MINLP: Undecidability and Hardness

A tutorial

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Section 1

Introduction

Mixed-Integer Nonlinear Programming

MINLP

- Formal declarative language
  sentences describe optimization problems
- Can encode pure feasibility problems
  by minimizing a constant function e.g. $\min\{0 \mid g(x) \le 0\}$
- Includes most other MP classes
  e.g. LP, MILP, NLP
- Interpreter = solver
  shifts focus from algorithmics to modelling
- Only consider single-objective MP

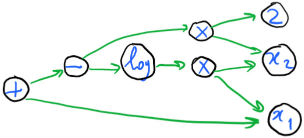
Syntax

Given functions $f, g_1, \ldots, g_m : \mathbb{Q}^n \to \mathbb{Q}$ **and** $Z \subseteq \{1, \ldots, n\}$

$$\left.\begin{array}{rcll} \min & f(x) & & \\ \forall i \le m & g_i(x) & \le & 0 \\ \forall j \in Z & x_j & \in & \mathbb{Z} \end{array}\right\}$$

- f, g_i **represented by** ***expression DAGs***

$x_1 + 2x_2 - \log(x_1 x_2)$

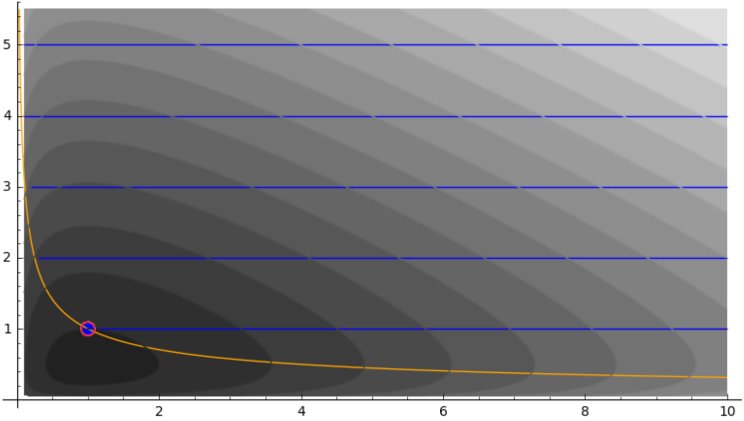


E.g. $\min\{x_1 + 2x_2 - \log(x_1 x_2) \mid x_1 x_2^2 \ge 1 \wedge x_1 \ge 0 \wedge x_2 \in \mathbb{N}\}$

Semantics

$$P \equiv \min\{x_1 + 2x_2 - \log(x_1 x_2) \mid x_1 x_2^2 \geq 1 \wedge x_1 \geq 0 \wedge x_2 \in \mathbb{N}\}$$

$$[\![P]\!] = (\mathsf{opt}(P), \mathsf{val}(P)) \qquad \mathsf{opt}(P) = (1, 1) \qquad \mathsf{val}(P) = 3$$



"Solving an MP"

- Given an MP P, there are three possibilities:
  1. $[\![P]\!]$ exists
  2. P is unbounded
  3. P is infeasible
- P *has a feasible solution* iff $[\![P]\!]$ exists or is unbounded

  otherwise it is infeasible
- P *has an optimum* iff $[\![P]\!]$ exists

  otherwise it is infeasible or unbounded
- Asymmetry between optimization and feasibility

$$P \equiv \min\{0 \mid g(x) \le 0 \wedge x \in X\} \quad \begin{cases} \text{YES} & \exists[\![P]\!] \vee \mathsf{unbnd}(P) \\ \text{NO} & \mathsf{infeas}(P) \end{cases}$$

$$Q \equiv \min\{f(x) \mid g(x) \le 0 \wedge x \in X\} \quad \begin{cases} \text{YES} & \exists[\![Q]\!] \\ \text{NO} & \mathsf{unbnd}(Q) \vee \mathsf{infeas}(Q) \end{cases}$$

Section 2

Undecidability

Formal systems (FS)

- *Formal System* $\mathcal{F}$
  - *alphabet* and *formal grammar*
    well-formed *formulæ* and *sentences*
  - *Axioms* $\mathcal{A}$
    (recursive[1] consistent set of sentences)
  - *Inference rules* $\mathcal{R}$
    derive new sentences from old ones
- *Language* $\mathcal{L}$
  set of all sentences of $\mathcal{F}$
- *Theory* $\mathcal{T}$
  sentences obtained by iterated application of $\mathcal{R}$ to $\mathcal{A}$

[1] M *recursive* if $\exists$ alg. solving "given a, is $a \in M$ or not?"

