FRIEDRICH-ALEXANDER UNIVERSITÄT ERLANGEN-NÜRNBERG
NATURWISSENSCHAFTLICHE FAKULTÄT

Mixed Integer Reformulations of Robust *b*-Matching Problems - Using Few Integer Variables

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WIRTSCHAFTS MATHEMATIK

Outline

Robust Bipartite *b*-Matching Problems in Air Traffic Management

Mixed Integer Reformulations of Integer Programs

Mixed Integer Reformulations for the Robust Bipartite *b*-Matching Problem

Computational Results

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Computational Results

Motivation: Air Traffic Management (ATM)

- continous growth of traffic demand
- possibilities of enlarging airport capacities are limited

source: tagaytayhighlands.net

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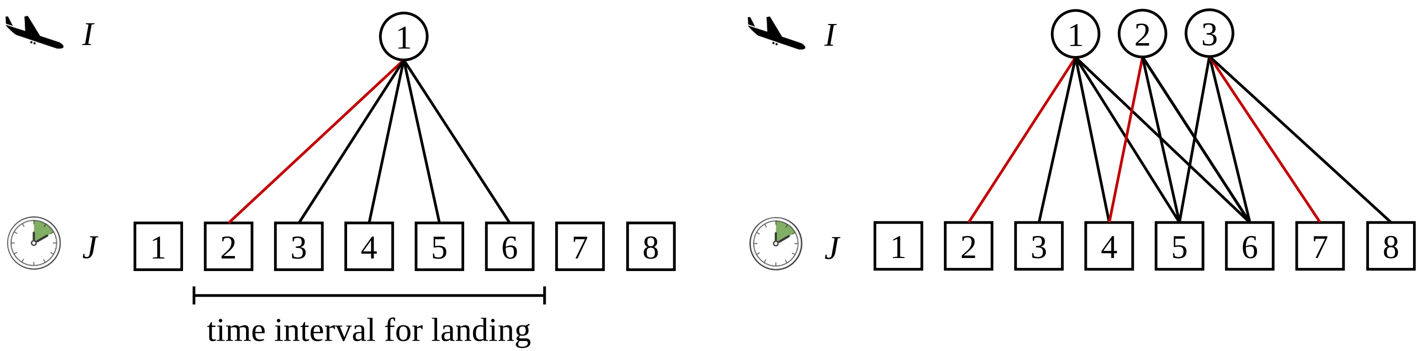
⇒ optimization of runway utilization is one of the main challenges in ATM

goal: runway schedules that are robust against uncertainties

Motivation: Airtraffic Management (ATM)

- assign aircraft to time windows
- for each aircraft: possible time interval for landing

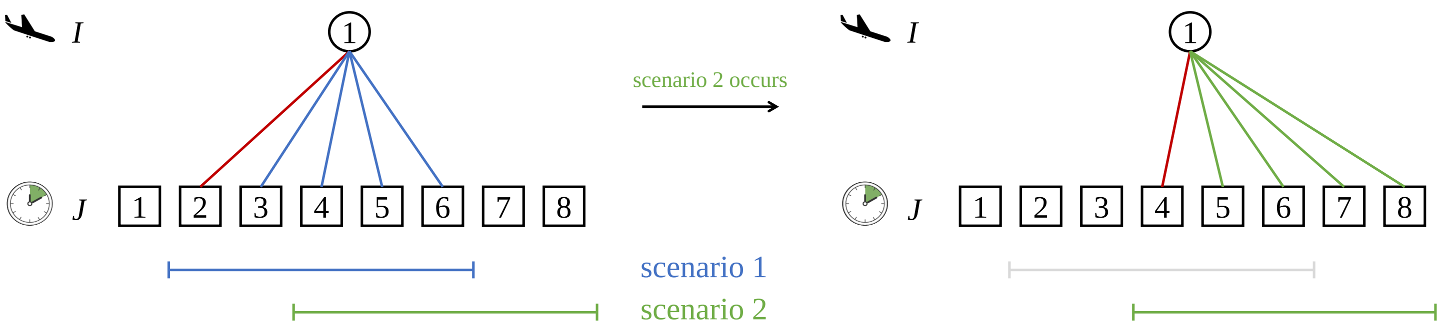


objective:

maximize punctuality i.e. minimize deviation from the (pubished) flight plan

Motivation: Airtraffic Management (ATM)

