On Rational Polytopes with Chvátal Rank 1

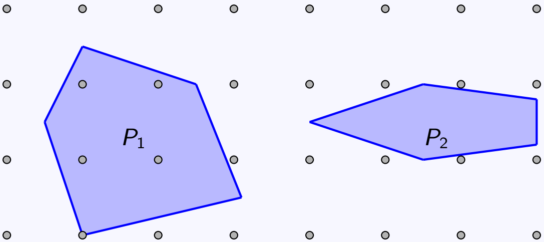
Dabeen Lee

Tepper School of Business, Carnegie Mellon University

January 9, 2017

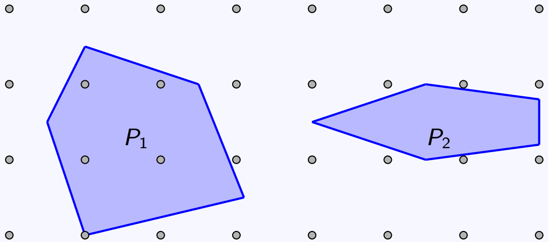
(Joint work with Gérard Cornuéjols and Yanjun Li)

Integer programming feasibility problem



Given a rational polyhedron $P := \{x \in \mathbb{R}^n : \; Ax \leq b\}$ where $A \in \mathbb{Q}^{m \times n}$ and $b \in \mathbb{Q}^m$, decide whether $P \cap \mathbb{Z}^n = \emptyset$.

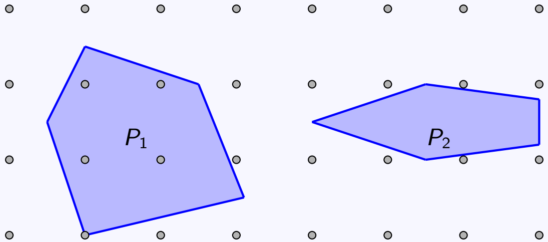
Integer programming feasibility problem



Given a rational polyhedron $P := \{x \in \mathbb{R}^n : \; Ax \leq b\}$ where $A \in \mathbb{Q}^{m \times n}$ and $b \in \mathbb{Q}^m$, decide whether $P \cap \mathbb{Z}^n = \emptyset$.

- It is a classical NP-complete problem!

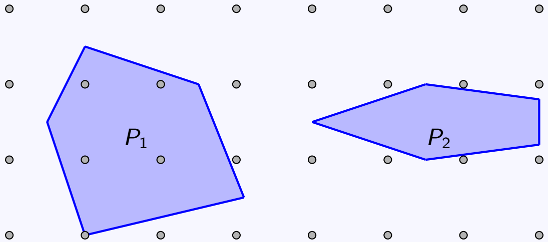
Integer programming feasibility problem



Given a rational polyhedron $P := \{x \in \mathbb{R}^n : \ Ax \leq b\}$ where $A \in \mathbb{Q}^{m \times n}$ and $b \in \mathbb{Q}^m$, decide whether $P \cap \mathbb{Z}^n = \emptyset$.

- It is a classical NP-complete problem!
- We still hope to find a "meaningful class" of easy instances.

Integer programming feasibility problem



Given a rational polyhedron $P := \{x \in \mathbb{R}^n : \; Ax \leq b\}$ where $A \in \mathbb{Q}^{m \times n}$ and $b \in \mathbb{Q}^m$, decide whether $P \cap \mathbb{Z}^n = \emptyset$.

- It is a classical NP-complete problem!
- We still hope to find a "meaningful class" of easy instances.
- What makes integer programming hard?

- The Chvátal closure of a rational polyhedron $P = \{x \in \mathbb{R}^n : \ Ax \leq b\}$ is defined as

$$P' := \bigcap_{c \in \mathbb{Z}^n} \{x \in \mathbb{R}^n : \ cx \leq \lfloor \max\{cy : \ y \in P\} \rfloor\}$$

- The Chvátal closure of a rational polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ is defined as

$$P' := \bigcap_{c \in \mathbb{Z}^n} \{x \in \mathbb{R}^n : cx \leq \lfloor \max\{cy : y \in P\} \rfloor\}$$

- The *k*th closure of P is defined as

$$P^{(k)} := \underbrace{((P')' \cdots)'}_{k}$$

- The Chvátal closure of a rational polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ is defined as

$$P' := \bigcap_{c \in \mathbb{Z}^n} \{x \in \mathbb{R}^n : cx \leq \lfloor \max\{cy : y \in P\} \rfloor\}$$

- The kth closure of P is defined as

$$P^{(k)} := \underbrace{((P')' \cdots)'}_{k}$$

Theorem [Chvátal, 1973, Schrijver, 1980]

Let P be a rational polyhedron. There exists a positive integer k such that $P^{(k)} = P_I$.

- The Chvátal closure of a rational polyhedron $P = \{x \in \mathbb{R}^n : \ Ax \leq b\}$ is defined as

$$P' := \bigcap_{c \in \mathbb{Z}^n} \{x \in \mathbb{R}^n : \ cx \leq \lfloor \max\{cy : \ y \in P\} \rfloor\}$$

- The kth closure of P is defined as

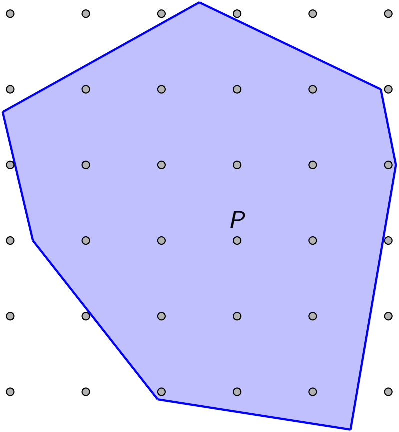
$$P^{(k)} := \underbrace{((P')' \cdots)'}_{k}$$

Theorem [Chvátal, 1973, Schrijver, 1980]

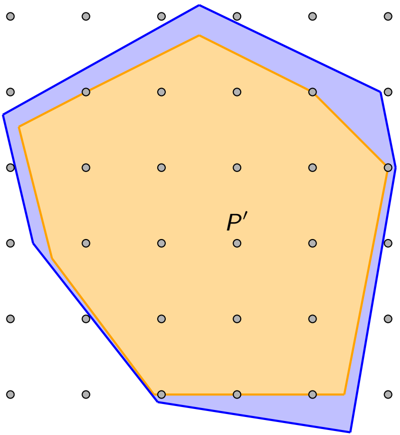
Let P be a rational polyhedron. There exists a positive integer k such that $P^{(k)} = P_I$.

