FRIEDRICH-ALEXANDER UNIVERSITÄT ERLANGEN-NÜRNBERG
NATURWISSENSCHAFTLICHE FAKULTÄT

Penalty Alternating Direction Methods for Mixed-Integer Optimization: A New View on Feasibility Pumps

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WIRTSCHAFTS MATHEMATIK

Feasibility Pumps for MIPs

Fischetti, Glover, Lodi (2005), Bertacco, Fischetti, Lodi (2007)

$$\begin{aligned} \min_x \quad & c^\top x \\ \text{s.t.} \quad & Ax \geq b \\ & x_i \in \mathbb{Z} \quad \text{for all } i \in I \end{aligned}$$

- Create two sequences
  - $(x^k)_k$ satisfies the inequalities,
  - $(y^k)_k$ is integral for all $i \in I$.
- Minimize distance of pairs x^k and y^k
- If you get stuck, perturb (at random).
- $x^k = y^k$: you are done.

Goal

Understand feasibility pump algorithms

- Does the procedure converge?
- Can we characterize the points to which it converges?

Specifically . . .

- What is the role of the perturbation step?
- Why randomize?

In this talk . . .

- A feasibility pump variant with no randomization for MIP and MINLP.
- Characterization of the points to which the method converges
- Observation: The standard feasibility pump only has "short" cycles (see also Dey et al. 2016)

The Basic Feasibility Pump for MIPs

Compute $x^0 \in \operatorname{argmin}\{c^\top x : x \in P\}$.

if x^0 *is integer feasible* then return x^0

$y^0 \leftarrow \lceil x^0 \rfloor$ and $k \leftarrow 0$.

while *not termination condition* do

- Compute $x^{k+1} \in \operatorname{argmin}\{\|x_I - y_I^k\|_1 : x \in P\}$.
- if x^{k+1} *is integer feasible* then return x^{k+1}
- $y^{k+1} \leftarrow \lceil x^{k+1} \rfloor$.
- if *algorithm stalls or cycles* then perturb y^{k+1}
- $k \leftarrow k + 1$

- "Stalling": $x^k = x^{k+1}$, $y^k = y^{k+1}$.
- "Cycling": $x^k = x^{k+\ell}$, $y^k = y^{k+\ell}$ for $\ell > 1$.

The *Idealized* Feasibility Pump for MIPs

Compute $x^0 \in \operatorname{argmin}\{c^\top x : x \in P\}$.

if x^0 *is integer feasible* then return x^0

$y^0 \leftarrow \lceil x^0 \rfloor$ and $k \leftarrow 0$.

while *True* do

Compute $x^{k+1} \in \operatorname{argmin}\{\|x_I - y_I^k\|_1 : x \in P\}$.

$y^{k+1} \leftarrow \lceil x^{k+1} \rfloor$.

$k \leftarrow k + 1$

How to analyze feasibility pumps?

Treat idealized FP as special case of other methods

- Frank-Wolfe method (De Santis et al. 2013, 2014, Eckstein and Nediak 2007)
- Proximal point method (Boland et al. 2012)
- Successive projection method (D'Ambrosio et al. 2012)

Change randomization step of basic FP

- Dey, Iroume, Molinaro, and Salvagnin (2016)

This talk

First approach: Interpret idealized FP as Alternating Direction Method

Mixed-Integer Nonlinear Problems

$$\begin{aligned} \min_x \quad & f(x) \\ \text{s.t.} \quad & h(x) \geq 0 \\ & x_i \in \mathbb{Z} \cap [l_i, u_i] \quad \text{for all } i \in I \end{aligned}$$

Convex MINLPs

- Bonami, Goncalves (2012): direct extension of feasibility pump for MIPs to convex MINLPs
- Bonami et al. (2009): rounding step replaced by MIP relaxation (OA) of the convex MINLP; high computational effort but “inheritance” of OA theory