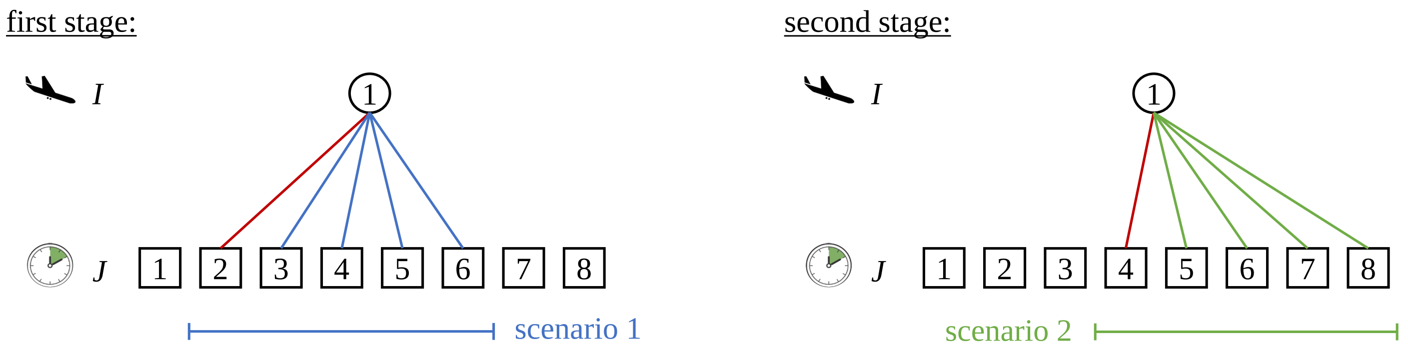
Uncertainty: time interval for landing



$\Rightarrow$ if scenario 2 occurs, plan of scenario 1 might become infeasible

$\Rightarrow$ the aircraft may need to be replanned

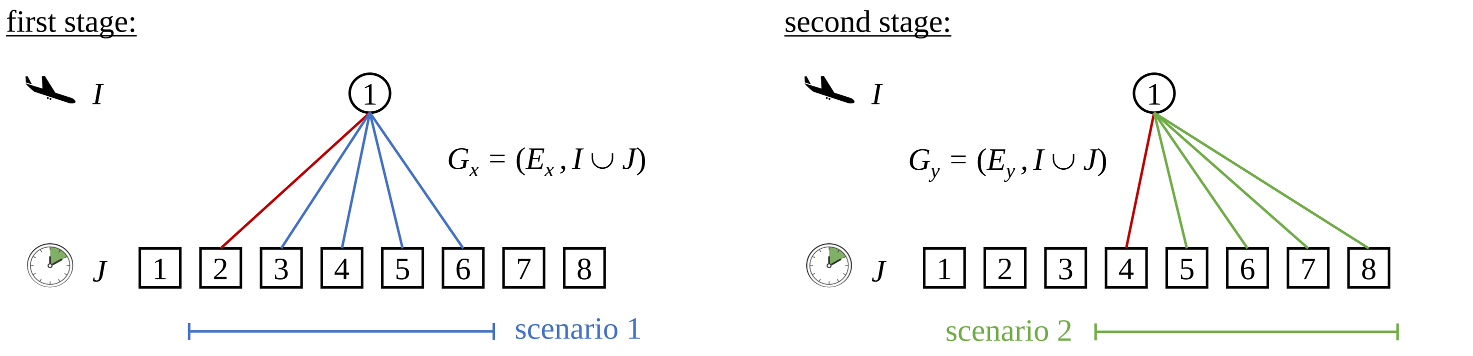
Motivation: Airtraffic Management (ATM)



Planning in two stages:

- Here and now: scheduling aircraft such that replanning is anticipated
- Uncertainty occurs: replanning

Motivation: Airtraffic Management (ATM)



Planning in two stages:

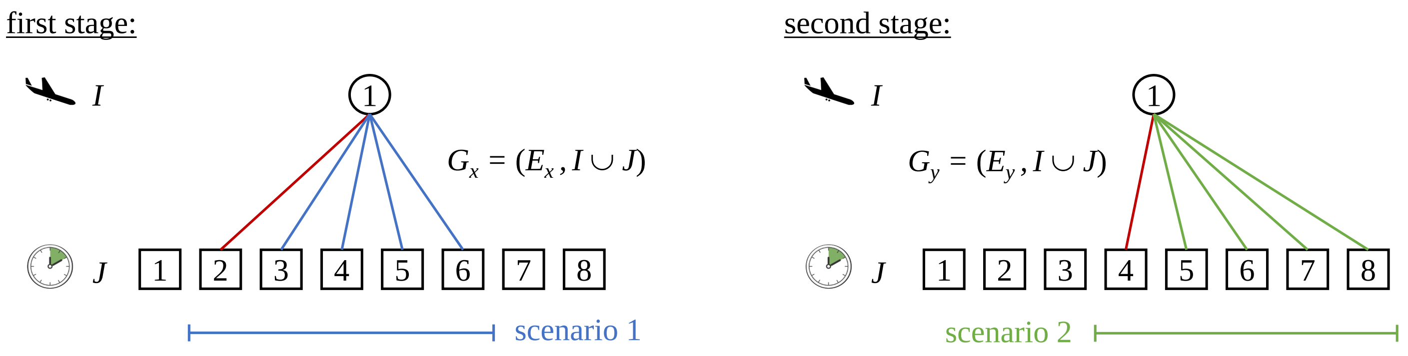
- Here and now: scheduling aircraft such that replanning is anticipated
- Uncertainty occurs: replanning

$$x_{ij} = \begin{cases} 1, & \text{if aircraft } i \text{ is assigned to time window } j \text{ on the } \textbf{first stage} \\ 0, & \text{otherwise} \end{cases}$$

$$y_{ij} = \begin{cases} 1, & \text{if aircraft } i \text{ is assigned to time window } j \text{ on the } \textbf{second stage} \\ 0, & \text{otherwise} \end{cases}$$

Motivation: Airtraffic Management (ATM)

For reasons of fairness: restrict replanning for each aircraft by at most *r*



Special knapsack constraint for **each** aircraft:

$$|2x_{12} + 3x_{13} + 4x_{14} + 5x_{15} + 6x_{16} - 4y_{14} - 5y_{15} - 6y_{16} - 7y_{17} - 8y_{18}| \leq r$$

$x_{ij} \in \{0, 1\}$ first stage variables, $y_{ij} \in \{0, 1\}$ second stage variables

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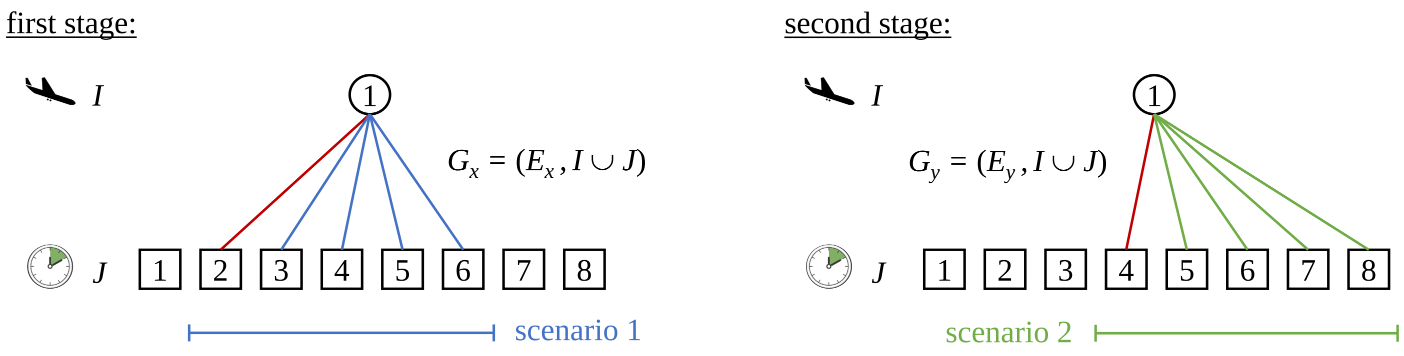