- The Chvátal rank of P is the smallest integer k such that $P^{(k)} = P_I$.

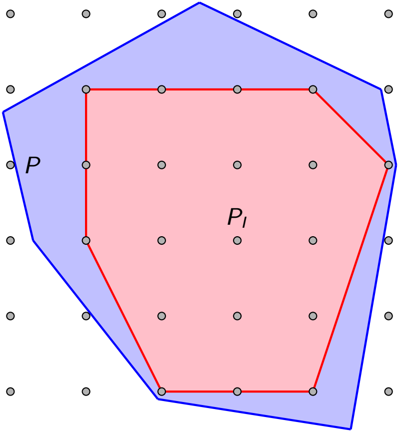
Chvátal rank measures how "close" a polyhedron is to its integer hull.



Chvátal rank measures how "close" a polyhedron is to its integer hull.



Chvátal rank measures how "close" a polyhedron is to its integer hull.



Can we use Chvátal rank to measure computational complexity of integer programming?

Can we use Chvátal rank to measure computational complexity of integer programming?

- Big Chvátal rank makes integer programming hard?

Can we use Chvátal rank to measure computational complexity of integer programming?

- Big Chvátal rank makes integer programming hard?
- Smaller Chvátal rank implies easier integer programming?

Can we use Chvátal rank to measure computational complexity of integer programming?

- Big Chvátal rank makes integer programming hard?
- Smaller Chvátal rank implies easier integer programming?
- Let's look at the rational polyhedra with Chvátal rank **1**.

Can we use Chvátal rank to measure computational complexity of integer programming?

- Big Chvátal rank makes integer programming hard?
- Smaller Chvátal rank implies easier integer programming?
- Let's look at the rational polyhedra with Chvátal rank **1**.

Question

Let $P \subset \mathbb{R}^n$ be a rational polyhedron with Chvátal rank 1. Can we decide whether $P \cap \mathbb{Z}^n = \emptyset$ in polynomial time?

Can we use Chvátal rank to measure computational complexity of integer programming?

- Big Chvátal rank makes integer programming hard?
- Smaller Chvátal rank implies easier integer programming?
- Let's look at the rational polyhedra with Chvátal rank **1**.

Question

Let $P \subset \mathbb{R}^n$ be a rational polyhedron with Chvátal rank 1. Can we decide whether $P \cap \mathbb{Z}^n = \emptyset$ in polynomial time?

- Does the Chvátal rank 1 condition make any differences?

Can we use Chvátal rank to measure computational complexity of integer programming?

- Big Chvátal rank makes integer programming hard?
- Smaller Chvátal rank implies easier integer programming?
- Let's look at the rational polyhedra with Chvátal rank **1**.

Question

Let $P \subset \mathbb{R}^n$ be a rational polyhedron with Chvátal rank 1. Can we decide whether $P \cap \mathbb{Z}^n = \emptyset$ in polynomial time?

- Does the Chvátal rank 1 condition make any differences?
- In fact, this problem belongs to NP $\cap$ co-NP. [Boyd and Pulleyblank, 2009]

Can we use Chvátal rank to measure computational complexity of integer programming?

- Big Chvátal rank makes integer programming hard?
- Smaller Chvátal rank implies easier integer programming?
- Let's look at the rational polyhedra with Chvátal rank **1**.

Question

Let $P \subset \mathbb{R}^n$ be a rational polyhedron with Chvátal rank 1. Can we decide whether $P \cap \mathbb{Z}^n = \emptyset$ in polynomial time?

- Does the Chvátal rank 1 condition make any differences?
- In fact, this problem belongs to NP $\cap$ co-NP. [Boyd and Pulleyblank, 2009]
- **This is the main motivation.**

Satisfiability problem with Chvátal rank 1

Formula in conjunctive normal form

Polytope in $[0,1]^n$

Satisfiability problem with Chvátal rank 1

Formula in conjunctive normal form

Polytope in $[0, 1]^n$

$$(x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_3 \vee x_4)$$

Satisfiability problem with Chvátal rank 1

Formula in conjunctive normal form

$$(x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_3 \vee x_4)$$

Polytope in $[0,1]^n$

$$\{x \in [0,1]^4 : x_1 + (1 - x_2) + x_3 \geq 1, \quad (1 - x_3) + x_4 \geq 1\}$$

Satisfiability problem with Chvátal rank 1

Formula in conjunctive normal form

$$(x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_3 \vee x_4)$$

Polytope in $[0,1]^n$

$$\{x \in [0,1]^4 : x_1 + (1 - x_2) + x_3 \geq 1, \; (1 - x_3) + x_4 \geq 1\}$$

- We say a formula has Chvátal rank 1 if the corresponding polytope does.