Some FSs

- **Peano Arithmetic (PA):** $\rightarrow \leftrightarrow, \wedge, \vee, \neg, \forall, \exists, +, \times, =$ and variable names; 1st order sentences about $\mathbb{N}$; A=PL+induction; *modus ponens* and generalization
  - *$\mathcal{T}$: provable sentences about $\mathbb{N}$*
- **Real-closed Fields (RLF):** like above and $>$; polynomials over $\mathbb{R}$; field axioms for $\mathbb{R}$, "basic operations on polynomials"
  - *$\mathcal{T}$: polynomial systems over $\mathbb{R}$ with solution in $\mathbb{R}$*
- **Diophantine Equations (DE):** existentially quantified subset of PA
  - *$\mathcal{T}$: polynomial systems over $\mathbb{Z}$ with solution in $\mathbb{N}$*
    - $\{[\exists x \in \mathbb{N}^n \; p(x) = 0] \mid p \in \mathbb{Z}[x]\} \equiv \{[f \wedge x \in \mathbb{N}^n] \mid f \in \mathcal{L}(\mathsf{RLF})\}$
    - *"between PA and RLF"*

What is decidability?

FS $\mathcal{F}$ is *decidable* if

$$\exists \textbf{ algorithm } A : \mathcal{L} \to \{0,1\}$$

$$\forall f \in \mathcal{L} \quad A(f) = \begin{cases} 1 & \textbf{if } f \in \mathcal{T} \\ 0 & \textbf{otherwise} \end{cases}$$

- PA: **does a sentence have a proof in PA or not?**
- RLF: **does a polynomial over $\mathbb{R}$ have a solution in $\mathbb{R}$ or not?**
- DE: **does a polynomial over $\mathbb{Z}$ have a solution in $\mathbb{N}$ or not?**

Only YES/NO answer required (rather than explicit proof)

Subsection 1

Polynomial systems in integer variables

DE: Relevance to MINLP

- DE ⊆ [MINLP feasibility] *obvious*
- DE ⊆ [MINLP optimality]

$$\left.\begin{array}{rcl} \forall i \le m \quad g_i(x) & = & 0 \\ x & \in & \mathbb{N}^n \end{array}\right\} \quad \textit{is feasible}$$

$$\Leftrightarrow \quad \left\{ \begin{array}{rcl} & u^* = \min u & \\ (1-u) \sum_{i \le m} (g_i(x))^2 & = & 0 \\ (x, u) & \in & \mathbb{N}^{n+1} \end{array} \right\} = 0$$

- *if* $u^* > 1$ *get* $-1 = 0$ *(contradiction)*
- *if* $u^* = 1$ *get* $g(x) \neq 0$ *(infeasible)*
- *if* $u^* = 0$ *get* $g(x) = 0$ *(feasible)*

Suppose $u^* = 1$ and $g(x) = 0$ feas., then $u = 0$ would also satisfy constr. and contradict minimality of u^*

Goal: MINLP is undecidable

- MINLP contains DE
- show DE is undecidable
  - any r.e.[2] subset of $\mathbb{N}$ can be encoded by a DE
  - $\{a \in \mathbb{N} \mid a \in \text{HALTING}\}$ is r.e.
  - *so* HALTING *can be represented by a DE*
  - if every DE were decidable, we could solve HALTING

[2] $M \subseteq \mathbb{N}$ is r.e. if $\exists$ alg. terminating on input a iff $a \in M$ (nonterm. for $a \notin M$)

DE and set membership

- DE sentence $[p(y) = 0 \wedge y \in \mathbb{N}^{n+1}]$
  $y = (a, x)$ where $a \in \mathbb{N}$ "encodes the instance"
- **DEs define subsets $M \subseteq \mathbb{N}$:**

$$a \in M \quad \leftrightarrow \quad \exists x \in \mathbb{N}^n \; p(a, x) = 0 \quad (\dagger)$$

- **Conversely, given $M \subseteq \mathbb{N}$,**
  is there $p(a, x) \in \mathbb{Z}[a, x]$ s.t. (†)?

