2)\right) = \emptyset$$

But

$$\textit{conv}\left(P^{LP} \setminus S_1\right) \cap \textit{conv}\left(P^{LP} \setminus S_2\right) = (1/2, 1/2)$$

t-branch split closure

- *Given* $t \in \mathcal{Z}$ *, a (rational) mixed integer set*

$$P^{IP} \; = \; \{x \in \mathcal{R}^{n_1+n_2} \; : \; Ax \geq b\} \cap (\mathcal{Z}^{n_1} \times \mathcal{R}^{n_2})$$

  and a subset $\mathcal{T} \subseteq \mathcal{T}^*$ *(a collection of* $T = \cup_{i=1}^{t} S_i$ *where* $S_i \in \mathcal{S}^*$ *),*

$$Cl(P^{LP}, \mathcal{T}) = \bigcap_{T \in \mathcal{T}} \text{conv}\left(P^{LP} \setminus T\right)$$

  is a polyhedron.

- *Furthermore, there exists a finite subset* $\mathcal{T}_F \subseteq \mathcal{T}$ *such that for all* $T \in \mathcal{T}$

$$\text{conv}\left(P^{LP} \setminus T'\right) \subseteq \; \text{conv}\left(P^{LP} \setminus T\right)$$

  for some $T' \in \mathcal{T}^F$ *(i.e. each* $T \in \mathcal{T}$ *is dominated by some* $T' \in \mathcal{T}^F$ *.)*
  . *[Dash/Gunluk/Moran '15]*

- *For all* t *there is a mixed-integer set with unbounded* t*-branch split rank.*
  . *[Dash/Gunluk '13]*

k-lattice closure:

$$Cl(P^{LP}, \mathcal{H}) = \bigcap_{(\pi_1,\ldots,\pi_k)\in\mathcal{H}} \textit{conv}\left(P^{LP} \cap \{x \ : \ \pi_i x \in \mathcal{Z} \ \ \textit{for } i = 1,\ldots,k\}\right)$$

where $\mathcal{H} \subseteq \mathcal{H}^*$ *and* $\mathcal{H}^*$ *is the collection of all* $(\pi_1, \ldots, \pi_k) \in \mathcal{Z}^{n\times k}$ *.*

The results:

- *Given a (rational) polyhedral set* P^{LP} *and* $\mathcal{H} \subseteq \mathcal{H}^*$ *, the k-lattice closure* $Cl(P^{LP}, \mathcal{H})$ *is a polyhedron.*
- *There is a finite subset* $\mathcal{H}_F \subseteq \mathcal{H}$ *that gives member-wise domination.*
- *There is a (rational) mixed integer set whose integer hull cannot be obtained by applying the k-lattice closure repeatedly.*

Why call it lattice closure?

For $k = 2$ *let* $\pi_1, \pi_2 \in \mathcal{Z}^n \setminus \{0\}$ *, and* $M(\pi_1, \pi_2) = \{x : \pi_1 x \in \mathcal{Z}, \pi_2 x \in \mathcal{Z}\}$.

Then for any $q \in \mathcal{Z} \setminus \{0\}$

$$M(\pi_1, \pi_2) = M(\pi_1, q\pi_1 + \pi_2).$$

$\Rightarrow$ *the pair* $\{\pi_1, \pi_2\}$ *does not uniquely define the* "mixed-lattice" $M(\pi_1, \pi_2)$.