Satisfiability problem with Chvátal rank 1

Formula in conjunctive normal form

$$(x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_3 \vee x_4)$$

Polytope in $[0,1]^n$

$$\{x \in [0,1]^4 : x_1 + (1 - x_2) + x_3 \geq 1, \quad (1 - x_3) + x_4 \geq 1\}$$

- We say a formula has Chvátal rank 1 if the corresponding polytope does.
- Given a Chvátal rank 1 formula, decide whether it has a satisfying assignment.

Satisfiability problem with Chvátal rank 1

Formula in conjunctive normal form

$$(x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_3 \vee x_4)$$

Polytope in $[0,1]^n$

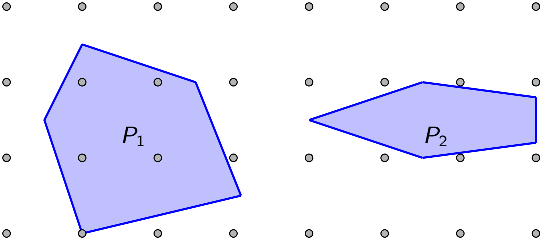
$$\{x \in [0,1]^4 : \begin{aligned} x_1 + (1 - x_2) + x_3 &\geq 1, \\ (1 - x_3) + x_4 &\geq 1\} \end{aligned}$$

- We say a formula has Chvátal rank 1 if the corresponding polytope does.
- Given a Chvátal rank 1 formula, decide whether it has a satisfying assignment.

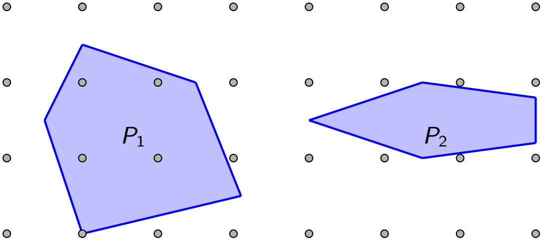
Theorem [Cornuéjols, Lee, and Li, 2016]

Let φ be a Chvátal rank 1 formula with at least 3 variables in each clause. There is a polynomial algorithm to decide satisfiability of φ.

Let us look at more general problem.

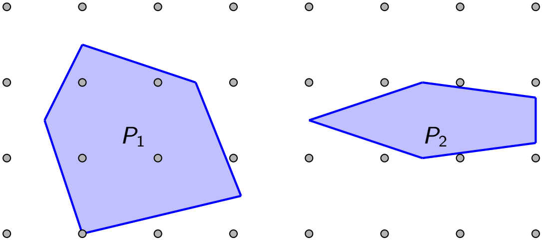


Let us look at more general problem.



If a polyhedron is not "flat", it will contain an integer point.

Let us look at more general problem.



If a polyhedron is not "flat", it will contain an integer point.

How do you measure "flatness"?

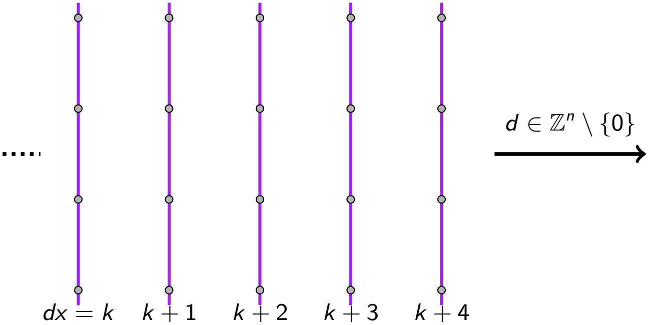
Let $d \in \mathbb{Z}^n \setminus \{0\}$ be a nonzero integer vector.

Let $d \in \mathbb{Z}^n \setminus \{0\}$ be a nonzero integer vector.

$\mathbb{Z}^n$ can be decomposed into parallel hyperplanes all orthogonal to d.

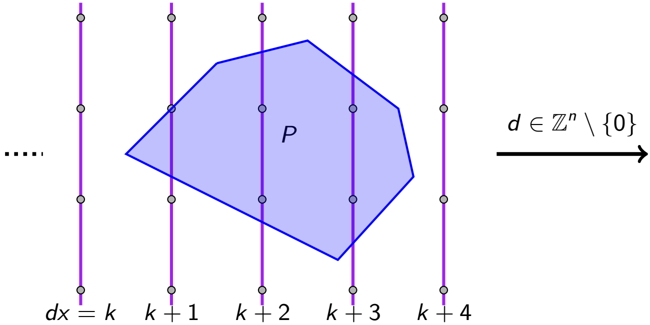
Let $d \in \mathbb{Z}^n \setminus \{0\}$ be a nonzero integer vector.

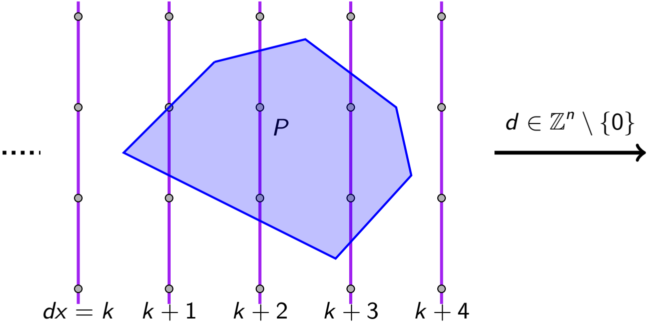
$\mathbb{Z}^n$ can be decomposed into parallel hyperplanes all orthogonal to d.



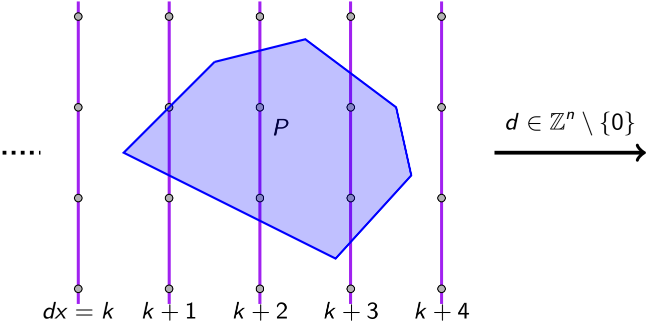
Let $d \in \mathbb{Z}^n \setminus \{0\}$ be a nonzero integer vector.