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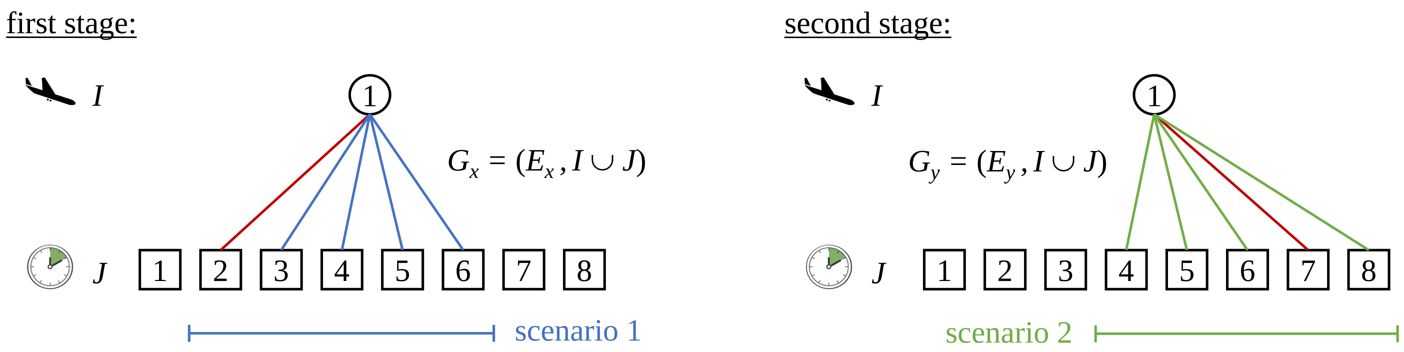

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$x_{ij} \in \{0,1\}$ first stage variables, $y_{ij} \in \{0,1\}$ second stage variables

Robust bipartite *b*-Matching Problem (RMP)

minimize deviation from scheduled times

$$x_e,\ y_f \quad \in \ \{0,1\} \qquad\qquad \forall e \in E_x,\ f \in E_y$$

Robust bipartite *b*-Matching Problem (RMP)

$$\min_{x,y} \quad \sum_{(i,j)\in E_x} c^x_{ij} x_{ij} + \sum_{(i,j)\in E_y} c^y_{ij} y_{ij}$$

$$x_e,\ y_f \quad \in \ \{0,1\} \qquad\qquad \forall e \in E_x,\ f \in E_y$$

Robust bipartite *b*-Matching Problem (RMP)

$$\min_{x,y} \sum_{(i,j)\in E_x} c^x_{ij} x_{ij} + \sum_{(i,j)\in E_y} c^y_{ij} y_{ij}$$

assign each aircraft to one time window on the first and second stage

$$x_e,\ y_f \quad \in \{0,1\} \qquad \forall e \in E_x,\ f \in E_y$$

Robust bipartite *b*-Matching Problem (RMP)

$$\min_{x,y} \quad \sum_{(i,j)\in E_x} c^x_{ij} x_{ij} + \sum_{(i,j)\in E_y} c^y_{ij} y_{ij}$$

$$\sum_{\substack{j\in J:\\(i,j)\in E_x}} x_{ij} = 1, \qquad \sum_{\substack{j\in J:\\(i,j)\in E_y}} y_{ij} = 1 \qquad \forall i \in I \qquad \text{(aircraft)}$$

$$x_e,\ y_f \quad \in \ \{0,1\} \qquad \forall e \in E_x,\ f \in E_y$$

Robust bipartite *b*-Matching Problem (RMP)

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assign at most *b* aircraft to a time window

$$x_e,\ y_f \quad \in \{0,1\} \qquad \forall e \in E_x,\ f \in E_y$$

Robust bipartite *b*-Matching Problem (RMP)

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$$\sum_{\substack{i\in I:\\(i,j)\in E_x}} x_{ij} \leq b, \qquad \sum_{\substack{i\in I:\\(i,j)\in E_y}} y_{ij} \leq b \qquad \forall j \in J \qquad \text{(time windows)}$$

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Robust bipartite b-Matching Problem (RMP)

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restrict replanning action

$$x_e,\ y_f \quad \in \ \{0,1\} \qquad \forall e \in E_x,\ f \in E_y$$

Robust bipartite b-Matching Problem (RMP)

$$\min_{x,y} \sum_{(i,j)\in E_x} c^x_{ij} x_{ij} + \sum_{(i,j)\in E_y} c^y_{ij} y_{ij}$$

$$\sum_{\substack{j\in J:\\(i,j)\in E_x}} x_{ij} = 1, \qquad \sum_{\substack{j\in J:\\(i,j)\in E_y}} y_{ij} = 1 \qquad \forall i \in I \qquad \text{(aircraft)}$$

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$$\left| \sum_{\substack{j\in J:\\(i,j)\in E_x}} j \cdot x_{ij} - \sum_{\substack{j\in J:\\(i,j)\in E_y}} j \cdot y_{ij} \right| \leq r_i \qquad \forall i \in I \qquad \text{(replanning constraints)}$$

$$x_e,\ y_f \quad \in \ \{0,1\} \qquad \forall e \in E_x,\ f \in E_y$$

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Robust Bipartite b-Matching Problems in Air Traffic Management

Mixed Integer Reformulations of Integer Programs

Mixed Integer Reformulations for the Robust Bipartite b-Matching Problem

Computational Results

Mixed Integer Reformulations

- Approach [Bader,Hildebrand,Weismantel,Zenklusen (2016)]
- Solve:

$$\begin{aligned} \max \quad & c^\top x \\ \text{s.t.} \quad & Ax \leq b \\ & x \in \mathbb{Z}^n \end{aligned}$$

Mixed Integer Reformulations

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- Solve:

$$\begin{aligned} \max \quad & c^\top x \\ \text{s.t.} \quad & Ax \leq b \\ & \cancel{x \in \mathbb{Z}^n} \quad x \in \mathbb{R}^n, Wx \in \mathbb{Z}^k \end{aligned}$$

Mixed Integer Reformulations

- Approach [Bader,Hildebrand,Weismantel,Zenklusen (2016)]
- Solve:

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Given: Polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$, A, b integral

Goal: Find $W \in \mathbb{Z}^{k \times n}$

$$\text{conv}(\{x \in P \mid x \in \mathbb{Z}^n\}) = \text{conv}(\{x \in P \mid Wx \in \mathbb{Z}^k\})$$

- k instead of n integrality constraints

Affine TU Decomposition of Matrix A:

$A, \overline{A}, U, W$ integer matrices

$$A = \overline{A} + UW$$

such that

$$\begin{pmatrix} \overline{A} \\ W \end{pmatrix}$$

is totally unimodular (TU).