The lattice generated by set of rational vectors $\{\pi_1, \ldots, \pi_k\}$ *in* $\mathcal{R}^n$ *is*

$$L = \{x \in \mathcal{R}^n : x = u_1\pi_1 + \cdots + u_k\pi_k,\ u \in \mathcal{Z}^k\}.$$

and its **dual lattice** *is*

$$\begin{aligned} L^* &= \{x \in \text{span}(L) : y^T x \in \mathcal{Z} \text{ for all } y \in L\}, \\ &= \{x \in \text{span}(L) : \pi_i^T x \in \mathcal{Z} \text{ for } i = 1, \ldots, k\}. \end{aligned}$$

The **mixed lattice** *in* $\mathcal{R}^n$ *generated by the dual lattice of* L *is*

$$M = L^* + \text{span}(L)^\perp$$

Recap of main result with lattice notation

For $\pi \in \mathcal{Z}^n \setminus \{0\}$ *, let* $M(\pi) = \{x \in \mathcal{R}^n \,:\, \pi^T x \in \mathcal{Z}\}$.

- *Define*

$$\mathcal{M}_n^1 = \{M(\pi) \,:\, \pi \in \mathcal{Z}^n \setminus \{0\}\}$$

- *and*

$$\mathcal{M}_n^k = \left\{ \cap_{j=1}^k M_j \,:\, M_j \in \mathcal{M}_n^1 \text{ for all } j \in \{1, \ldots, k\} \right\}.$$

As $\mathcal{Z}^n \subset M(\pi)$ *for all* $\pi \in \mathcal{Z}^n \setminus \{0\}$ *, any* $M \in \mathcal{M}_n^k$ *contains* $\mathcal{Z}^n$ *.*

Given a rational polyhedron $P \subset \mathcal{R}^n$ *and* $\mathcal{M} \subseteq \mathcal{M}_n^k$ *, the lattice closure*

$$Cl(P, \mathcal{M}) = \bigcap_{M \in \mathcal{M}} \text{conv}\,(P \cap M)\,.$$

is a polyhedron and there exists finite $\mathcal{M}_F \subseteq \mathcal{M}$ *that dominates* $\mathcal{M}$ *elementwise.*

($k = 1$ $\Rightarrow$ *split closure;* $k = 2$ $\Rightarrow$ *crooked-cross closure)*

Polytopes vs polyhedra: Integer hulls and lattice closure

- *We know that if* P^{LP} is an unbounded rational polyhedron; then

$$P^{LP} = \text{conv}\,(v_1, \ldots, v_m) + \text{cone}\,(r_1, \ldots, r_t)$$

where all rays r_i are integral. (can be done by scaling)

- Letting $P^{LP} = Q^{LP} + C$ define

$$\bar{Q}^{LP} = Q^{LP} + \left\{ \sum_{i=1}^{t} \lambda_i r_i : 0 \le \lambda_i \le 1 \text{ for } i = 1, \ldots t \right\}$$

- **Meyer's theorem:** If nonempty, $P^{IP} = \bar{Q}^{IP} + C$

- **Extension:** If nonempty, $\text{conv}\,\left(P^{LP} \cap M\right) = \text{conv}\,\left(\bar{Q}^{LP} \cap M\right) + C$ for $M \in \mathcal{M}_n^k$

M^1 *dominates* M^2 *on* P^{LP} *if and only if* M^1 *dominates* M^2 *on* $\bar{Q}^{LP}$

Fairly well ordered qosets

Given $P \subset \mathcal{R}^n$ *and* $M^1, M^2 \in \mathcal{M}_n^k$ *define the binary relation* $\preceq_P$ *as*

$$M^1 \preceq_P M^2 \qquad \text{if and only if} \qquad \textit{conv}\left(P \cap M^1\right) \subseteq \textit{conv}\left(P \cap M^2\right).$$

Note that $\preceq_P$ *defines a* **quasi-order** *on* $\mathcal{M}_n^k$ *as it is*

1. *reflexive (i.e.,* $M \preceq_P M$ *for all* $M \in \mathcal{M}_n^k$ *), and*
2. *transitive (i.e., if* $M^1 \preceq_P M^2$ *and* $M^2 \preceq_P M^3$ *, then* $M^1 \preceq_P M^3$ *).*

   (Not a partial order as it is not antisymmetric)