$\mathbb{Z}^n$ can be decomposed into parallel hyperplanes all orthogonal to d.



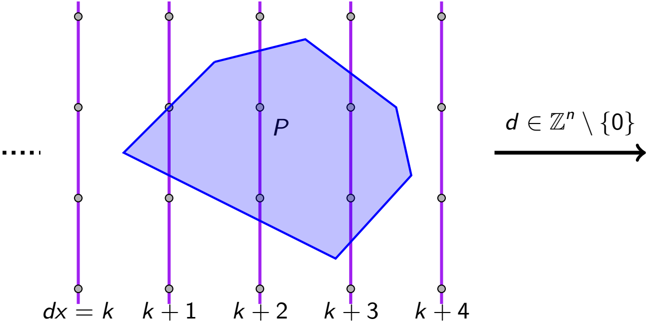


P
$d \in \mathbb{Z}^n \setminus \{0\}$
$dx = k$
$k+1$
$k+2$
$k+3$
$k+4$



The integer width of P along d is

$$\lfloor \max\{dx : x \in P\} \rfloor - \lceil \min\{dx : x \in P\} \rceil + 1.$$

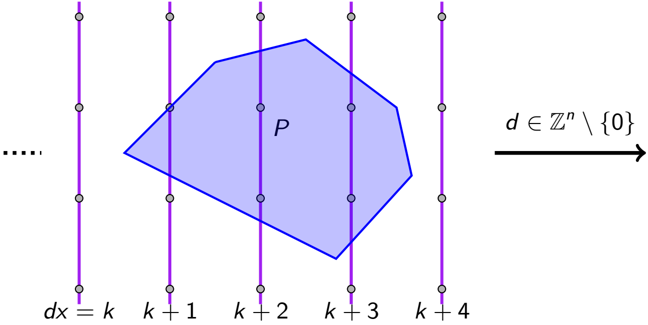


The integer width of P along d is

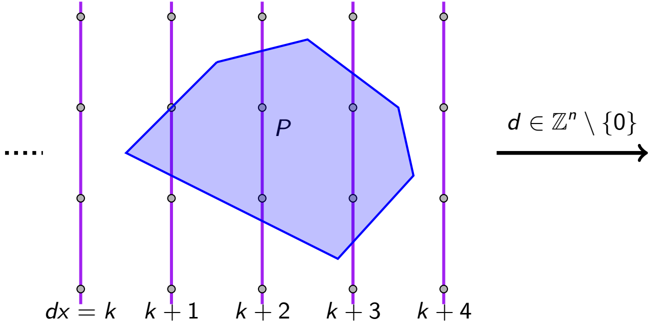
$$\lfloor \max\{dx : x \in P\} \rfloor - \lceil \min\{dx : x \in P\} \rceil + 1.$$

The integer width of P is

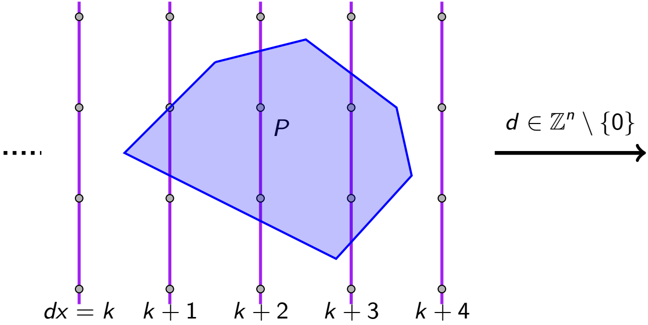
$$\inf_{d \in \mathbb{Z}^n \setminus \{0\}} \lfloor \max\{dx : x \in P\} \rfloor - \lceil \min\{dx : x \in P\} \rceil + 1.$$



- In this figure, the integer width of P along d is 3.



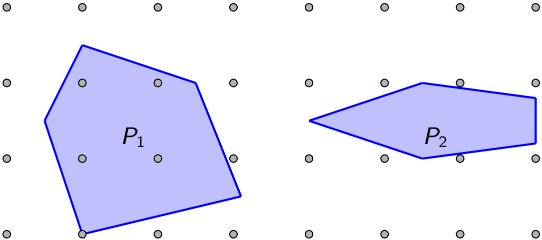
- In this figure, the integer width of P along d is 3.
- You can check that the integer width of P is 2.



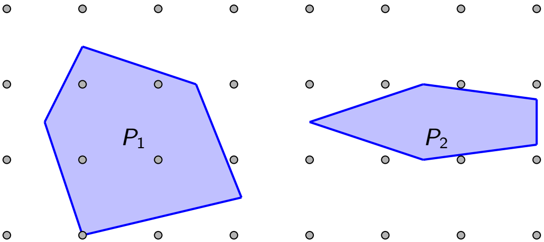
- In this figure, the integer width of P along d is 3.
- You can check that the integer width of P is 2.
- Integer width indeed measures “flatness” of a polyhedron.

Is a polyhedron containing no integer point "flat" (in at least one direction)?

Is a polyhedron containing no integer point "flat" (in at least one direction)?



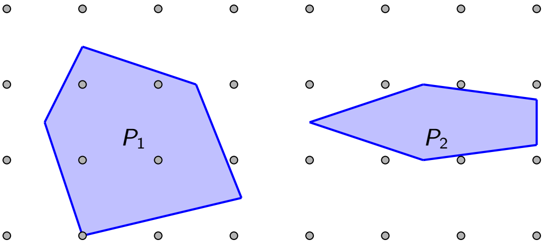
Is a polyhedron containing no integer point "flat" (in at least one direction)?



