MUUC - Multiobjective Unconstrained Combinatorial Optimization
Weight Space Decomposition, Arrangements of Hyperplanes and Zonotopes

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Multiobjective Unconstrained Combinatorial Optimization

\[
v_{\text{max}} \quad f(x) = \left( \sum_{i=1}^{n} p^1_i x_i, \sum_{i=1}^{n} p^2_i x_i, \ldots, \sum_{i=1}^{n} p^m_i x_i \right) \quad (\text{mo.0c})
\]

s.t. \( x_i \in \{0, 1\} \quad \forall i \in \{1, \ldots, n\}. \)

Data: \( p_i = (p^1_i, \ldots, p^m_i)^\top \in \mathbb{Z}^m \setminus \{0\}, \forall i \in \{1, \ldots, n\} \)
MUCO

Multiobjective Unconstrained Combinatorial Optimization

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v_{\text{max}} \quad f(x) = \left( \sum_{i=1}^{n} p_i^1 x_i, \sum_{i=1}^{n} p_i^2 x_i, \ldots, \sum_{i=1}^{n} p_i^m x_i \right) \quad \text{(mo.0c)}
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Multiobjective Unconstrained Combinatorial Optimization

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Data: \[ p_i = (p_1^i, \ldots, p_m^i)^\top \in \mathbb{Z}^m \setminus \{0\}, \quad \forall i \in \{1, \ldots, n\} \]

A solution \( \bar{x} \in \{0, 1\}^n \) is called efficient or Pareto optimal if there is no other solution \( x \in \{0, 1\}^n \) such that \[ f_j(x) \geq f_j(\bar{x}) \quad \forall j \in \{1, \ldots, m\} \quad \text{and} \quad f(x) \neq f(\bar{x}) \]
MUOCO

Multiobjective Unconstrained Combinatorial Optimization

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The image \( f(\bar{x}) \in \mathbb{Z}^m \) is called nondominated
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- Multiobjective Combinatorial Optimization (MOCO) is almost always intractable: The number of Pareto optimal solutions may grow exponentially with the problem size
  - Shortest paths [Hansen, 79]
  - Minimum spanning trees [Hamacher and Ruhe, 94]
  - MUOCO [Ehrgott, 2005]
  - and many others
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- Multiobjective Combinatorial Optimization (MOCO) is almost always intractable: The number of Pareto optimal solutions may grow exponentially with the problem size.
- Pareto optimal solutions are almost always non-connected.
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- Multiobjective Combinatorial Optimization (MOCO) is almost always intractable: The number of Pareto optimal solutions may grow exponentially with the problem size.

- Pareto optimal solutions are almost always non-connected:
  - Shortest paths and spanning trees [Ehrgott & K, 97]
  - MUCO [Gorski, K & Ruzika, 2011]
  - and many others
Why is this interesting?

- Multiobjective Combinatorial Optimization (MOCO) is almost always intractable: The number of Pareto optimal solutions may grow exponentially with the problem size.
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- Question: Are there “easy” problems and/or cases?
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- Good news for MUCO: Polynomial number of extreme supported solutions!
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- Pareto optimal solutions are almost always non-connected
- Question: Are there “easy” problems and/or cases?
- Good news for MOCO: Polynomial number of extreme supported solutions!
- Applications:
  - Relaxation of multiobjective and multidimensional knapsack problems
  - Positive and negative cost coefficients
  - Preprocessing and efficient bound computations
Notation

- $\mathcal{X} = \{0, 1\}^n$ set of feasible solutions
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- $\mathcal{Y}_{eN}$ set of extreme supported nondominated points
Computation of Extreme Supported Solutions

**Weighted sum scalarization**

\[
\begin{align*}
\text{max} & \quad \sum_{j=1}^{m} \lambda_j \cdot \sum_{i=1}^{n} p_i^j x_i \\
\text{s.t.} & \quad x_i \in \{0, 1\} \quad \forall i \in \{1, \ldots, n\}
\end{align*}
\]

with \( \lambda \in \tilde{\mathcal{N}}^0 = \left\{ (\lambda_1, \ldots, \lambda_m)^T \in \mathbb{R}^m : \sum_{j=1}^{m} \lambda_j = 1 \right\} \)