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is totally unimodular (TU).

Theorem [Bader,Hildebrand,Weismantel,Zenklusen (2016)]:

Let

$P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$,

$A = \overline{A} + UW$ be affine TU decomposition,

then

$$\text{conv}(\{x \in P \mid x \in \mathbb{Z}^n\}) = \text{conv}(\{x \in P \mid Wx \in \mathbb{Z}^k\}).$$

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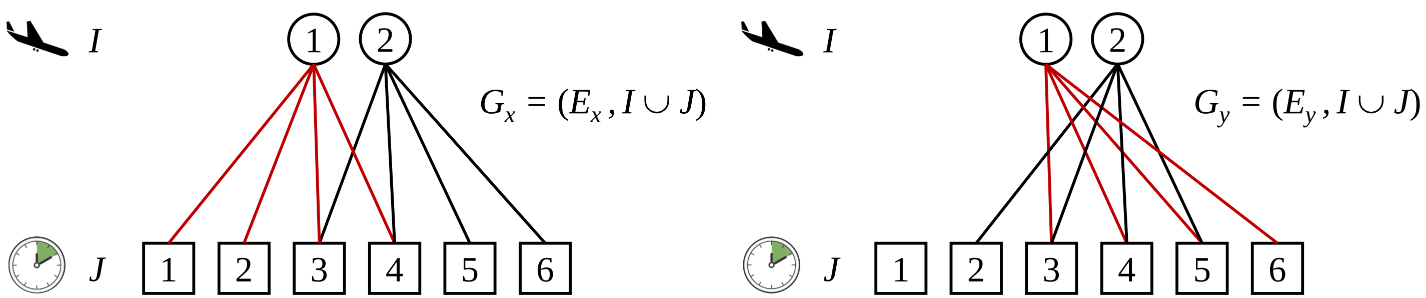
Computational Results

Robust bipartite *b*-Matching Problem with One Replanning Constraint for All Aircraft

Variant: restrict the whole replanning action



One special knapsack constraint for **all** aircraft:

$$\Big|(1x_{11} + 2x_{12} + 3x_{13} + 4x_{14}) + (3x_{23} + 4x_{24} + 5x_{25} + 6x_{26}) + \\ - (3y_{13} + 4y_{14} + 5y_{15} + 6y_{16}) - (2y_{22} + 3y_{23} + 4y_{24} + 5y_{25})\Big| \leq b$$

$x_{ij} \in \{0, 1\}$ first stage variables, $\quad y_{ij} \in \{0, 1\}$ second stage variables

Robust bipartite b-Matching Problem with One Re-planning Constraint for All Aircraft (RMP1)

$$\min_{x,y} \quad \sum_{ij\in E_x} c^x_{ij} x_{ij} + \sum_{ij\in E_y} c^y_{ij} y_{ij}$$

$$\sum_{\substack{j\in J:\\(i,j)\in E_x}} x_{ij} = 1, \qquad \sum_{\substack{j\in J:\\(i,j)\in E_y}} y_{ij} = 1 \qquad \forall i \in I \qquad \text{(aircraft)}$$

$$\sum_{\substack{i\in I:\\(i,j)\in E_x}} x_{ij} \leq b, \qquad \sum_{\substack{i\in I:\\(i,j)\in E_y}} y_{ij} \leq b \qquad \forall j \in J \qquad \text{(time windows)}$$

sum of the replanning actions over all aircraft

$$x_e,\ y_f \quad \in \ \{0,1\} \qquad \forall e \in E_x,\ f \in E_y$$

Robust bipartite b-Matching Problem with One Re-planning Constraint for All Aircraft (RMP1)

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$$\sum_{\substack{i \in I: \\ (i,j) \in E_x}} x_{ij} \leq b, \qquad \sum_{\substack{i \in I: \\ (i,j) \in E_y}} y_{ij} \leq b \qquad \forall j \in J \qquad \text{(time windows)}$$

$$\left| \sum_{i \in I} \sum_{\substack{j \in J: \\ (i,j) \in E_x}} j \cdot x_{ij} - \sum_{i \in I} \sum_{\substack{j \in J: \\ (i,j) \in E_y}} j \cdot y_{ij} \right| \leq r \qquad \text{(1 replanning constraint for all aircraft)}$$

$$x_e,\ y_f \quad \in \ \{0, 1\} \qquad \forall e \in E_x,\ f \in E_y$$

Robust bipartite b-Matching Problem with One Re-planning Constraint for All Aircraft (RMP1)

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$\longrightarrow M$

$$\left| \sum_{i \in I} \sum_{\substack{j \in J: \\ (i,j) \in E_x}} j \cdot x_{ij} - \sum_{i \in I} \sum_{\substack{j \in J: \\ (i,j) \in E_y}} j \cdot y_{ij} \right| \leq r$$

ONE replanning constraint! $\longrightarrow R$

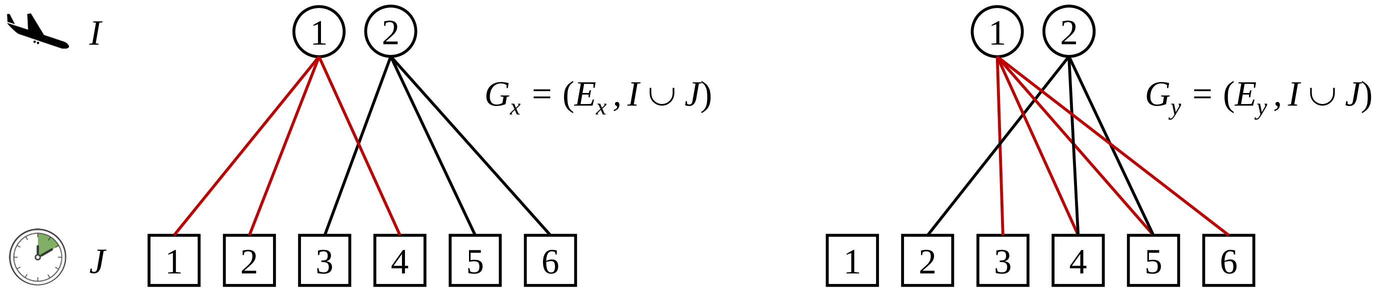
$$x_e, \; y_f \quad \in \quad \{0, 1\} \qquad \forall e \in E_x, \; f \in E_y$$