Theorem [Rudelson, 2000]

Let $K \subset \mathbb{R}^n$ be a compact convex set.

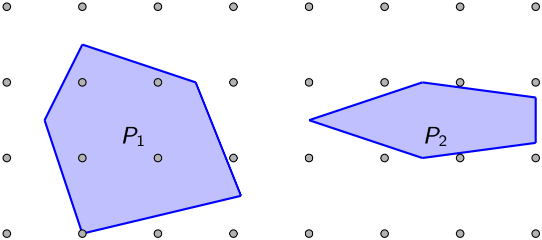
Is a polyhedron containing no integer point "flat" (in at least one direction)?



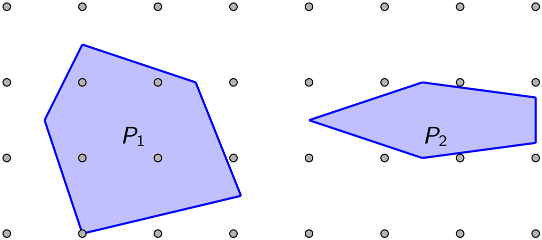
Theorem [Rudelson, 2000]

Let $K \subset \mathbb{R}^n$ be a compact convex set. Then, either K contains an integer point or the integer width of K is $O\left(n^{\frac{4}{3}} polylog(n)\right)$.

Is a Chvátal rank 1 polyhedron containing no integer point "more flat" than general lattice-free convex sets?



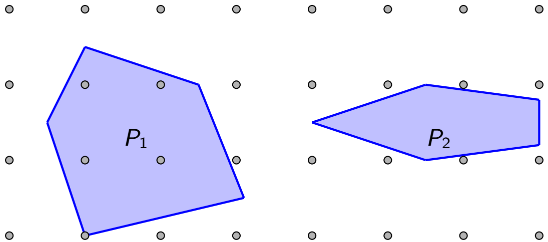
Is a Chvátal rank 1 polyhedron containing no integer point "more flat" than general lattice-free convex sets?



Theorem [Cornuéjols, Lee, and Li, 2016]

Let $K \subset \mathbb{R}^n$ be a closed convex set whose Chvátal closure is equal to its integer hull.

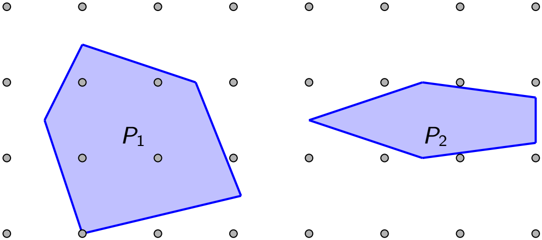
Is a Chvátal rank 1 polyhedron containing no integer point "more flat" than general lattice-free convex sets?



Theorem [Cornuéjols, Lee, and Li, 2016]

Let $K \subset \mathbb{R}^n$ be a closed convex set whose Chvátal closure is equal to its integer hull. Then, K contains an integer point or the integer width of K is at most n.

Is a Chvátal rank 1 polyhedron containing no integer point "more flat" than general lattice-free convex sets?



Theorem [Cornuéjols, Lee, and Li, 2016]

Let $K \subset \mathbb{R}^n$ be a closed convex set whose Chvátal closure is equal to its integer hull. Then, K contains an integer point or the integer width of K is at most n.

- There exists a polytope whose Chvátal closure is empty with its integer width exactly $n-1$.

Remark

- The flatness theorem implies the existence of a $2^{O(n)}n^n$ time Lenstra-type algorithm to solve the integer feasibility problem over a given rational polyhedron with Chvátal rank 1.

Remark

- The flatness theorem implies the existence of a $2^{O(n)}n^n$ time Lenstra-type algorithm to solve the integer feasibility problem over a given rational polyhedron with Chvátal rank 1.
- It does not improve Dadush's algorithm for integer programming over general convex sets.

Remark

- The flatness theorem implies the existence of a $2^{O(n)}n^n$ time Lenstra-type algorithm to solve the integer feasibility problem over a given rational polyhedron with Chvátal rank 1.
- It does not improve Dadush's algorithm for integer programming over general convex sets.
- We cannot improve the bound on the integer width of a closed convex set whose Chvátal closure is empty (It is really tight).

Remark

- The flatness theorem implies the existence of a $2^{O(n)}n^n$ time Lenstra-type algorithm to solve the integer feasibility problem over a given rational polyhedron with Chvátal rank 1.
- It does not improve Dadush's algorithm for integer programming over general convex sets.
- We cannot improve the bound on the integer width of a closed convex set whose Chvátal closure is empty (It is really tight).
- Is there another way to get a better algorithm?

We are analyzing difference between IP over a general polyhedron and IP over a Chvátal rank 1 polyhedron.

We are analyzing difference between IP over a general polyhedron and IP over a Chvátal rank 1 polyhedron.

The only information about Chvátal rank 1 polyhedra we have is a better bound on the integer width.

We are analyzing difference between IP over a general polyhedron and IP over a Chvátal rank 1 polyhedron.

The only information about Chvátal rank 1 polyhedra we have is a better bound on the integer width.

- Can we use more information about Chvátal rank 1 polyhedra?

We are analyzing difference between IP over a general polyhedron and IP over a Chvátal rank 1 polyhedron.

The only information about Chvátal rank 1 polyhedra we have is a better bound on the integer width.

- Can we use more information about Chvátal rank 1 polyhedra?
- Can we even characterize Chvátal rank 1 polyhedra?

We are analyzing difference between IP over a general polyhedron and IP over a Chvátal rank 1 polyhedron.