Nonconvex MINLPs

- D’Ambrosio et al. (2010): first feasibility pump for nonconvex MINLPs
- Solving nonconvex projection step NLP via a multistart heuristic using local NLP solvers

Alternating Direction Methods

$$\min_{x,y} \quad f(x,y) \quad \text{s.t.} \quad x \in X, \quad y \in Y, \quad g(x,y) = 0, \quad h(x,y) \geq 0$$

Choose initial values $(x^0, y^0) \in X \times Y$.

for $k = 0, 1, \ldots$ do

Compute $x^{k+1} \in \text{argmin}_x\{f(x, y^k) : g(x, y^k) = 0,\ h(x, y^k) \geq 0, x \in X\}$

Compute

$y^{k+1} \in \text{argmin}_y\{f(x^{k+1}, y) : g(x^{k+1}, y) = 0,\ h(x^{k+1}, y) \geq 0, y \in Y\}$

ADMs: Convergence Theory

A point $(x^*, y^*) \in \Omega$ is called a partial minimum if

$$\begin{aligned} f(x^*, y^*) &\leq f(x, y^*) \quad \text{for all } (x, y^*) \in \Omega, \\ f(x^*, y^*) &\leq f(x^*, y) \quad \text{for all } (x^*, y) \in \Omega \end{aligned}$$

holds.

Theorem (see e.g. Gorski et al. 2007)

Let f, g, h be continuous, X, Y non-empty, compact, and disjoint and $(x^k, y^k)_{k=0}^{\infty}$ a sequence generated by ADM. If the solution of one optimization problem is always unique, then every convergent subsequence of $\left\{(x^k, y^k)\right\}_{k=0}^{\infty}$ converges to a partial minimum and the objective values of all these limit points are equal.

Idealized Feasibility Pumps are ADMs

- Consider the MIP

$$\begin{aligned} \min_x \quad & c^\top x \\ \text{s.t.} \quad & Ax \geq b \\ & x_i \in \mathbb{Z} \quad \text{for all } i \in I \end{aligned}$$

Idealized Feasibility Pumps are ADMs

- Consider the MIP

$$\begin{aligned} \min_{x} \quad & c^\top x \\ \text{s.t.} \quad & Ax \geq b \\ & x_i \in \mathbb{Z} \quad \text{for all } i \in I \end{aligned}$$

- Duplicate variables x_I using the new variable vector $y \in \{0, 1\}^I$

$$\begin{aligned} \min_{x,y} \quad & c^\top x \\ \text{s.t.} \quad & x \in X := \{x \in \mathbb{R}^n : Ax \geq b, x_I \in [0, 1]^I\} \\ & y \in Y := \{0, 1\}^I, \quad g(x, y) = x_I - y = 0 \end{aligned}$$

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Idealized Feasibility Pumps are ADMs

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- ℓ_1 penalization of coupling condition $y = x_I$

$$\begin{aligned} \min_{x,y} \quad & \|x_I - y\|_1 \\ \text{s.t.} \quad & x \in X := \{x \in \mathbb{R}^n : Ax \geq b, x_I \in [0, 1]^I\} \\ & y \in Y := \{0, 1\}^I \end{aligned}$$

ADM Theory for Convex MINLP Feasibility Pumps

Lemma (Geißler, Morsi, LS, Schmidt (2016))

The ADM does not cycle . . . and so does not the idealized feasibility pump.

Theorem (Geißler, Morsi, LS, Schmidt (2016))

The idealized feasibility pump for convex MINLPs is equivalent to the ADM algorithm applied to the reformulated problem above. Thus, it terminates at a partial minimum (x^*, y^*) after a finite number of iterations. If this partial minimum has objective value $\|x_I^* - y^*\|_1 = 0$, the point (x^*, y^*) is feasible for the original problem.

ADM Theory for Feasibility Pumps

Positive Case

Idealized feasibility pump converges to a MI(NL)P-feasible partial minimum of the reformulated problem.

Negative Case

Idealized feasibility pump converges to a partial minimum of the reformulated problem that is not MI(NL)P-feasible.