---

*Arthur M. Geoffrion*

*Proper efficiency and the theory of vector maximization*

*Journal of Mathematical Analysis and Applications, 1968*
Computation of Extreme Supported Solutions

Weighted sum scalarization

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\]

s.t. \( x_i \in \{0, 1\} \quad \forall i \in \{1, \ldots, n\} \)

with \( \lambda \in \mathcal{N}^0 = \left\{ (\lambda_2, \ldots, \lambda_m)^T \in \mathbb{R}^{m-1} : \sum_{j=2}^{m} \lambda_j < 1 \right\} \) and

\[
\lambda_1 = \left( 1 - \sum_{j=2}^{m} \lambda_j \right)
\]

Arthur M. Geoffrion
Proper efficiency and the theory of vector maximization
Weight Space Decomposition

Let \( \bar{y} \in \mathcal{Y}_{sN} \).

\[
\mathcal{W}^0(\bar{y}) = \left\{ (\lambda_2, \ldots, \lambda_m)^T \in \mathcal{W}^0 : \lambda_1 = (1 - \sum_{j=2}^{m} \lambda_j), \right. \\
\left. \sum_{j=1}^{m} \lambda_j \bar{y}_j \geq \sum_{j=1}^{m} \lambda_j y_j, \forall y \in \mathcal{Y} \right\}
\]
Weight Space Decomposition

Let $\tilde{y} \in \mathcal{Y}_{\text{sN}}$.

$$\mathcal{W}^{0}(\tilde{y}) = \left\{ (\lambda_{2}, \ldots, \lambda_{m})^{T} \in \mathcal{W}^{0} : \lambda_{1} = (1 - \sum_{j=2}^{m} \lambda_{j}), \right.$$  

$$\left. \sum_{j=1}^{m} \lambda_{j} \tilde{y}_{j} \geq \sum_{j=1}^{m} \lambda_{j} y_{j}, \forall y \in \mathcal{Y} \right\}$$

Anthony Przybylski, Xavier Gandibleux, and Matthias Ehrgott
A recursive algorithm for finding all nondominated extreme points in the outcome set of a multiobjective integer programme
INFORMS Journal on Computing, 2010
Weight Space Decomposition

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Weight Space Decomposition

Let $\bar{y} \in \mathcal{V}_{sN}$.

$$\mathcal{W}^0(\bar{y}) = \left\{ (\lambda_2, \ldots, \lambda_m)^T \in \mathcal{W}^0 : \lambda_1 = (1 - \sum_{j=2}^m \lambda_j), \right. \\
\left. \sum_{j=1}^m \lambda_j \bar{y}_j \geq \sum_{j=1}^m \lambda_j y_j, \forall y \in \mathcal{V} \right\}$$

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Weight Space Decomposition

Let $\bar{y} \in \mathcal{Y}_{sN}$.

$$\mathcal{W}^0(\bar{y}) = \left\{ (\lambda_2, ..., \lambda_m)^T \in \mathcal{W}^0 : \lambda_1 = (1 - \sum_{j=2}^{m} \lambda_j), \sum_{j=1}^{m} \lambda_j \bar{y}_j \geq \sum_{j=1}^{m} \lambda_j y_j, \forall y \in \mathcal{Y} \right\}$$
Weight Space Decomposition

Let $\bar{y} \in \mathcal{Y}_{sN}$.