The only information about Chvátal rank 1 polyhedra we have is a better bound on the integer width.

- Can we use more information about Chvátal rank 1 polyhedra?
- Can we even characterize Chvátal rank 1 polyhedra?

Question

Is there a characterization of Chvátal rank 1 polyhedra which can be efficiently checked?

We are analyzing difference between IP over a general polyhedron and IP over a Chvátal rank 1 polyhedron.

The only information about Chvátal rank 1 polyhedra we have is a better bound on the integer width.

- Can we use more information about Chvátal rank 1 polyhedra?
- Can we even characterize Chvátal rank 1 polyhedra?

Question

Is there a characterization of Chvátal rank 1 polyhedra which can be efficiently checked?

- **This is the second topic of this talk.**

However, the answer is probably **NO**!

However, the answer is probably **NO**!

Theorem [Cornuéjols and Li, 2016]

Let $P \subset \mathbb{R}^n$ be a rational polytope. Even when $P_I = \emptyset$, the problem of deciding whether $P' = P_I$ is NP-hard.

However, the answer is probably **NO**!

Theorem [Cornuéjols and Li, 2016]

Let $P \subset \mathbb{R}^n$ be a rational polytope. Even when $P_I = \emptyset$, the problem of deciding whether $P' = P_I$ is NP-hard.

In fact,

However, the answer is probably **NO**!

Theorem [Cornuéjols and Li, 2016]

Let $P \subset \mathbb{R}^n$ be a rational polytope. Even when $P_I = \emptyset$, the problem of deciding whether $P' = P_I$ is NP-hard.

In fact,

Theorem [Cornuéjols, Lee, and Li, 2016]

- *Let $P \subseteq [0,1]^n$ be a rational polytope. Even when $P_I = \emptyset$, the problem of deciding whether $P' = P_I$ is NP-hard.*

However, the answer is probably **NO**!

Theorem [Cornuéjols and Li, 2016]

Let $P \subset \mathbb{R}^n$ be a rational polytope. Even when $P_I = \emptyset$, the problem of deciding whether $P' = P_I$ is NP-hard.

In fact,

Theorem [Cornuéjols, Lee, and Li, 2016]

- *Let $P \subseteq [0,1]^n$ be a rational polytope. Even when $P_I = \emptyset$, the problem of deciding whether $P' = P_I$ is NP-hard.*
- *Let $P \subset \mathbb{R}^n$ be a rational simplex. Even when $P_I = \emptyset$, the problem of deciding whether $P' = P_I$ is NP-hard.*

As a result,

As a result,

Corollary [Cornuéjols, Lee, and Li, 2016]

- *Given a rational polytope* $P \subseteq [0,1]^n$,

As a result,

Corollary [Cornuéjols, Lee, and Li, 2016]

- *Given a rational polytope $P \subseteq [0,1]^n$, the separation and optimization problems of the Chvátal closure over P are NP-hard.*

As a result,

Corollary [Cornuéjols, Lee, and Li, 2016]

- *Given a rational polytope $P \subseteq [0,1]^n$, the separation and optimization problems of the Chvátal closure over P are NP-hard.*
- *Given a rational simplex $P \subset \mathbb{R}^n$,*

As a result,

Corollary [Cornuéjols, Lee, and Li, 2016]

- *Given a rational polytope $P \subseteq [0,1]^n$, the separation and optimization problems of the Chvátal closure over P are NP-hard.*
- *Given a rational simplex $P \subset \mathbb{R}^n$, the separation and optimization problems of the Chvátal closure over P are NP-hard.*

Theorem [Cornuéjols, Lee, and Li, 2016]

Let $P \subseteq [0,1]^n$ be a rational polytope. Even when $P_I = \emptyset$, the problem of deciding whether $P' = P_I$ is NP-complete.

Proof.

Let $a_1, \ldots, a_n, b$ be positive integers such that $1 \leq a_1, \ldots, a_n < b$.

Theorem [Cornuéjols, Lee, and Li, 2016]

Let $P \subseteq [0,1]^n$ be a rational polytope. Even when $P_I = \emptyset$, the problem of deciding whether $P' = P_I$ is NP-complete.

Proof.

Let $a_1, \ldots, a_n, b$ be positive integers such that $1 \leq a_1, \ldots, a_n < b$.

Equality Knapsack Problem

Let $Q := \{x \in \mathbb{Z}^n : \sum_{i=1}^n a_i x_i = b,\ x \geq 0\}$. Decide whether $Q \neq \emptyset$.

Theorem [Cornuéjols, Lee, and Li, 2016]

Let $P \subseteq [0,1]^n$ be a rational polytope. Even when $P_I = \emptyset$, the problem of deciding whether $P' = P_I$ is NP-complete.

Proof.

Let $a_1, \ldots, a_n, b$ be positive integers such that $1 \le a_1, \ldots, a_n < b$.

Equality Knapsack Problem

Let $Q := \{x \in \mathbb{Z}^n : \sum_{i=1}^n a_i x_i = b, \ x \ge 0\}$. Decide whether $Q \neq \emptyset$.

- This problem is NP-complete.
- We are going to construct a 0,1 polytope $P \subseteq [0,1]^{n+4}$ such that $P_I = \emptyset$ using the data $a_1, \ldots, a_n, b$.
- Then, we will prove that $P' = \emptyset$ if and only if $Q \neq \emptyset$.