Random restarts can be seen as an attempt to escape MI(NL)P-infeasible partial minima

Another way to escape infeasibility ...

Penalty methods

- ℓ_1 penalty function

$$\phi_1(x, y; \mu, \rho) := f(x, y) + \sum_{i=1}^{m} \mu_i \, |g_i(x, y)| + \sum_{i=1}^{p} \rho_i [h_i(x, y)]^-$$

with

$$[\alpha]^- := \begin{cases} 0, & \text{if } \alpha \geq 0 \\ -\alpha, & \text{if } \alpha < 0 \end{cases}$$

- $\mu = (\mu_i)_{i=1}^m, \rho = (\rho_i)_{i=1}^p \geq 0$: penalty parameters
- Penalty problem

$$\min_{x,y} \quad \phi_1(x, y; \mu, \rho) \quad \text{s.t.} \quad x \in X, y \in Y$$

The ℓ_1 Penalty Alternating Direction Method

Choose initial values $(x^{0,0}, y^{0,0}) \in X \times Y$ and penalty parameters $\mu^0, \rho^0 \geq 0$

for $k = 0, 1, \ldots$ do

  $l \leftarrow 0$

  while $(x^{k,l}, y^{k,l})$ *is not a partial minimum of the penalty problem with* $\mu = \mu^k$ *and* $\rho = \rho^k$ do

    Compute $x^{k,l+1} \in \operatorname{argmin}_x \{\phi_1(x, y^{k,l}; \mu^k, \rho^k) : x \in X\}$

    Compute $y^{k,l+1} \in \operatorname{argmin}_y \{\phi_1(x^{k,l+1}, y; \mu^k, \rho^k) : y \in Y\}$

    $l \leftarrow l + 1$

  Choose new penalty parameters $\mu^{k+1} \geq \mu^k$ and $\rho^{k+1} \geq \rho^k$

Penalty ADM: Convergence Theory

Weighted ℓ_1 feasibility measure

$$\chi_{\mu,\rho}(x,y) := \sum_{i=1}^{m} \mu_i \, |g_i(x,y)| + \sum_{i=1}^{p} \rho_i [h_i(x,y)]^-$$

Theorem (Geißler, Morsi, LS, Schmidt (2016))

Suppose that the assumptions hold and that $\mu_i^k \nearrow \infty$ for all $i = 1, \dots, m$ and $\rho_i^k \nearrow \infty$ for all $i = 1, \dots, p$. Moreover, let (x^k, y^k) be a sequence of partial minima of the penalty problems (for $\mu = \mu^k$ and $\rho = \rho^k$) generated by PADM with $(x^k, y^k) \to (x^*, y^*)$. Then there exist weights $\bar{\mu}, \bar{\rho} \geq 0$ such that (x^*, y^*) is a partial minimizer of the feasibility measure $\chi_{\bar{\mu},\bar{\rho}}$.
If (x^*, y^*) is feasible for the original problem, then (x^*, y^*) is a partial minimum of the original problem.

The latter case can be improved if more regularity of the problem is assumed.

ADM-Exactness of ℓ_1 Penalty Functions

Theorem

Let (x^*, y^*) be a partial minimizer of

$$\min_{x,y} \quad f(x,y) \quad \text{s.t.} \quad g(x,y) = 0, x \in X, y \in Y, \tag{1}$$

and suppose that the Assumptions [...] hold. Then there exists a constant $\bar{\mu} > 0$ such that (x^*, y^*) is a partial minimizer of

$$\min_{x,y} \quad \phi_1(x,y;\mu) \quad \text{s.t.} \quad x \in X, y \in Y$$

for all $\mu \geq \bar{\mu}$ and

$$\phi_1(x,y;\mu) := f(x,y) + \sum_{i=1}^{m} \mu_i \left| g_i(x,y) \right| .$$

Back to Feasibility Pumps

- Take the MINLP and duplicate the integer components x_I of x

$$\min_{x,y} \quad f(x) \quad \text{s.t.} \quad h(x) \geq 0, \quad x_I = y, \quad y \in \mathbb{Z}^I \cap [l_I, u_I]$$