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$\mathcal{W}^0(\bar{y})$ is a convex polytope
Weight Space Decomposition

Let $\bar{y} \in Y_{sN}$.

\[
W^0(\bar{y}) = \left\{ (\lambda_2, \ldots, \lambda_m)^T \in W^0 : \lambda_1 = (1 - \sum_{j=2}^{m} \lambda_j), \right. \\
\left. \sum_{j=1}^{m} \lambda_j \bar{y}_j \geq \sum_{j=1}^{m} \lambda_j y_j, \forall y \in Y \right\}
\]

- $W^0(\bar{y})$ is a convex polytope
- $\bar{y}$ is extreme supported $\Leftrightarrow W^0(\bar{y})$ has dimension $m - 1$
Weight Space Decomposition

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- $\mathcal{W}^0(\bar{y})$ is a convex polytope
- $\bar{y}$ is extreme supported $\iff \mathcal{W}^0(\bar{y})$ has dimension $m - 1$
- $\mathcal{W}^0 = \bigcup_{\bar{y} \in \mathcal{Y}_{eN}} \mathcal{W}^0(\bar{y})$
Weight Space Decomposition ($m = 3$)

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Weight Space Decomposition \((m = 3)\)

\[W^0\]

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Definitions

Closed line segment

Let \( u \) and \( v \) be vectors in \( \mathbb{R}^m \). The closed line segment \([u, v]\) is given by

\[
[u, v] = \{ y \in \mathbb{R}^m : y = u + \mu \cdot (v - u), \mu \in [0, 1]\}.
\]
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\]

Minkowski sum

Let \( A, B \subset \mathbb{R}^m \). The Minkowski sum of \( A \) and \( B \) is

\[
A + B = \{ a + b : a \in A, b \in B \}.
\]
Definitions

Zonotope

A zonotope $Z$ is defined as the Minkowski sum of a finite set of closed line segments (generators) $[u_i, v_i]$ with $u_i, v_i \in \mathbb{R}^m$, $i \in \{1, \ldots, n\}$. 
Definitions

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![Diagram of zonotope with generators $u_1$, $v_1$, $u_2$, $v_2$]
Definitions

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![Diagram of a zonotope](image)
A zonotope \( \mathcal{Z} \) is defined as the Minkowski sum of a finite set of closed line segments \((\text{generators})\) \([u_i, v_i]\) with \(u_i, v_i \in \mathbb{R}^m, i \in \{1, \ldots, n\}\).
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In the following: $u_i = 0$ for all $i \in \{1, \ldots, n\}$. 
Definitions

Arrangement of hyperplanes

Given a finite set of hyperplanes $h_i$ in $\mathbb{R}^m$, the hyperplanes subdivide $\mathbb{R}^m$ into connected polyhedra of different dimensions.
**Definitions**

**Position vector**

Let $y \in \mathbb{R}^m$. The position vector $P(y) = (P_1(y), \ldots, P_n(y))$ of $y$ is defined by

$$P_i(y) = \begin{cases} 
-1 & \text{if } y \in h_i^- \\
0 & \text{if } y \in h_i \\
+1 & \text{if } y \in h_i^+ 
\end{cases}$$

$\forall i \in \{1, \ldots, n\}$. 
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\end{cases} \quad \forall i \in \{1, ..., n\}.$$ 

$m$-face or cell with

$$P(y) = (+1, +1, -1, +1)$$
Definitions

Position vector

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\end{cases} \quad \forall i \in \{1, \ldots, n\}.
$$

$k$-face ($k = 0$) with

$$
P(y) = (0, 0, 0, +1)$$
Definitions

Simple arrangement of hyperplanes

Hyperplanes $h_1, \ldots, h_n$ in $\mathbb{R}^m$, $m \leq n$
Definitions

Simple arrangement of hyperplanes

Hyperplanes $h_1, ..., h_n$ in $\mathbb{R}^m$, $m \leq n$

- Intersection of any subset of $m$ hyperplanes is a unique point
Simple arrangement of hyperplanes

Hyperplanes $h_1, \ldots, h_n$ in $\mathbb{R}^m$, $m \leq n$

- Intersection of any subset of $m$ hyperplanes is a unique point
- Intersection of any subset of $(m + 1)$ hyperplanes is empty
Central arrangement of hyperplanes

Hyperplanes $h_1, \ldots, h_n$ in $\mathbb{R}^m$, $m \leq n$
Central arrangement of hyperplanes