□

Proof.

$$
\begin{array}{rcllllllll}
 & & 1 & \cdots & n & n+1 & n+2 & n+3 & n+4 & \\
v^1 & = & (\; \frac{1}{2b}, & \cdots, & 0, & 0, & \frac{1}{2b}, & 0, & 0 &) \\
 & & & & \vdots & & & & & \\
v^n & = & (\; 0, & \cdots, & \frac{1}{2b}, & 0, & \frac{1}{2b}, & 0, & 0 &) \\
v^{n+1} & = & (\; 0, & \cdots, & 0, & 0, & 1/2, & 1/2, & 1/2 &) \\
v^{n+2} & = & (\; 1, & \cdots, & 1, & 1, & 1/2, & 1/2, & 1/2 &) \\
v^{n+3} & = & (\; 1/2, & \cdots, & 1/2, & 1/2, & 1, & 1, & 1 &) \\
v^{n+4} & = & (\; 1/4, & \cdots, & 1/4, & 1/4, & 1/4, & 1/4, & 1/4 &) \\
v^{n+5} & = & (\; 1/2, & \cdots, & 1/2, & 1/2, & 1, & 1, & 1/2 &) \\
v^{n+6} & = & (\; 1/2, & \cdots, & 1/2, & 1/2, & 0, & 0, & 1/2 &) \\
v^{n+7} & = & (\; 1/2, & \cdots, & 1/2, & 1/2, & 1/2, & 1, & 1 &) \\
v^{n+8} & = & (\; 1/2, & \cdots, & 1/2, & 1/2, & 1/2, & 0, & 0 &) \\
v^{n+9} & = & (\; \frac{a_1}{2b}, & \cdots, & \frac{a_n}{2b}, & 0, & 0, & \frac{1}{2} - \frac{1}{4b}, & 0 &) \\
v^{n+10} & = & (\; 1 - \frac{a_1}{2b}, & \cdots, & 1 - \frac{a_n}{2b}, & 1, & \frac{1}{2} + \frac{1}{4b}, & 0, & 0 &)
\end{array}
$$

Proof.

$$
\begin{array}{rcllllllll}
 & & 1 & \cdots & n & n+1 & n+2 & n+3 & n+4 & \\
v^1 = & (& \frac{1}{2b}, & \cdots, & 0, & 0, & \frac{1}{2b}, & 0, & 0 &) \\
 & & & & \vdots & & & & & \\
v^n = & (& 0, & \cdots, & \frac{1}{2b}, & 0, & \frac{1}{2b}, & 0, & 0 &) \\
v^{n+1} = & (& 0, & \cdots, & 0, & 0, & 1/2, & 1/2, & 1/2 &) \\
v^{n+2} = & (& 1, & \cdots, & 1, & 1, & 1/2, & 1/2, & 1/2 &) \\
v^{n+3} = & (& 1/2, & \cdots, & 1/2, & 1/2, & 1, & 1, & 1 &) \\
v^{n+4} = & (& 1/4, & \cdots, & 1/4, & 1/4, & 1/4, & 1/4, & 1/4 &) \\
v^{n+5} = & (& 1/2, & \cdots, & 1/2, & 1/2, & 1, & 1, & 1/2 &) \\
v^{n+6} = & (& 1/2, & \cdots, & 1/2, & 1/2, & 0, & 0, & 1/2 &) \\
v^{n+7} = & (& 1/2, & \cdots, & 1/2, & 1/2, & 1/2, & 1, & 1 &) \\
v^{n+8} = & (& 1/2, & \cdots, & 1/2, & 1/2, & 1/2, & 0, & 0 &) \\
v^{n+9} = & (& \frac{a_1}{2b}, & \cdots, & \frac{a_n}{2b}, & 0, & 0, & \frac{1}{2} - \frac{1}{4b}, & 0 &) \\
v^{n+10} = & (& 1 - \frac{a_1}{2b}, & \cdots, & 1 - \frac{a_n}{2b}, & 1, & \frac{1}{2} + \frac{1}{4b}, & 0, & 0 &)
\end{array}
$$

Let $P := \text{conv}\{v^1, \ldots, v^{n+10}\}$.

Proof.

$$\begin{array}{rcllllllll}
 & & 1 & \cdots & n & n+1 & n+2 & n+3 & n+4 & \\
v^1 & = & (\ \frac{1}{2b}, & \cdots, & 0, & 0, & \frac{1}{2b}, & 0, & 0 &) \\
 & & & & \vdots & & & & & \\
v^n & = & (\ 0, & \cdots, & \frac{1}{2b}, & 0, & \frac{1}{2b}, & 0, & 0 &) \\
v^{n+1} & = & (\ 0, & \cdots, & 0, & 0, & 1/2, & 1/2, & 1/2 &) \\
v^{n+2} & = & (\ 1, & \cdots, & 1, & 1, & 1/2, & 1/2, & 1/2 &) \\
v^{n+3} & = & (\ 1/2, & \cdots, & 1/2, & 1/2, & 1, & 1, & 1 &) \\
v^{n+4} & = & (\ 1/4, & \cdots, & 1/4, & 1/4, & 1/4, & 1/4, & 1/4 &) \\
v^{n+5} & = & (\ 1/2, & \cdots, & 1/2, & 1/2, & 1, & 1, & 1/2 &) \\
v^{n+6} & = & (\ 1/2, & \cdots, & 1/2, & 1/2, & 0, & 0, & 1/2 &) \\
v^{n+7} & = & (\ 1/2, & \cdots, & 1/2, & 1/2, & 1/2, & 1, & 1 &) \\
v^{n+8} & = & (\ 1/2, & \cdots, & 1/2, & 1/2, & 1/2, & 0, & 0 &) \\
v^{n+9} & = & (\ \frac{a_1}{2b}, & \cdots, & \frac{a_n}{2b}, & 0, & 0, & \frac{1}{2} - \frac{1}{4b}, & 0 &) \\
v^{n+10} & = & (\ 1 - \frac{a_1}{2b}, & \cdots, & 1 - \frac{a_n}{2b}, & 1, & \frac{1}{2} + \frac{1}{4b}, & 0, & 0 &)
\end{array}$$

Let $P := \text{conv}\{v^1, \ldots, v^{n+10}\}$. We get a linear description for P in polytime.