- The sets

$$X := \{x : h(x) \geq 0\}, \quad Y := \mathbb{Z}^I \cap [l_I, u_I]$$

  are compact
- Additional equality constraints

$$g(x, y) = x_I - y = 0$$

- Apply penalty ADM algorithm

Mixed-Integer Linear Problems

Computational Setup

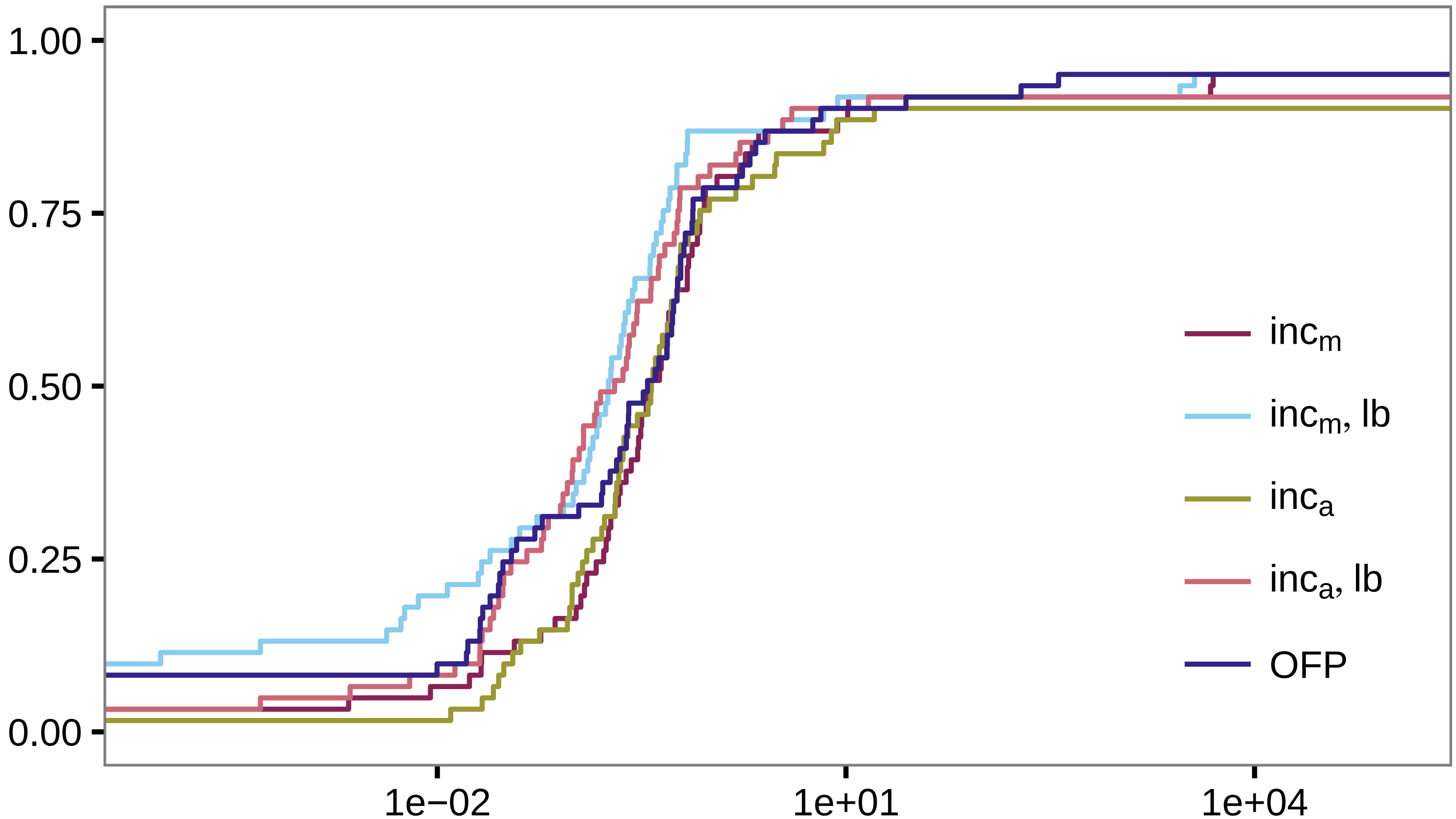
- C++ implementation; compiled with gcc 4.8.4 using flag o3
- LP solver: Gurobi 6.5.0
- Performance profiles (Dolan, Moré 2002)
  - Running time; time limit 1 h
  - Performance measure: primal-dual gap