Hyperplanes $h_1, \ldots, h_n$ in $\mathbb{R}^m$, $m \leq n$

- $(0, \ldots, 0)^\top$ is contained in every hyperplane
Interrelations

(mo.0c)
Interrelations

\[(\text{mo.0c})\]

\[
\begin{align*}
\text{Weight space} \\
\text{decomposition}
\end{align*}
\]
Interrelations

\[ (\text{mo.0c}) \]

\[ \text{Zonotope} \]

\[ \text{Weight space decomposition} \]
Interrelations

(weight space decomposition

Zonotope

Arrangement of hyperplanes

(mo.0c)
Interrelations

(mo.0c)  Zonotope

Weight space decomposition  Arrangement of hyperplanes
Arrangements and Zonotopes

Zonotope:

\[ \ell_i = [0, p_i], \ p_i \in \mathbb{R}^m, \ i \in \{1, \ldots, n\} \]
Arrangements and Zonotopes

Zonotope:

\[ \ell_i = [0, p_i], \quad p_i \in \mathbb{R}^m, \quad i \in \{1, \ldots, n\} \]

Associated arrangement of hyperplanes:

\[ h_i = \{ y \in \mathbb{R}^m : \langle p_i, y \rangle = 0 \} \text{ with } \]
\[ h_i^+ = \{ y \in \mathbb{R}^m : \langle p_i, y \rangle > 0 \} \]
\[ h_i^- = \{ y \in \mathbb{R}^m : \langle p_i, y \rangle < 0 \} \]
This duality has an order reversing characteristic: A $k$-face of a zonotope in $\mathbb{R}^m$ corresponds to an $(m-k)$-face of the associated arrangement of hyperplanes.
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Interrelations

(mo.0c)

Weight space decomposition

Zonotope

Arrangement of hyperplanes
Interrelations

(mo.0c) ↔ Zonotope

Weight space decomposition

Arrangement of hyperplanes
(mo.0c) and Zonotopes

Instance of (mo.0c):

\[ p_i = (p_i^1, p_i^2, \ldots, p_i^m)^\top, \ i \in \{1, \ldots, n\}. \]
(mo.0c) and Zonotopes

Instance of (mo.0c):

\[ p_i = (p_i^1, p_i^2, \ldots, p_i^m)^\top, \ i \in \{1, \ldots, n\}. \]

Associated zonotope:

\[ \ell_i = [0, p_i] \quad i \in \{1, \ldots, n\} \]
(mo.0c) and Zonotopes

Instance of (mo.0c):

$$\text{conv}(\mathcal{Y}) = \left\{ \sum_{i=1}^{n} p_i x_i : x_i \in [0, 1], i = 1, \ldots, n \right\}$$

Associated zonotope:

$$\mathcal{Z} = \left\{ \sum_{i=1}^{n} \mu_i p_i : \mu_i \in [0, 1], i = 1, \ldots, n \right\}$$
**Interrelations**

\[(\text{mo.0c}) \leftrightarrow \text{Zonotope}\]

- Weight space decomposition
- Arrangement of hyperplanes
Interrelations

\[ Y_{eN} \]

of (mo.0c) \[ \xleftarrow{} \]

Zonotope

\[ \downarrow \]

Weight space decomposition

\[ \xrightarrow{} \]

Arrangement of hyperplanes
Interrelations

\[ Y_{eN} \]

of (mo.0c)

\[ \uparrow \]

Weight space decomposition

\[ \downarrow \]

Arrangement of hyperplanes

\[ \uparrow \]

set of nondominated extreme points of Zonotope

\[ \downarrow \]
Interrelations

\[ \mathcal{V}_{eN} \text{ of } (\text{mo.}0\text{c}) = \text{set of nondominated extreme points of Zonotope} \]

- Weight space decomposition
- Arrangement of hyperplanes
Interrelations

\[ \mathcal{V}_{eN} \text{ of } (\text{mo.0c}) = \text{set of nondominated extreme points of Zonotope} \]

\[ \text{Weight space decomposition} \]

\[ \text{Intersection of Arrangement of hyperplanes with } \mathbb{R}_+^m \]
Interrelations

\[ \mathcal{Y}_{eN} \quad \text{of (mo.0c)} \quad = \quad \text{set of nondominated extreme points of Zonotope} \]

\[ \text{Weight space decomposition} \quad = \quad \text{Intersection of Arrangement of hyperplanes with } \tilde{\mathcal{W}}^0 \subset \mathbb{R}_\succ^m \]
(mo.0c), Arrangements and Weight Space
(mo.0c), Arrangements and Weight Space
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(mo.0c), Arrangements and Weight Space
Complexity