Proof.

$$
\begin{array}{rcllllllll}
 & & 1 & \cdots & n & n+1 & n+2 & n+3 & n+4 & \\
v^1 & = & (\; \frac{1}{2b}, & \cdots, & 0, & 0, & \frac{1}{2b}, & 0, & 0 &) \\
 & & & & \vdots & & & & & \\
v^n & = & (\; 0, & \cdots, & \frac{1}{2b}, & 0, & \frac{1}{2b}, & 0, & 0 &) \\
v^{n+1} & = & (\; 0, & \cdots, & 0, & 0, & 1/2, & 1/2, & 1/2 &) \\
v^{n+2} & = & (\; 1, & \cdots, & 1, & 1, & 1/2, & 1/2, & 1/2 &) \\
v^{n+3} & = & (\; 1/2, & \cdots, & 1/2, & 1/2, & 1, & 1, & 1 &) \\
v^{n+4} & = & (\; 1/4, & \cdots, & 1/4, & 1/4, & 1/4, & 1/4, & 1/4 &) \\
v^{n+5} & = & (\; 1/2, & \cdots, & 1/2, & 1/2, & 1, & 1, & 1/2 &) \\
v^{n+6} & = & (\; 1/2, & \cdots, & 1/2, & 1/2, & 0, & 0, & 1/2 &) \\
v^{n+7} & = & (\; 1/2, & \cdots, & 1/2, & 1/2, & 1/2, & 1, & 1 &) \\
v^{n+8} & = & (\; 1/2, & \cdots, & 1/2, & 1/2, & 1/2, & 0, & 0 &) \\
v^{n+9} & = & (\; \frac{a_1}{2b}, & \cdots, & \frac{a_n}{2b}, & 0, & 0, & \frac{1}{2}-\frac{1}{4b}, & 0 &) \\
v^{n+10} & = & (\; 1-\frac{a_1}{2b}, & \cdots, & 1-\frac{a_n}{2b}, & 1, & \frac{1}{2}+\frac{1}{4b}, & 0, & 0 &)
\end{array}
$$

Let $P := \text{conv}\{v^1, \ldots, v^{n+10}\}$. We get a linear description for P in polytime. $P' = \emptyset$ if and only if $Q \neq \emptyset$.

In addition,

In addition,

Theorem [Cornuéjols, Lee, and Li, 2016]

- *Given a rational polytope* $P \subseteq [0,1]^n$,

In addition,

Theorem [Cornuéjols, Lee, and Li, 2016]

- *Given a rational polytope $P \subseteq [0,1]^n$, deciding whether adding at most 1 Chvátal inequality is sufficient to describe its integer hull is NP-hard.*

In addition,

Theorem [Cornuéjols, Lee, and Li, 2016]

- *Given a rational polytope $P \subseteq [0,1]^n$, deciding whether adding at most 1 Chvátal inequality is sufficient to describe its integer hull is NP-hard.*
- *Given a rational simplex $P \subset \mathbb{R}^n$,*

In addition,

Theorem [Cornuéjols, Lee, and Li, 2016]

- *Given a rational polytope $P \subseteq [0,1]^n$, deciding whether adding at most 1 Chvátal inequality is sufficient to describe its integer hull is NP-hard.*
- *Given a rational simplex $P \subset \mathbb{R}^n$, deciding whether adding at most 1 Chvátal inequality is sufficient to describe its integer hull is NP-hard.*

In addition,

Theorem [Cornuéjols, Lee, and Li, 2016]

- *Given a rational polytope $P \subseteq [0,1]^n$, deciding whether adding at most 1 Chvátal inequality is sufficient to describe its integer hull is NP-hard.*
- *Given a rational simplex $P \subset \mathbb{R}^n$, deciding whether adding at most 1 Chvátal inequality is sufficient to describe its integer hull is NP-hard.*

- These hardness results imply that we should not expect to find a "simple" characterization of polyhedra with Chvátal rank 1.

- However, meaningful algorithmic "properties" of Chvátal rank 1 polyhedra have not been studied (widely open).

- However, meaningful algorithmic "properties" of Chvátal rank 1 polyhedra have not been studied (widely open).
- More information will lead to better algorithm.

Final remark

- However, meaningful algorithmic "properties" of Chvátal rank 1 polyhedra have not been studied (widely open).
- More information will lead to better algorithm.
- We need better understanding about Chvátal rank 1 polyhedra.

Thank you for your attention!

Paper available on (integer.tepper.cmu.edu)

Boyd, S. and Pulleyblank, W. R. (2009).
Facet Generating Techniques, pages 33–55.
Springer Berlin Heidelberg, Berlin, Heidelberg.

Chvátal, V. (1973).
Edmonds polytopes and a hierarchy of combinatorial problems.
Discrete Mathematics, 4(4):305 – 337.

Cornuéjols, G. and Li, Y. (2016).
Deciding emptiness of the Gomory-Chvátal closure is NP-complete, even for a rational polyhedron containing no integer point.
In Louveaux, Q. and Skutella, M., editors, *Proceedings of the 18th International Conference on Integer Programming and Combinatorial Optimization*, pages 387–397. Springer.

Dadush, D., Peikert, C. P., and Vempala, S. (2011).
Enumerative lattice algorithms in any norm via M-ellipsoid coverings.
In *Proceedings of the 52nd Annual IEEE Symposium on Foundations of Computer Science*, pages 580–589. IEEE.

Rudelson, M. (2000).
Distances between non-symmetric convex bodies and the mm^*-estimate.
Positivity, 4(2):161 – 178.

Schrijver, A. (1980).

On cutting planes.
Annals of Discrete Mathematics, 9:291 – 296.