- **Focus on r.e. sets $M \subseteq \mathbb{N}$**
- Matyiasevich-Davis-Putnam-Robinson thm. (MDPR, 1970):
  For each r.e. set M there is a DE $p(a, x)$ s.t. (†)

MDPR implies undecidability

- Suppose DE is decidable
  - $\Rightarrow$ *given DE, can decide feasibility/infeas.*
  - $\Rightarrow$ *given r.e. set M, can decide $a \in M$ and $a \notin M$*
- HALTING: given TM T and input ι, will $T(\iota)$ terminate?
  undecidable by [Turing 1936] *(diagonal argument)*
- Let $H = \{(T, \iota) \mid T(\iota) \downarrow\}$
  H is r.e.: simulate T with input ι, terminates iff $T(\iota) \downarrow$
  M_H = encoding of H in $\mathbb{N} \Rightarrow M_H$ is r.e.
- $\Rightarrow$ can decide M_H and solve HALTING, contradiction
- Hence DE undecidable

- $\exists$ Universal Diophantine Equations (UDE)
  encoding the dynamics of a UTM
  $\exists$ UDE deg d and n vars where $(d, n) \in \{(4, 58), (1.6 \times 10^{45}, 9)\}$

Structure of the MDPR theorem

- **Proof of Gödel's 1st incompleteness thm.**

  r.e. sets $\equiv$ DE with $< \infty$ $\exists$ and bounded $\forall$ quantifiers

- **Davis' normal form**

  one bounded quantifier suffices: $\exists x_0 \forall a \leq x_0 \exists x \; p(a,x) = 0$

  - (2 bnd qnt $\equiv$ 1 bnd qnt on pairs) and induction

- **Robinson's idea**

  get rid of universal quantifier by using exponent vars

  - *idea:* $[\exists x_0 \forall a \leq x_0 \exists x \; p(a,x) = 0]$ "$\rightarrow$" $\left[\exists x \prod_{a \leq x_0} p(a,x)\right]$
  - precise encoding needs variables in exponents

- **Matyiasevic's contribution**

  express $c = b^a$ using polynomials

  - use Pell's equation $x^2 - dy^2 = 1$
  - solutions (x_n, y_n) satisfy $x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n$
  - x_n, y_n grow exponentially with n

Subsection 2

Polynomial systems in continuous variables

RLF: Relevance to MINLP

- RLF $\supseteq$ [poly NLP feasibility] *obvious*
- RLF $\subseteq$ [poly NLP optimality]

$$\forall i \le m \quad g_i(x) = 0 \quad \textit{is feasible}$$

$$\Leftrightarrow \left\{ \begin{array}{c} \alpha^* = \min u^2 \\ (1-u^2) \sum_{i \le m} (g_i(x))^2 \quad = \quad 0 \end{array} \right\} = 0$$

- *if* $\alpha^* > 1$ *get* $-1 = 0$ *(contradiction)*
- *if* $\alpha^* > 0$ *get* $g(x) \neq 0$ *(infeasible)*
- *if* $\alpha^* = 0$ *get* $g(x) = 0$ *(feasible)*

Goal 1: [poly NLP feas.] is decidable

- RLF contains [poly NLP feasibility]
- RLF decidable
- $\Rightarrow$ [poly NLP feasibility] is decidable

Goal 2: [poly NLP] is decidable

- **[poly NLP]:**
  we need to tell optimality and unboundedness apart
- RLF **also includes universal quantifiers**
- $P \equiv \min\{f(x)\ g(x) \leq 0\}$ **unbounded:**

$$\forall y \quad f(x) = y \wedge g(x) \leq 0$$

- $[\![P]\!]$ **exists:**

$$\exists y \quad f(x) = y \wedge g(x) \leq 0 \wedge \neg \textbf{unbounded}(P)$$