$$\text{gap} = \frac{b_\text{p} - b_\text{d}}{\inf\{|z| : z \in [b_\text{d}, b_\text{p}]\}}$$

- Test instances: MIPLIB 2003, 2010

Mixed-Integer Linear Problems

PADM w/o local branching compared to OFP by Achterberg, Berthold

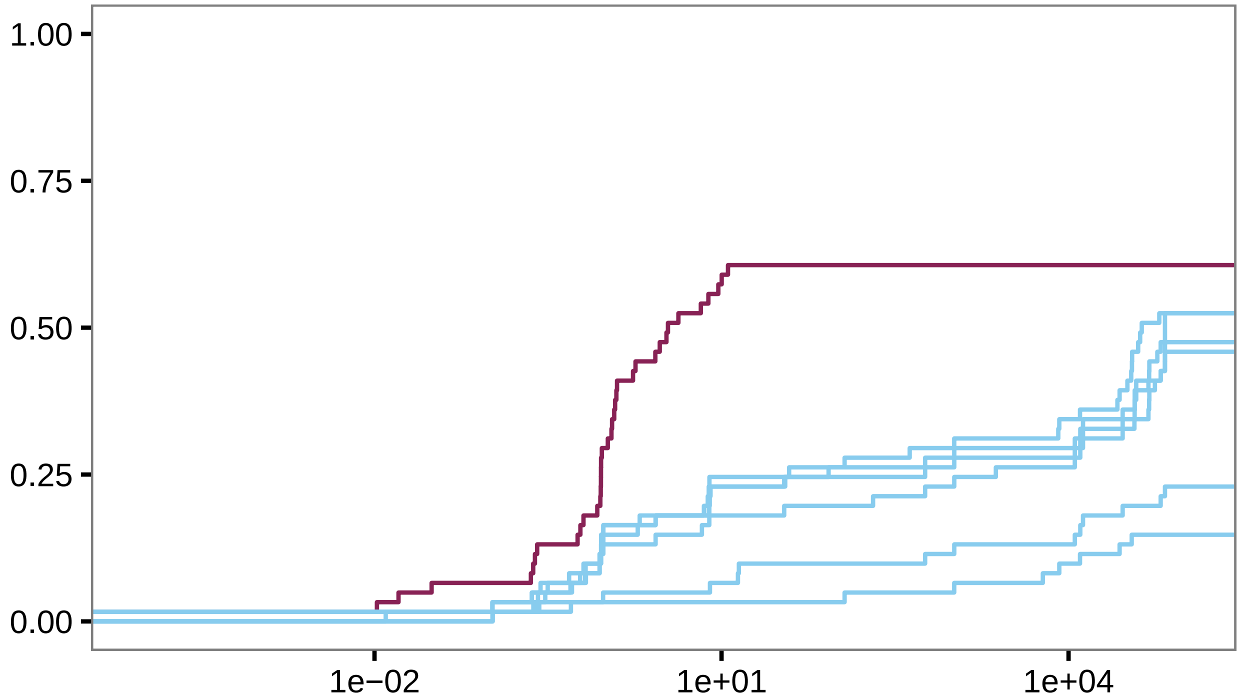


Mixed-Integer Nonlinear Problems

Computational Setup

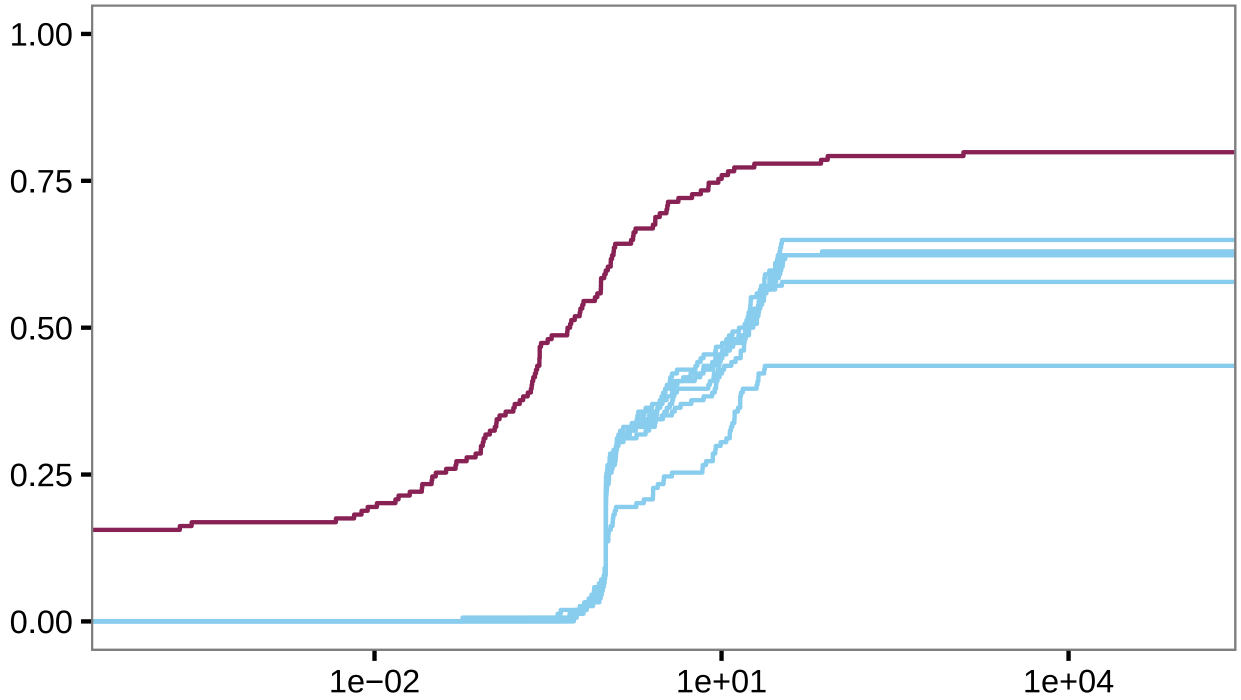
- C++ implementation using the GAMS Expert-Level API
- GAMS 24.5.4
- NLP solver: CONOPT 3.17A
- Test instances: MINLPLib and MINLPLib2

Mixed-Integer Nonlinear Problems



Red: penalty ADM based feasibility pump
Blue: six feasibility pump variants of D'Ambrosio et al. (2012)

Mixed-Integer Nonlinear Problems



Red: penalty ADM based feasibility pump
Blue: feasibility pump variants by Berthold (2014)

Summary

- Idealized feasibility pumps are alternating direction methods
- Convergence towards partial minima of a reformulated problem
- Random restarts: attempt to escape MI(NL)P-infeasible partial minima
- Random restarts $\rightarrow$ penalty framework
- New penalty ADM with convergence theory
- Very encouraging numerical results

Geißler, Morsi, Schewe, Schmidt (2016):

“Penalty Alternating Direction Methods for Mixed-Integer Optimization: A New View on Feasibility Pumps”

http://www.optimization-online.org/DB_HTML/2016/04/5399.html

Thanks!