$\Rightarrow$ $(\mathcal{M}_n^k, \preceq_P)$ *is a qoset.*

A **qoset** $(X, \preceq)$ *is called* **fairly well-ordered** *if each* $X' \subseteq X$ *has a finite subset* $X'_F \subseteq X'$ *such that for all* $x \in X'$ *, there exists* $y \in X'_F$ *such that* $y \preceq x$ *.*

Recap of main result in qoset language

A qoset $(X^*, \preceq)$ is called fairly well-ordered if each $X \subseteq X^*$ has a finite subset $X_F \subseteq X$ such that for all $x \in X$, there exists $y \in X_F$ such that $y \preceq x$.

Main result:

For any rational polyhedra $P \subset \mathcal{R}^n$, the qoset $(\mathcal{M}_n^k, \preceq_P)$ is fairly well-ordered.

proof:

- *Given $\mathcal{M} \subseteq \mathcal{M}_n^k$, induction on the dimension dim(P).*
- *If the result holds for lower dimensional polyhedra, then it holds for each facet of P.*
- *Then $(\mathcal{M}, \preceq_{F_i})$ is fairly well-ordered for the facets $\{F_i\}$ of P.*
- *Using [Higmans'52], $(\mathcal{M}, \preceq_Q)$ where $Q = \cup_i F_i$ is fairly well-ordered.*

  i.e. for some finite $\mathcal{M}_F \subseteq \mathcal{M}$, $Cl(Q, \mathcal{M}) = Cl(Q, \mathcal{M}_F)$
- *Remaining $M \in \mathcal{M} \setminus \mathcal{M}_F$ have an effect on P only if they exclude a ball $B(\delta)$...*

Lemma : *Let* $P \subseteq \mathcal{R}^n$ *be a polytope and* $M' \in \mathcal{M}_n^k$ *be a mixed-lattice. Let* $M \in \mathcal{M}_n^k$ *be such that* $P \cap M \neq \emptyset$ *, and* M *is dominated by* M' *on all facets of* P *but not on* P *. Then there is a constant* κ *, that depends only on* P *and* M'*, such that there is an* $\tilde{M} \in \mathcal{M}_n^k$ *that satisfies (i)* aff$(P) \cap M =$ aff$(P) \cap \tilde{M}$ *, (ii)* $\tilde{M} = M(\pi) \cap M^2$ *where* $\|\pi\| \leq \kappa$ *and* $M^2 \in \mathcal{M}_n^{k-1}$ *, and (iii)* $P \not\subset M(\pi)$.

$\|\pi\| \leq \kappa$ *implies:*

- *All such* π *form a finite set*
- *Each such* $M(\pi) \cap P$ *is the union of a finite number of slices*
- *Each slice is a lower dimensional polytope.*
- *Finite dominance as the result holds for lower dimensional polytopes.*

■

Rank result

Let $P \subset \mathcal{R}^n$ *be a rational polyhedron and*

$$P^I = \{x \in \mathcal{R}^n : x \in P,\ x_i \in \mathcal{Z} \text{ for } i = 1, \ldots, k\}$$

Consider

$$\mathcal{M} = \{M \in \mathcal{M}_n^k : M \supseteq \mathcal{Z}^k \times \mathcal{R}^{n-k}\}.$$

Clearly

$$Cl(P, \mathcal{M}) = \bigcap_{M \in \mathcal{M}} \text{conv}\,(P \cap M) = \text{conv}(P^I)$$

as $(\mathcal{Z}^k \times \mathcal{R}^{n-k}) \in \mathcal{M}$.

Now consider $\mathcal{M} \cap \mathcal{M}^{k-1}$ *, mixed-lattices with lattice dimension at most* $k - 1$.

> *There exists a rational polyhedron* $P \subset \mathcal{R}^n$ *for which the repeated closure*
>
> $$Cl^q(P, \mathcal{M} \cap \mathcal{M}^{k-1}) \neq \text{conv}(P^I),$$
>
> *for any finite* $q > 0$.