- P **infeasible:**

$$\neg \exists y \quad f(x) = y \wedge g(x) \leq 0$$

Decidability of RLF

- RLF **sentence** $[p(x) \mathscr{R}\ 0]$: $p \in \mathbb{R}[x], x \in \mathbb{R}^n, \mathscr{R} \in \{=, <\}$

$$\begin{array}{rrcl} & p(x) < 0 & \longleftrightarrow & p(x) - y = 0 \wedge y < 0 \\ & p(x) \le 0 & \longleftrightarrow & p(x) - y^2 = 0 \\ \forall i \le m & p_i(x) = 0 & \longleftrightarrow & \sum_{i \le m} (p_i(x))^2 = 0 \end{array}$$

open constraints $y < 0$ invalid in MP, need not bother

- **$\exists$? alg. for deciding if any $p(x) = 0$ solves $\mathbb{R}$ or not?**
- RLF **is decidable by** ***quantifier elimination*** **[Tarski 1948]**
- **Quantifier elimination:**
  - *constructs solution sets (YES) or derives contradictions (NO)*
  - $\Rightarrow$ RLF is complete, too
  - *think of Fourier-Motzkin elimination for linear RLF*

Example: quantifier elimination in $\mathbb{R}^1$

- DLO (Dense Linear Order):

  $0, 1, \neg, \vee, \wedge, \exists, \forall, <, =$, vars; quantifiers over $\mathbb{R}$

- Reduce to form $\exists x \bigwedge_{i \leq m} q_i$

  where all q_i's have form $x = v$, $x < v$, or $x > v$ (v var/const)

  - $x = x$ can be removed from conjunction
  - $x < x$: sentence is false (*and there's a proof!*)
  - if v x differ, rewrite $\exists x\ x = v \wedge r(x, v) \leftrightarrow \exists x\ r(x, x)$

    back to previous case

  - remaining case: $\bigwedge q_i$ is

    $$\bigwedge_i (u_i < x) \wedge \bigwedge_i (x < v_i)$$

    rewrite as $\bigwedge_i u_i < v_i$

  - *get* [$\exists x \bar{q}$ where $\bar{q}$ does not involve x] *or* contradiction

- repeat until only constants in $\mathbb{R}$ left

  get proof of YES and proof of NO

  DLO is decidable and complete

Rationals

- **[Robinson 1949]:**
  RT (1st ord. theory over $\mathbb{Q}$) is undecidable
- **[Pheidas 2000]:** *existential* theory of $\mathbb{Q}$ (ERT) is open
  can we decide wether $p(x) = 0$ *has solutions in* $\mathbb{Q}$*?* Boh!
- **[Matyiasevich 1993]:**
  - equivalence between DEH and ERT
  - DEH = [DE restricted to homogeneous polynomials]
  - *but we don't know whether DEH is decidable*

Note that Diophantus solved DE in positive rationals

Subsection 3

Digressions

Proof complexity bounds with UDEs

The following surprising bound is due to [Jones 1982]

For any axiomatizable theory $\mathcal{T}$ in PA1 and any sentence $p \in \mathcal{T}$, if p has a proof in $\mathcal{T}$, then it has a proof consisting of 100 additions and multiplications of integers

- Gödel numbering: $\mathcal{T} \longrightarrow$ r.e. subset of $\mathbb{N}$
- Search for proofs $\longleftrightarrow$ search for DE solutions
  solution encodes whole proof
- $\exists$ UDE the evaluation of which takes 100 $+, \times$ operations
- Any solution of the UDE can be verified in at most 100 operations

Common misconception 1

"Since $\mathbb{N}$ is contained in $\mathbb{R}$, how is it possible that RLF is decidable but DE (= RLF $\cap$ $\mathbb{N}$) is not?"

After all, if a problem contains a hard subproblem, it's hard by inclusion, right?

- **Can you express** DE $p(x) = 0 \wedge x \in \mathbb{N}$ **in** RLF?
  - $p(x) = 0$ **belongs to both** DE **and** RLF, **OK**
  - "$x \in \mathbb{N}$" **in** RLF?

    $\Leftarrow$ find poly $q(x)$ s.t. $\exists x\, q(x) = 0$ **iff** $x \in \mathbb{N}^n$
  - $q(x) = x(x-1)\cdots(x-a)$ **only good for** $\{0, 1, \ldots, a\}$

    $q(x) = \prod_{i \in \omega} (x - i)$ is ∞ly long, invalid
  - **IMPOSSIBLE!**

    if it were possible, DE would be decidable, contradiction

Common misconception 2

"Decidability implies completeness"

For any sentence f we can decide whether $\exists$ proof or not, so either f or $\neg f$ must be provable, right?