The number of cells in a central arrangement of hyperplanes in $\mathbb{R}^m$, $m$ fixed, is bounded by

$$2 \cdot \sum_{i=0}^{m-1} \binom{n-1}{i}.$$
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The number of cells in a central arrangement of hyperplanes in $\mathbb{R}^m$, $m$ fixed, is bounded by

$$2 \cdot \sum_{i=0}^{m-1} \binom{n-1}{i}.$$ 

- The same bound holds for the number of extreme supported solutions of (mo.0c).
Weight Space Decomposition

Each item $i$ defines a hyperplane

$$h_i = \left\{ (\lambda_2, \ldots, \lambda_m) \in \mathcal{W}^0 : \left(1 - \sum_{j=2}^m \lambda_j\right)p_i^1 + \sum_{j=2}^m \lambda_j p_i^j = 0 \right\}$$
Weight Space Decomposition

Each item $i$ defines a hyperplane

$$h_i = \left\{ (\lambda_2, ..., \lambda_m) \in \mathcal{W}^0 : \left( 1 - \sum_{j=2}^{m} \lambda_j \right) p_i^1 + \sum_{j=2}^{m} \lambda_j p_i^j = 0 \right\}$$

and half spaces

$$h_i^- = \left\{ (\lambda_2, ..., \lambda_m) \in \mathcal{W}^0 : \left( 1 - \sum_{j=2}^{m} \lambda_j \right) p_i^1 + \sum_{j=2}^{m} \lambda_j p_i^j < 0 \right\} \quad (x_i = 0)$$

$$h_i^+ = \left\{ (\lambda_2, ..., \lambda_m) \in \mathcal{W}^0 : \left( 1 - \sum_{j=2}^{m} \lambda_j \right) p_i^1 + \sum_{j=2}^{m} \lambda_j p_i^j > 0 \right\} \quad (x_i = 1)$$
Weight Space Decomposition

\[ \lambda_3 \]

\[ \lambda_2 \]

\[ \lambda_1 \]

1

1
Weight Space Decomposition

\[ P(\lambda) = (-1, -1, -1, +1)^T \]
\[ x = (0, 0, 0, 1)^T \]

\[ P(\lambda) = (+1, -1, -1, +1)^T \]
\[ x = (1, 0, 0, 1)^T \]

\[ P(\lambda) = (+1, -1, +1, +1)^T \]
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Weight Space Decomposition

\[ P(\lambda) = (-1, -1, -1, +1)^T \]
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\[ x = (0, 0, 1, 1)^T \]

\[ P(\lambda) = (0, -1, 0, +1)^T \]
Solution Approach

**Input:** hyperplanes $h_i$, $i \in \{1, ..., n\}$, and $h_{n+j}$, $j \in \{1, ..., m\}$.

1: for all intersection points $\lambda$ of $(m-1)$ hyperplanes do
2: if $P_{n+j}(\lambda), ..., P_{n+m}(\lambda)$ have the correct sign then
3: generate and save all solutions/points corresponding to $\lambda$

**Output:** set of supported solutions
Solution Approach

**Input:** hyperplanes $h_i$, $i \in \{1, \ldots, n\}$, and $h_{n+j}$, $j \in \{1, \ldots, m\}$.

1. for all intersection points $\lambda$ of $(m-1)$ hyperplanes do
2. if $P_{n+j}(\lambda), \ldots, P_{n+m}(\lambda)$ have the correct sign then
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**Output:** set of supported solutions