Affine TU Decomposition for RMP1 with **One** Replanning Constraint

Goal: affine TU decomposition

$$\begin{pmatrix} M \\ R \end{pmatrix} = \overline{A} + UW \quad \text{and} \quad \begin{pmatrix} \overline{A} \\ W \end{pmatrix} \quad \text{is TU.}$$

Example: Aggregating Variables



$$(1x_{11} + 2x_{12} + 3x_{13} + 4x_{14}) + (3x_{23} + 4x_{24} + 5x_{25} + 6x_{26}) +$$
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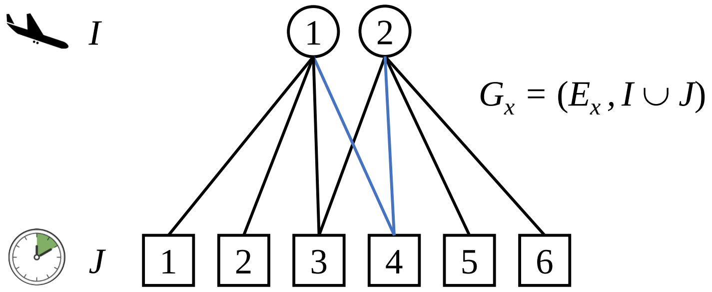

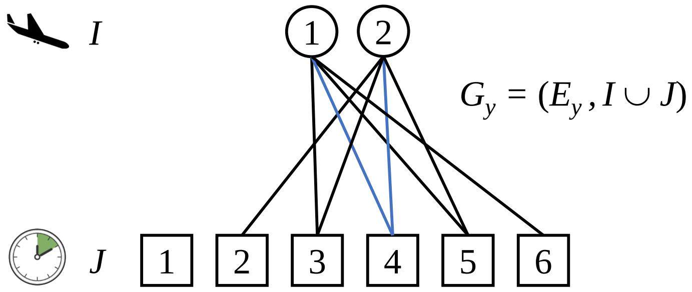

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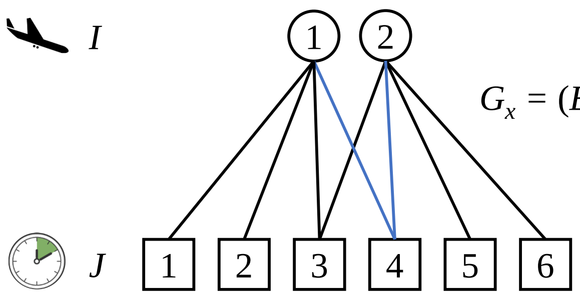

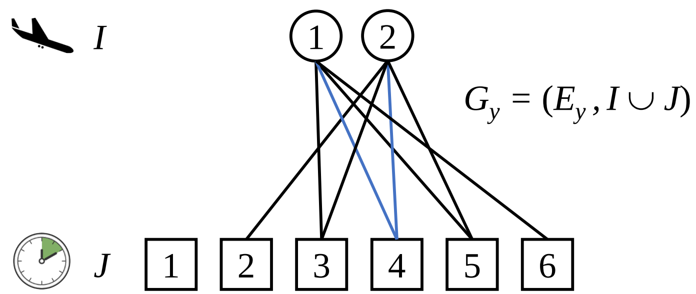

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$$-(3y_{13} + 4y_{14} + 5y_{15} + 6y_{16}) - (2y_{22} + 3y_{23} + 4y_{24} + 5y_{25}) \leq b$$

Aggregation:

$$z_j = \sum_{\substack{i \in I \\ (i,j) \in E_x}} x_{ij} - \sum_{\substack{i \in I \\ (i,j) \in E_y}} y_{ij} \qquad \forall j \in J$$

$$\begin{pmatrix} z_1 & \cdots & z_{|J|} \end{pmatrix}^T = W \begin{pmatrix} x \\ y \end{pmatrix}$$

Aggregating Variables

$$(1x_{11} + 2x_{12} + 3x_{13} + 4x_{14}) + (3x_{23} + 4x_{24} + 5x_{25} + 6x_{26})$$
$$- (3y_{13} + 4y_{14} + 5y_{15} + 6y_{16}) - (2y_{22} + 3y_{23} + 4y_{24} + 5y_{25}) \leq b$$

Aggregation:

$$R = \begin{pmatrix} 1 & 2 & \dots & |J| \end{pmatrix} \cdot W = \overline{U} \cdot W$$

Example:

$$\overline{U} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$$

$$W = \left(\begin{array}{cccccccc|cccccccc}
11 & 12 & 13 & 14 & 23 & 24 & 25 & 26 & 13 & 14 & 15 & 16 & 22 & 23 & 24 & 25 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0
\end{array}\right)$$

Affine TU Decomposition

$$\begin{pmatrix} M \\ R \end{pmatrix} = \overline{A} + UW$$

Affine TU Decomposition

$$\begin{pmatrix} M \\ R \end{pmatrix} = \overline{A} + UW$$
$$= \begin{pmatrix} M \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \hline U \end{pmatrix} W$$

Affine TU Decomposition

$$\begin{pmatrix} M \\ R \end{pmatrix} = \overline{A} + UW$$
$$= \begin{pmatrix} M \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \overline{U} \end{pmatrix} W$$

Lemma

The matrix $\begin{pmatrix} M \\ W \end{pmatrix}$ is TU.