- **Algebraically Closed Fields (ACF):**
  field axioms + "every polynomial splits" schema
- *ACF is **decidable** by quantifier elimination*
- $\mathsf{C}_p \equiv [\sum_{j \leq p} 1 = 0]$ **(for any prime p): independent of ACF**
  - *$\exists$ fields of every prime characteristic p*
  - *each different field satisfies C_p and negates C_q for $q \neq p$*
- **ACF is incomplete: neither C_p nor $\neg\mathsf{C}_p$ is provable in ACF**
- **Decision algorithm for ACF returns NO for both**

$\Rightarrow$ *theories can be decidable and incomplete*

Subsection 4

Application to MINLP

MIQCP is undecidable

- **[Jeroslow 1973]: MIQCP:**

$$\left.\begin{array}{rrcl} \min & c^\top x & & \\ \forall i \le m & x^\top Q^i x + {a_i}^\top x + b_i & \ge & 0 \\ & x & \in & \mathbb{Z}^n \end{array}\right\} \qquad (\dagger)$$

is undecidable

Proof:

- **Let $U(a,x) = 0$ be an UDE**
- $P(a) \equiv \min\{u \in \mathbb{N} \mid (1-u)U(a,x) = 0 \wedge x \in \mathbb{Z}^n\}$

  $P(a)$ describes an undecidable problem
- **Linearize every product $x_i x_j$ by y_{ij} and add $y_{ij} = x_i x_j$**

  until only degree 1 and 2 left
- **Obtain MIQCP (†)**

Some MIQCQPs are decidable

- If each Q_i is diagonal PSD, decidable [Witzgall 1963]

- If x are bounded in $[x^L, x^U] \cap \mathbb{Z}^n$, decidable

  can express $x \in \{x^L, x^L+1, \ldots, x^U\}$ by polynomial

$$\forall i \le m \qquad \prod_{x_i^L \le i \le x_i^U} (x - i) = 0$$

  turn into poly system in $\mathbb{R}$ (in RLF, decidable)

- $\Rightarrow$ *Bounded* (vars) easier than *unbounded* (for $\mathbb{Z}$)

- [MIQP decision vers.] is decidable

$$\left.\begin{array}{rrcl} & x^\top Q x + c^\top x & \le & \gamma \\ & Ax & \ge & b \\ \forall j \in Z & x_j & \in & \mathbb{Z} \end{array}\right\} \qquad \text{(in NP [Del Pia et al. 2014])}$$

NLP is undecidable

We can't represent unbounded subsets of $\mathbb{N}$ by polynomials
But we can if we allow some transcendental functions

$$x \in \mathbb{Z} \quad \longleftrightarrow \quad \sin(\pi x) = 0$$

- **Constrained NLP is undecidable:**

$$\min\{0 \mid U(a,x) = 0 \wedge \forall j \le n \ \sin(\pi x_j) = 0\}$$

- **Even with just one nonlinear constraint:**

$$\min\{0, \mid (U(a,x))^2 + \sum_{j \le n} (\sin(\pi x_j))^2 = 0\}$$

- **Unconstrained NLP is undecidable:**

$$\min (U(a,x))^2 + \sum_{j \le n} (\sin(\pi x_j))^2$$

- **Box-constrained NLP is undecidable** *(boundedness doesn't help)***:**

$$\min\{(U(a, \tan x_1, \ldots, \tan x_n))^2 + \sum_{j \le n} (\sin(\pi \tan x_j))^2 \mid -\frac{\pi}{2} \le x \le \frac{\pi}{2}\}$$

Some NLPs are decidable

- All polynomial NLPs are **decidable**

  by decidability of RLF

- Quadratic Programming (QP) is **decidable over $\mathbb{Q}$**

$$\left.\begin{array}{rrcl} \min & x^\top Q x & + & c^\top x \\ & Ax & \geq & b \end{array}\right\} \qquad (P)$$