- For simple arrangements: Complexity $\mathcal{O}(n^m)$. 
Solution Approach

**Input:** hyperplanes $h_i$, $i \in \{1, \ldots, n\}$, and $h_{n+j}$, $j \in \{1, \ldots, m\}$.

1: for all intersection points $\lambda$ of $(m - 1)$ hyperplanes do
2: if $P_{n+j}(\lambda), \ldots, P_{n+m}(\lambda)$ have the correct sign then
3: generate and save all solutions/points corresponding to $\lambda$

**Output:** set of supported solutions

- For simple arrangements: Complexity $O(n^m)$.
- For nonsimple arrangements: Complexity $O(2^n)$. 
Nonextreme Supported Solutions

Nonextreme supported solutions occur if more than $m - 1$ hyperplanes intersect in one point.
Nonextreme Supported Solutions

Nonextreme supported solutions occur if more than $m - 1$ hyperplanes intersect in one point.

$$P(\lambda) = (0, 0, 0, +1)^T$$
Nonextreme Supported Solutions

Nonextreme supported solutions occur if more than $m - 1$ hyperplanes intersect in one point.

$$P(\lambda) = (0, 0, 0, +1)^\top$$

$$x = (0, 0, 0, 1)^\top$$
$$x = (1, 0, 0, 1)^\top$$
$$x = (0, 1, 0, 1)^\top$$
$$x = (0, 0, 1, 1)^\top$$
$$x = (1, 1, 0, 1)^\top$$
$$x = (1, 0, 1, 1)^\top$$
$$x = (0, 1, 1, 1)^\top$$
$$x = (1, 1, 1, 1)^\top$$
Multiobjective Knapsack Problem (mo.1c)

\[
v_{\text{max}} \quad f(x) = \left( \sum_{i=1}^{n} p_{i1} x_i, \sum_{i=1}^{n} p_{i2} x_i, \ldots, \sum_{i=1}^{n} p_{im} x_i \right)
\]

s.t. \[ \sum_{i=1}^{n} w_i x_i \leq W \]

\[ x_i \in \{0, 1\} \quad \forall i \in \{1, \ldots, n\}. \]

with \[ p_i = (p_{i1}, \ldots, p_{im})^\top \in \mathbb{Z}^m \setminus \{0\}, \forall i \in \{1, \ldots, n\}. \]
Solution Approach

Iterative weighted sum scalarizations: Variants of weight space decomposition and dichotomic search

Fritz Bökler and Petra Mutzel
Output-sensitive algorithms for enumerating the extreme nondominated points of multiobjective combinatorial optimization problems

Anthony Przybylski, Xavier Gandibleux, and Matthias Ehrgott
A recursive algorithm for finding all nondominated extreme points in the outcome set of a multiobjective integer programme
INFORMS Journal on Computing, 2010

Özgür Özpeynirci and Murat Köksalan
An exact algorithm for finding extreme supported nondominated points of multiobjective mixed integer programs
Management Science, 2010
Weight Space Decomposition: Initialize with (mo.0c)
Weight Space Decomposition:
Initialize with (mo.0c)
Weight Space Decomposition: Initialize with (mo.0c)
Weight Space Decomposition: Initialize with \((\text{mo.}0\text{c})\)

- Refine using dichotomic search when infeasible for \((\text{mo.}1\text{c})\)
Summary
Summary
Summary

The diagrams illustrate the multiobjective knapsack problem with two objectives, $f_1$ and $f_2$, and constraints on the decision variables $x_1$ and $x_2$. The feasible region is represented by the set $\mathcal{W}^0$ defined by the constraints $h_2, h_3, h_4, 0, 1, \lambda_2$. The diagrams highlight the interrelations and the trade-offs between the objectives, emphasizing the concept of a Pareto front.
Summary

\[ f_1 \] \hspace{1cm} \[ f_2 \]

\[ x_1 \] \hspace{1cm} \[ x_2 \]

\[ \mathcal{W}^0 \]

\[ h_1 \] \hspace{1cm} \[ h_2 \] \hspace{1cm} \[ h_3 \] \hspace{1cm} \[ h_4 \] \hspace{1cm} \[ h_5 \]

\[ \hat{\mathcal{W}}^0 \]
Summary
Summary
Thank you for your attention!
Summary

Thank you for your attention!

Questions?