One Replanning Constraint for All Aircraft (RMP1)

Lemma

Let

$$P = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{|E_x|+|E_y|} \;\middle|\; \begin{pmatrix} M \\ R \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \leq \begin{pmatrix} \tilde{b} \\ r \end{pmatrix} \right\}$$

Then

$$\operatorname{conv}\left(\begin{pmatrix} x \\ y \end{pmatrix} \in P \;\middle|\; \begin{pmatrix} x \\ y \end{pmatrix} \in \{0,1\}^{|E_x|+|E_y|} \right) = \operatorname{conv}\left(\begin{pmatrix} x \\ y \end{pmatrix} \in P \;\middle|\; W \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{Z}^{|J|} \right)$$

One Replanning Constraint for All Aircraft (RMP1)

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$\Rightarrow$ $|J|$ instead of $|E_x| + |E_y|$ integrality constraints

Affine TU Decomposition for RMP with a Replanning Constraint for Each Aircraft?

$$\min_{x,y} \sum_{(i,j)\in E_x} c^x_{ij} x_{ij} + \sum_{(i,j)\in E_y} c^y_{ij} y_{ij}$$

$$\sum_{\substack{j\in J:\\(i,j)\in E_x}} x_{ij} = 1, \quad \sum_{\substack{j\in J:\\(i,j)\in E_y}} y_{ij} = 1 \qquad \forall i \in I$$

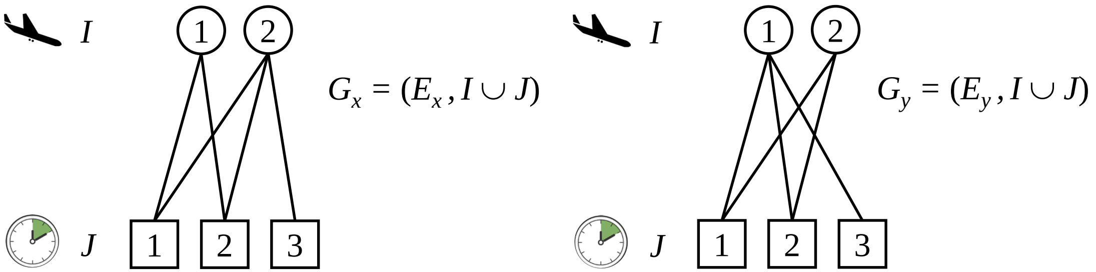
$$\sum_{\substack{i\in I:\\(i,j)\in E_x}} x_{ij} \le b, \quad \sum_{\substack{i\in I:\\(i,j)\in E_y}} y_{ij} \le b \qquad \forall j \in J \qquad \longrightarrow M$$

$$\left| \sum_{\substack{j\in J:\\(i,j)\in E_x}} j \cdot x_{ij} - \sum_{\substack{j\in J:\\(i,j)\in E_y}} j \cdot y_{ij} \right| \le r_i \qquad \forall \mathbf{i} \in \mathbf{I} \qquad \longrightarrow R$$

for each aircraft one replanning constraint

$$x_e,\ y_f \in \{0,1\} \qquad \forall e \in E_x,\ f \in E_y$$

Example for Replanning Constraints

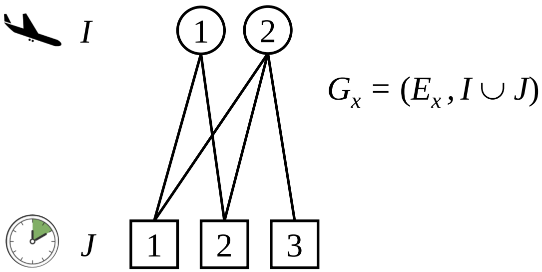


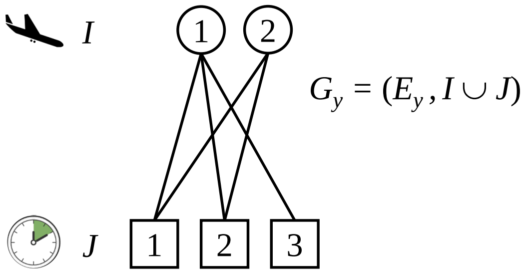
$$
\begin{aligned}
1x_{11} \quad +2x_{12} \quad\quad\quad\quad - 1y_{11} \quad -2y_{12} \quad -3y_{13} &\le r_1 \\
1x_{21} \quad +2x_{22} \quad +3x_{23} - 1y_{21} \quad -2y_{22} \quad\quad\quad &\le r_2
\end{aligned}
$$

Aggregation:

$$
\begin{aligned}
z_{ij} &= x_{ij} - y_{ij} && \forall j \in J : (i,j) \in E_x \ \wedge \ (i,j) \in E_y \\
z_{ij} &= x_{ij} && \forall j \in J : (i,j) \in E_x \ \wedge \ (i,j) \notin E_y \\
z_{ij} &= -y_{ij} && \forall j \in J : (i,j) \notin E_x \ \wedge \ (i,j) \in E_y
\end{aligned}
$$

Example for Replanning Constraints





$$1x_{11} + 2x_{12} - 1y_{11} - 2y_{12} - 3y_{13} \leq r_1$$
$$1x_{21} + 2x_{22} + 3x_{23} - 1y_{21} - 2y_{22} \leq r_2$$

Aggregation:

$$z_{ij} = x_{ij} - y_{ij} \qquad \forall j \in J : (i,j) \in E_x \wedge (i,j) \in E_y$$
$$z_{ij} = x_{ij} \qquad \forall j \in J : (i,j) \in E_x \wedge (i,j) \notin E_y$$
$$z_{ij} = -y_{ij} \qquad \forall j \in J : (i,j) \notin E_x \wedge (i,j) \in E_y$$

$$z = W \binom{x}{y}$$

Example for Replanning Constraints

$$\begin{array}{llllll} 1x_{11} & +2x_{12} & & -1y_{11} & -2y_{12} & -3y_{13} \leq r_1 \\ 1x_{21} & +2x_{22} & +3x_{23} & -1y_{21} & -2y_{22} & \leq r_2 \end{array}$$

$$R = \overline{U} \cdot W = \begin{pmatrix} 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Example for Replanning Constraints

$$
\begin{array}{llllll}
1x_{11} & +2x_{12} & & -1y_{11} & -2y_{12} & -3y_{13} \leq r_1 \\
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\end{array}
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$$

$$
\begin{pmatrix} M \\ R \end{pmatrix} = \overline{A} + UW = \begin{pmatrix} M \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \overline{U} \end{pmatrix} W
$$

Example for Replanning Constraints

$$
\begin{array}{llllll}
1x_{11} & +2x_{12} & & -1y_{11} & -2y_{12} & -3y_{13} \leq r_1 \\
1x_{21} & +2x_{22} & +3x_{23} & -1y_{21} & -2y_{22} & \leq r_2
\end{array}
$$

$$
R = \overline{U} \cdot W = \begin{pmatrix} 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
$$

$$
\begin{pmatrix} M \\ R \end{pmatrix} = \overline{A} + UW = \begin{pmatrix} M \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \overline{U} \end{pmatrix} W
$$

In general the matrix $\begin{pmatrix} M \\ W \end{pmatrix}$ is not TU.