- *Bricks of the proof*
  - if Q is PSD, $[\![P]\!] \in \mathbb{Q}$
    1. *replace components using active constraints at opt*
    2. *work out KKT conditions, they are linear in rational coefficients*
    3. *$\Rightarrow$ solution is rational*
  - $\exists$ polytime IPM for solving P [Renegar&Shub 1992]
  - *unbounded case treated in [Vavasis 1990]*
- $\Rightarrow$ [QP decision version] is in NP

  $\Rightarrow$ QP is decidable over $\mathbb{Q}$

Section 3

Hardness

MILP is NP-hard

- SAT is NP-hard by Cook's theorem

  Reduce from SAT in CNF

$$\bigwedge_{i \le m} \bigvee_{j \in C_i} \ell_j$$

  where ℓ_j is either x_j or $\bar{x}_j \equiv \neg x_j$

- Polynomial reduction ρ

SAT	x_j	$\bar{x}_j$	$\vee$	$\wedge$
MILP	x_j	$1 - x_j$	$+$	≥ 1

- E.g. ρ maps $(x_1 \vee x_2) \wedge (\bar{x}_2 \vee x_3)$ to MILP

$$\min\{0 \mid x_1 + x_2 \ge 1 \wedge x_3 - x_2 \ge 0 \wedge x \in \{0,1\}^3\}$$

- SAT is YES iff MILP is feasible

  trivially shows MINLP is NP*-hard*

Focus on continuous problems

- Hardness of MINLP due to integer variables
  question hardness <u>without</u> integrality constrs.

- **Interesting issue:** **hardness in fixed dimension**
  - $[ax^2 + by - c = 0 \wedge (x, y) \in \mathbb{N}^2]$ **is NP-complete**
    [Manders& Adleman 1978], from 3SAT → KNAPSACK → ·
  - $\Rightarrow [x^2 \equiv a \pmod{b}]$ **is NP-complete**
  - $\Rightarrow \min\{(ax^2 + by - c)^2 \mid x, y \in \mathbb{N}\}$ **is NP-hard**

Subsection 1

Quadratic Programming

QP is NP-hard

- By reduction from SAT, let σ be an instance
- $\hat{\rho}(\sigma, x) \geq 1$: linear constraints from SAT $\to$ MILP reduction
- Consider QP

$$\left.\begin{array}{ll} \min & f(x) = \sum_{j \leq n} x_j(1 - x_j) \\ & \hat{\rho}(\sigma, x) \geq 1 \\ & 0 \leq x \leq 1 \end{array}\right\} \quad (\dagger)$$

- **Claim:** σ is YES iff $\mathsf{val}(\dagger) = 0$
- *Proof:*
  - assume σ YES with soln. x^*, then $x^* \in \{0, 1\}^n$, hence $f(x^*) = 0$, since $f(x) \geq 0$ for all x, $\mathsf{val}(\dagger) = 0$
  - assume σ NO, suppose $\mathsf{val}(\dagger) = 0$, then $(\dagger)$ feasible with soln. x', since $f(x') = 0$ then $x' \in \{0, 1\}$, feasible in SAT hence σ is YES, contradiction

Box-constrained QP is NP-hard

- **Add surplus vars v to SAT→MILP constraints:**
  $\hat{\rho}(\sigma, x) - 1 - v = 0$
  (denote by $\forall i \leq m \ (a_i^\top x - b_i - v_i = 0)$**)**
- **Now sum them on the objective**

$$\left.\begin{array}{ll} \min & \sum\limits_{j \leq n} x_j(1 - x_j) + \sum\limits_{i \leq m} (a_i^\top x - b_i - v_i)^2 \\ & 0 \leq x \leq 1, v \geq 0 \end{array}\right\}$$

- **Issue: v not bounded above**
- **Reduce from 3SAT, get ≤ 3 literals per clause**
  $\Rightarrow$ ***can consider*** $0 \leq v \leq 2$

cQKP is NP-hard

- CONTINUOUS QUADRATIC KNAPSACK PROBLEM (cQKP)

$$\left.\begin{array}{rcl} \min \quad f(x) = x^\top Q x & + & c^\top x \\ \sum_{j \le n} a_j x_j & = & \gamma \\ x & \in & [0,1]^n, \end{array}\right\}$$