The set $P(h)$

Consider the n-dimensional simplex

$$S = \{x \in \mathcal{R}^n : \sum_{i=1}^{n} x_i \leq n, x_i \geq 0 \text{ for } i = 1, \ldots, n\}.$$

(S is integral and maximal lattice-free.)

For $x \in S$, let $d(x)$ denote its distance from closest facet.

$$d(x) = \min\{x_1, \ldots, x_n, (n - \sum_{i=1}^{n} x_i)/\sqrt{n}\}$$

Let

$$p = (1/2, \ldots, 1/2) \in S$$

and for any $h > 0$

$$P(h) = conv\,(S \times \{0\}\ ,\ \{(p, h)\}) \subset \mathcal{R}^{n+1}$$

Note:

$$P(h) \cap (\mathcal{Z}^n \times \mathcal{R}) = S \times \{0\}$$

Proof of the rank result

Thm: $Cl^q(P(1),\mathcal{M}) \neq P^I$ *where* $\mathcal{M} = \{M \in \mathcal{M}_{n+1}^{n-1} : M \supseteq \mathcal{Z}^n \times \mathcal{R}\},\ q \in \mathcal{Z}$.

proof. *Consider a mixed-lattice* $M = \cap_{i=1}^{n-1} M(\pi_i)$.

- *All* $\pi_i = \begin{pmatrix} \pi_i' \\ 0 \end{pmatrix}$ *where* $\pi_i' \in \mathcal{Z}^n$ *for* $i = 1, \ldots, n-1$.
- $\text{span}(\pi_1', \ldots, \pi_{n-1}')$ *has dimension strictly less than* n.
- *There exists a rational* $v \in \mathcal{R}^n$ *is orthogonal to all* π_i
- *For all* $y \in \mathcal{Z}^n$ *and* $\alpha \in \mathcal{R}$ *, the point* $x = y + \alpha v$ *satisfies* $\pi_i' x \in \mathcal{Z}$ *for all* i
- $\exists x \in S \cap (\mathcal{Z}^n + \text{span}(v))$ *with* $d(x) \geq 1/2n$.
- *For* $x \in S$ *if* $d(x) \geq \gamma$, *then* $(x, 2\gamma h/n) \in P(h)$.
- *Therefore* $(x, 2\gamma h/n) \in P(h) \cap M$.
- *[By the Height Lemma] For some* $\epsilon > 0$, $(p, \epsilon) \in P(h) \cap M$ *for all* $M \in \mathcal{M}_{n+1}^{n-1}$.

$P(\epsilon) \subseteq Cl(P(1),\mathcal{M})$ *for some* $\epsilon > 0 \ \Rightarrow\ P(\epsilon^q) \subseteq Cl^q(P(1),\mathcal{M})$

Cuts from lattice-free sets

Let $P^I = \{x \in \mathcal{R}^n : x \in P^{LP},\ x_i \in \mathcal{Z} \text{ for } i = 1, \ldots, k\}$

Consider a lattice-free set $F \subset \mathcal{R}^k$ i.e. $F \cap \mathcal{Z}^k = \emptyset$.

Clearly

$$P^I \subseteq \text{conv}\left(P^{LP} \setminus (F \times \mathcal{R}^{n-k})\right) \subseteq P^{LP}.$$

If the lineality space of F contains a non-zero rational vector, then we can show that

$$P \cap M = \emptyset$$

for some mixed lattice $M \in \mathcal{M}_k^{k-1}$ such that $M \supset \mathcal{Z}^k$.

Therefore, cuts generated by $F \times \mathcal{R}^{n-k}$ can also be generated by $M' = M \times \mathcal{R}^{n-k}$.

Iterated closure of $P(1)$ using such lattice-free sets does not give the integer hull.

(Example: t -branch split sets for $t < k$ and unbounded (max) convex lattice-free sets.)

thank you