Can we nevertheless apply MIP reformulations?

Polyhedron corresponding to RMP:

$$P = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{|E_x|+|E_y|} \;\middle|\; \begin{pmatrix} M \\ R \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \leq \begin{pmatrix} \widetilde{b} \\ r \end{pmatrix} \right\}$$

Can we nevertheless apply MIP reformulations?

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Use decomposition:

$$\begin{pmatrix} M \\ R \end{pmatrix} = \overline{A} + UW = \begin{pmatrix} M \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \overline{U} \end{pmatrix} W$$

$$W \begin{pmatrix} x \\ y \end{pmatrix} = z, \;\; z \in \mathbb{Z}^k$$

Can we nevertheless apply MIP reformulations?

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$$P = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{|E_x|+|E_y|} \;\middle|\; \begin{pmatrix} M \\ R \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \leq \begin{pmatrix} \widetilde{b} \\ r \end{pmatrix} \right\}$$

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$$W \begin{pmatrix} x \\ y \end{pmatrix} = z, \quad z \in \mathbb{Z}^k$$

For fixed $z \in \mathbb{Z}^k$

$$P_z = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{|E_x|+|E_y|} \;\middle|\; \begin{pmatrix} M \\ 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \leq \begin{pmatrix} \widetilde{b} \\ r \end{pmatrix} - \begin{pmatrix} 0 \\ \overline{U} \end{pmatrix} z,\; W \begin{pmatrix} x \\ y \end{pmatrix} = z \right\}$$

RMP with a Replanning Constraint for Each Aircraft

Theorem

Let $b = 1$ and

$$P = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{|E_x|+|E_y|} \;\middle|\; \begin{pmatrix} M \\ R \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \leq \begin{pmatrix} 1 \\ r \end{pmatrix} \right\}$$

Then

$$\operatorname{conv}\left(\begin{pmatrix} x \\ y \end{pmatrix} \in P \;\middle|\; \begin{pmatrix} x \\ y \end{pmatrix} \in \{0,1\}^{|E_x|+|E_y|} \right) = \operatorname{conv}\left(\begin{pmatrix} x \\ y \end{pmatrix} \in P \;\middle|\; W \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{Z}^k \right).$$

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$\Rightarrow$ k instead of $|E_x| + |E_y|$ integrality constraints

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$\Rightarrow$ k instead of $|E_x| + |E_y|$ integrality constraints

$\Rightarrow$ for complete bipartite graphs $k = \frac{|E_x|+|E_y|}{2}$

Outline

Robust Bipartite *b*-Matching Problems in Air Traffic Management

Mixed Integer Reformulations of Integer Programs

Mixed Integer Reformulations for the Robust Bipartite *b*-Matching Problem

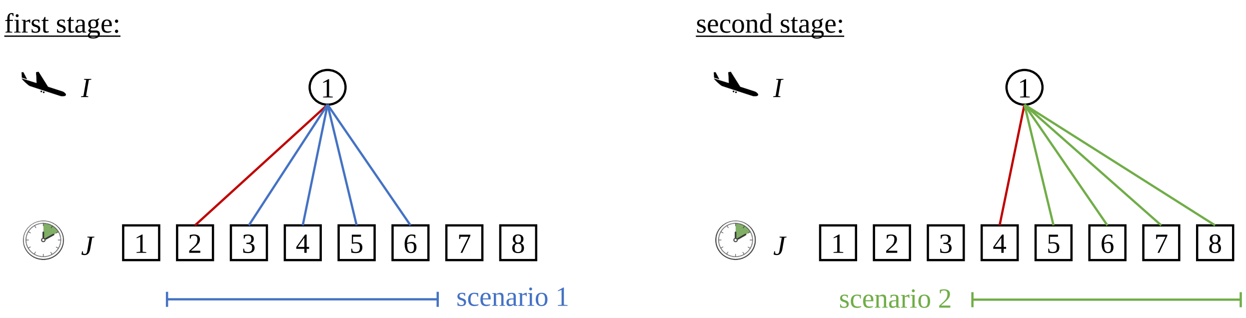
Computational Results

Implementation and Instances

- Gurobi Optimizer 7.0

Benchmark Sets of Instances:

- random graphs from ATM application,



- $|I| \gg |J|$ (more aircraft than time windows),
- randomly choose b_i^x, b_i^y for the b-matching constraints.

Results for RMP with ONE Replanning Constraint for All Aircraft

MIP/IP Methods:

- **Rec**: IP formulation with one replanning constraint
  - variables: $\begin{pmatrix} x \\ y \end{pmatrix} \in \{0,1\}^{|E_x|+|E_y|}$
- **RecZ**: IP formulation with new *z* variables and aggregation constraints
  - variables: $\begin{pmatrix} x \\ y \end{pmatrix} \in \{0,1\}^{|E_x|+|E_y|}$,
  - $z \in \mathbb{Z}^{|J|}$
- **RecZ-BP**: RecZ with branching priority on the *z* variables
- **RecZCont**: RecZ but with continuous x, y variables
  - variables: $\begin{pmatrix} x \\ y \end{pmatrix} \in [0,1]^{|E_x|+|E_y|}$,
  - $z \in \mathbb{Z}^{|J|}$

Results for Graphs From the ATM Application

Instances			Rec (IP)		RecZ		RecZ-BP		RecZCont	
|I|	|J|	r	cpu [s]	solved	cpu [s]	solved	cpu [s]	solved	cpu [s]	solved
816	120	5	127	(1/5)	**31**	(5/5)	211	(5/5)	3402	(5/5)
816	120	10	-	(0/5)	**14**	(5/5)	33	(5/5)	1038	(5/5)
816	120	100	25	(2/5)	**10**	(5/5)	36	(5/5)	2167	(5/5)
3696	120	5	15,128	(2/5)	85	(5/5)	**81**	(5/5)	880	(4/5)
3696	120	10	216	(2/5)	83	(5/5)	**81**	(5/5)	971	(4/5)
3696	120	100	-	(0/5)	85	(4/5)	**504**	(5/5)	1595	(4/5)