- Reduction from SUBSET-SUM

  given list $a \in \mathbb{Q}^n$ and γ, is there $J \subseteq \{1, \ldots, n\}$ s.t. $\sum_{j \in J} a_j = \gamma$?

  reduce to $f(x) = \sum_j x_j(1 - x_j)$
- σ is a YES instance of SUBSET-SUM
  - let $x^*_j = 1$ iff $j \in J$, $x^*_j = 0$ otherwise
  - feasible by construction
  - f is non-negative on $[0,1]^n$ and $f(x^*) = 0$: optimum
- σ is a NO instance of SUBSET-SUM
  - suppose $\mathsf{opt}(\text{cQKP}) = x^*$ s.t. $f(x^*) = 0$
  - then $x^* \in \{0,1\}^n$ because $f(x^*) = 0$
  - feasibility of $x^* \to \mathsf{supp}(x^*)$ solves σ, contradiction, hence $f(x^*) > 0$

One negative eigenvalue: hard

- Convex QP is in P

  *need some negative eigenvalue for hardness: **how many?***

- QP with 1 Negative Eigenvalue (QP1NE) is NP-hard

  reduction from k-CLIQUE instance $(G = (V, E), k)$

- Reduce to the following QP:

$$
\begin{array}{rrcll}
 & \min \quad z & - & w^2 & (1)\\
 & \sum_{j\in V} 4^j x_j & = & w & (2)\\
 & \sum_{j\in V} 4^{2j} x_j + 2\sum_{i<j} 4^{i+j} y_{ij} & = & z & (3)\\
\forall i<j\in V & \max(0, x_i + x_j - 1) & \le & y_{ij} & (4)\\
\forall \{i,j\}\notin E & x_i + x_j & \le & 1 & (5)\\
 & \sum_{j\in V} x_j & = & k & (6)\\
 & 0 \le x & \le & 1. & (7)
\end{array}
$$

- $z = w^2$ iff $y_{ij} = x_i x_j$, integrality nontrivial, Eq. (5)-(7) encode k-clique

QP on a simplex is NP-hard

$$\left.\begin{array}{rrcl} \min & f(x) = x^\top Q x & + & c^\top x \\ & \sum\limits_{j \le n} x_j & = & 1 \\ \forall j \le n & x_j & \ge & 0 \end{array}\right\}$$

- **Reduce MAX CLIQUE to subclass** $f(x) = -\sum\limits_{\{i,j\} \in E} x_i x_j$

  Motzkin-Straus formulation (MSF)

- **Theorem [Motzkin& Straus 1964]**

  Let C be the maximum clique of the instance $G = (V, E)$ of MAX CLIQUE

  $\exists x^* \in \mathsf{opt}\,(\mathbf{MSF}) \qquad f^* = f(x^*) = \frac{1}{2}\left(1 - \frac{1}{\omega(G)}\right)$

  $\forall j \in V \qquad x_j^* = \begin{cases} \frac{1}{\omega(G)} & \text{if } j \in C \\ 0 & \text{otherwise} \end{cases}$

Proof of the Motzkin-Straus theorem

$$x^* = \mathsf{opt}\Big(\max_{\substack{x\in[0,1]^n \\ \sum_j x_j = 1}} \sum_{ij\in E} x_i x_j\Big) \textbf{ s.t. } |C = \{j \in V \mid; x_j^* > 0\}| \textbf{ smallest } (\ddagger)$$

1. *C is a clique*
   - Suppose $1,2 \in C$ but $\{1,2\} \not\in E[C]$, then $x_1^*, x_2^* > 0$, can perturb by small $\epsilon \in [-x_1^*, x_2^*]$, get $x^\epsilon = (x_1^* + \epsilon, x_2^* - \epsilon, \ldots)$, feasible w.r.t. simplex and bounds
   - $\{1,2\} \not\in E \Rightarrow x_1 x_2$ does not appear in $f(x) \Rightarrow f(x^\epsilon)$ depends linearly on ϵ; by optimality of x^*, f achieves max for $\epsilon = 0$, in interior of its range $\Rightarrow f(\epsilon)$ constant
   - set $\epsilon = -x_1^*$ or $= x_2^*$ yields global optima with more zero components than x^*, against assumption $(\ddagger)$, hence $\{1,2\} \in E[C]$, by relabeling C is a clique