Table: Small instances

Rec: IP formulation, **RecZ**: IP formulation with new z variables,
RecZ-BP: RecZ with branching priority on z variables,
RecZCont: RecZ but with continuous x, y

Results for Graphs From the ATM Application

Instances			Rec (IP)		RecZ		RecZ-BP		RecZCont	
$\|I\|$	$\|J\|$	r	cpu [s]	solved	cpu [s]	solved	cpu [s]	solved	cpu [s]	solved
816	120	5	127	(1/5)	**31**	(5/5)	211	(5/5)	3402	(5/5)
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Table: Small instances

- IP is outperformed by RecZ, RecZ-BP, RecZCont

Rec: IP formulation, **RecZ**: IP formulation with new z variables,
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Results for Graphs From the ATM Application

Instances			Rec (IP)		RecZ		RecZ-BP		RecZCont	
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Table: Small instances

- IP is outperformed by RecZ, RecZ-BP, RecZCont
- RecZ and RecZ-BP often perform better than RecZCont

Rec: IP formulation, **RecZ**: IP formulation with new z variables,
RecZ-BP: RecZ with branching priority on z variables,
RecZCont: RecZ but with continuous x, y

Results for Graphs From the ATM Application

Instances			Rec (IP)		RecZ		RecZ-BP		RecZCont	
$\|I\|$	$\|J\|$	r	cpu [s]	solved	cpu [s]	solved	cpu [s]	solved	cpu [s]	solved
10,000	320	5	-	(0/10)	**5796**	(10/10)	9754	(9/10)	2027	(9/10)
10,000	320	10	4309	(1/10)	**3892**	(10/10)	7331	(9/10)	2089	(9/10)
10,000	320	100	5154	(1/10)	4630	(8/10)	4233	(8/10)	**1283**	(9/10)
20,000	320	5	8586	(3/10)	**8962**	(9/10)	7226	(8/10)	2116	(7/10)
20,000	320	10	11,607	(4/10)	14,047	(9/10)	**10,577**	(9/10)	1631	(7/10)
20,000	320	100	20,285	(3/10)	**14,205**	(8/10)	15,863	(8/10)	3033	(7/10)

Table: Big instances

Rec: IP formulation, **RecZ**: IP formulation with new z variables,
RecZ-BP: RecZ with branching priority on z variables,
RecZCont: RecZ but with continuous x, y

Results for Graphs From the ATM Application

Instances			Rec (IP)		RecZ		RecZ-BP		RecZCont	
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Table: Big instances

- RecZ and RecZ-BP solve more instances to optimality than RecZCont

Rec: IP formulation, **RecZ**: IP formulation with new z variables,
RecZ-BP: RecZ with branching priority on z variables,
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Results for Graphs From the ATM Application

Instances			Rec (IP)		RecZ		RecZ-BP		RecZCont	
|I|	|J|	r	cpu [s]	solved	cpu [s]	solved	cpu [s]	solved	cpu [s]	solved
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Table: Big instances

- RecZ and RecZ-BP solve more instances to optimality than RecZCont
- In many cases RecZCont solves fastest an instance to optimaltiy

Rec: IP formulation, **RecZ**: IP formulation with new z variables,
RecZ-BP: RecZ with branching priority on z variables,
RecZCont: RecZ but with continuous x, y

Cpu Times for Big Instances

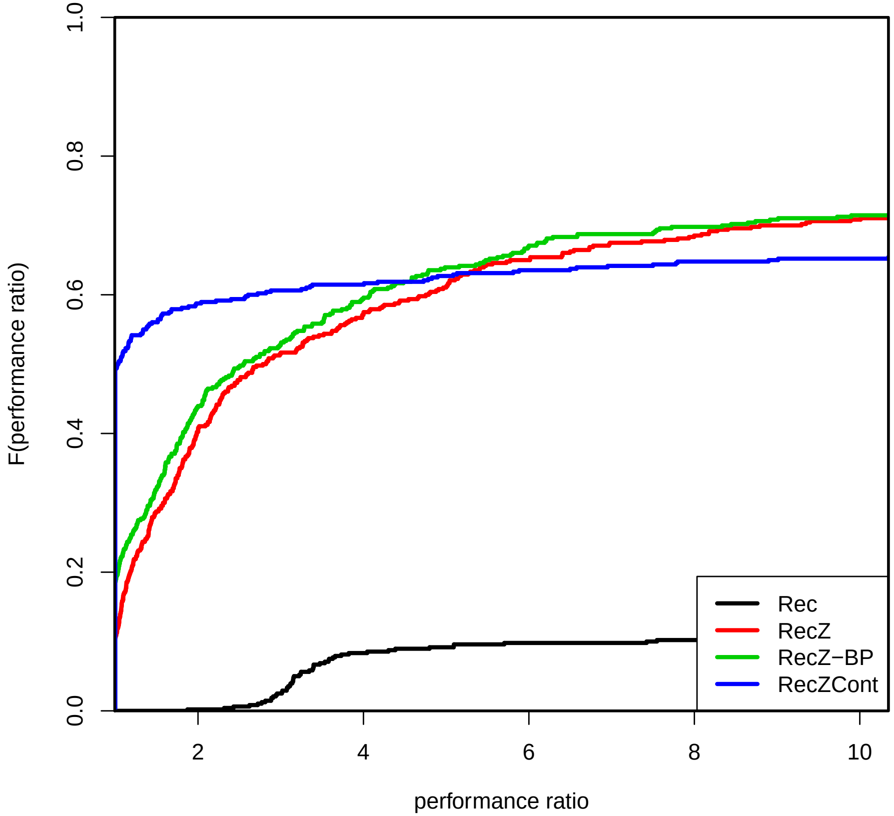

Figure: cpu

Conclusions and Further Results

Conclusions

- Idea of affine TU decompositions reduces number of integer variables for RMP with one replanning constraint for all aircraft
- Approach can be extended to RMP with a replanning constraint for each aircraft
- Computational results show that new MIP reformulations outperform IP formulation

Further Results

MIP reformulations can also be applied to

- two stage matching problems with more general knapsack constraints,
- general matching problems with one or several knapsack constraints with a quite general structure.

Thank you very much!