Proof of the Motzkin-Straus theorem

$$x^* = \mathsf{opt}(\max_{\substack{x\in[0,1]^n \\ \sum_j x_j = 1}} \sum_{ij\in E} x_i x_j) \text{ s.t. } |C = \{j \in V \,|; x_j^* > 0\}| \text{ smallest } (\ddagger)$$

2. $\boxed{|C| = \omega(G)}$

- square simplex constraint $\sum_j x_j = 1$, get

$$\sum_{j\in V} x_j^2 + 2\sum_{i<j\in V} x_i x_j = 1$$

- by construction $x_j^* = 0$ for $j \notin C \Rightarrow$

$$\psi(x^*) = \sum_{j\in C}(x_j^*)^2 + 2\sum_{i<j\in C} x_j^* x_j^* = \sum_{j\in C}(x_j^*)^2 + 2f(x^*) = 1$$

- $\psi(x) = 1$ for all feasible x, so $f(x)$ achieves maximum when $\sum_{j\in C}(x_j^*)^2$ is minimum, i.e. $x_j^* = \frac{1}{|C|}$ for all $j \in C$
- again by simplex constraint

$$f(x^*) = 1 - \sum_{j\in C}(x_j^*)^2 = 1 - |C|\frac{1}{|C|^2} \leq 1 - \frac{1}{\omega(G)}$$

so $f(x^*)$ attains maximum $1 - 1/\omega(G)$ when $|C| = \omega(G)$

Bilinear Programming

- **Separable BPP is NP-hard**

$$\left.\begin{array}{ll} \min & \sum\limits_{j\le n} x_j y_j \\ & Ax \ge b \\ & By \ge c \end{array}\right\} \qquad (\dagger)$$

- **Reduction from 2LS [Bennett & Mangasarian 1993]**

  geometric piecewise-linear point partition problem

- ~~(apparently still)~~ ~~Open~~ research problem: *Settled in [Matsui 1996]*

$$\left.\begin{array}{ll} \min & (c^\top x + \gamma)(d^\top x + \delta) \\ & Ax \ge b \end{array}\right\} (\ddagger)$$

Is ($\ddagger$) NP-hard? [Vavasis 1995]

- Thanks to S. Iwata for updating me on the status of this problem!

Subsection 2

Tractable cases of QP

Unconstrained QP

$$\min x^\top Qx$$

- **Check constant Hessian**
  - *If PSD, attains minimum*
  - *If $\exists$ 1 negative eigenvalue, unbounded direction*
- **Always feasible**

Trust Region Subproblems

- ℓ_2-TRS

  $$\min\{x^\top Qx + c^\top x \mid \|x\|_2 \leq 1\}$$

  can be solved in polynomial time
- Many variants [Bienstock & Michalka, 2013]
  - $+1$ *linear constraint* $a^\top x \leq b$
  - $+1$ ℓ_2 *norm constr.* $\|x - x^0\|_2 \leq r_0$
  - $+2$ *linear constraints* $a_i^\top x \leq b_i$ *(*$i \leq 2$*)*
    s.t. $(a_1^\top x - b_1)(a_2^\top x - b_2) = 0$
  - $+2$ *linear affinely parallel constraints*
  - $+m$ *linear constraints*
    (no two simultaneously binding in feas. reg.)
  - $\ldots$
- Unfortunately, not much practical use
  Applications mostly use ℓ_∞ *norm, over which QP is hard*

  ...but see [Buchheim et al. 2013 & 2015]

Piggy-backing on [HKLW 2010]

- Decidability of cvx constr. minimization in $\mathbb{Z}^n$
  $\Leftarrow$ *turns out it's bounded*
- Fixed Param. Tractable (FPT) cases of hard MINLP
- Integer convex maximization
  - *fixed number of objective fun. arguments*
  - *some fixed data (e.g. constraint matrix)*
  - *fixed number of variables*
- Integer quasicvx constrained minimization
  fixed number of variables
- *…and more*

- QP over $\mathbb{Z}^2$ is in P [Del Pia & Weismantel 2014]

The end