Complete Description of Matching Polytopes with One Linearized Quadratic Term for Bipartite Graphs

Matthias Walter

Aussois 2017
Polyhedral method:

- Feasible sets: certain subsets \( F \subseteq E \) of ground set \( E \).
- Linear case: minimize \( \sum_{e \in F} c_e \) for given costs \( c : E \to \mathbb{R} \).
- Identify \( F \) with \( \chi(F) \in \{0, 1\}^E \) (with \( \chi(F)_e = 1 \iff e \in F \)) & LP.
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Quadratic objective:
- Replace objective by $\sum_{e \in F} c_e + \sum_{e,f \in F} q_{e,f} \cdot x_e \cdot x_f$.
- This yields $\sum_{e \in E} c_e x_e + \sum_{e,f \in E} q_{e,f} \cdot x_e \cdot x_f$ (not linear in $x$!)
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- Trivial: $y_{e, f} := x_e \cdot x_f \in \{0, 1\}$.
- Linearization of this product and $x_e, x_f \in [0, 1]$ yields
  
  \begin{align*}
  y_{e, f} &\leq x_e, \\
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  x_e + x_f &\leq y_{e, f} + 1 \quad \text{and} \\
  y_{e, f} &\geq 0.
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  $y_{e,f} \leq x_e$,  
  $y_{e,f} \leq x_f$,  
  $x_e + x_f \leq y_{e,f} + 1$ and  
  $y_{e,f} \geq 0$.

- Usual starting point: add all $y$-variables and the above constraints, and use LP/IP methods.
Stronger Linearizations

**Weak relaxation:** We only used *bounds* on $x$ and *one* $y$-variable.
Stronger Linearizations

Weak relaxation: We only used bounds on $x$ and one $y$-variable.

Using all $y$-variables (Buchheim, Liers & Oswald ’10):

- Inequalities valid for all $(x, y) \in \{0, 1\}^n \times \{0, 1\}^{\binom{n}{2}}$ with $y_{i,j} = x_i \cdot x_j$.
- Advantage: very general, since independent of underlying problem!
- Disadvantage: this is the CUT polytope (NP-hard)!
Stronger Linearizations

**Weak relaxation:** We only used bounds on $x$ and one $y$-variable.

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- Advantage: very general, since independent of underlying problem!
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Using all $x$-variables, and one $y$-variable (Buchheim & Klein ’13):
- Problem-specific approach, but as tractable as the underlying problem!
- We can solve problem with one quadratic term:
  1. Find optimum with $x_e = x_f = 1$ (and $y_{e,f} = 1$).
  2. Find optimum with $x_e = 0$ (and $y_{e,f} = 0$).
  3. Find optimum with $x_f = 0$ (and $y_{e,f} = 0$).
- For problems in P we can hope for complete descriptions.
One-Term-Linearization: Recent Work

2013 Buchheim & Klein: suggested approach on CTW with conjecture for spanning tree problem.

2013 Fischer & Fischer: proved conjecture.

2014 Buchheim & Klein: independent proof of conjecture, together with computational study.

Both proofs by constructing dual LP solution.

2014 Klein: investigation of quadratic assignment problem, i.e., perfect matching problem on bipartite graphs.

Two inequality classes are facet-defining, and conjectured to be sufficient (together with original matching constraints).

2015 Hupp, Klein & Liers: facet-proofs for general (nonbipartite) matching and computational study.

2016 Fischer, Fischer & McCormick: generalized to matroids, even with more monomials that are nested in a certain way.

today Bipartite matching (including a proof of Klein's conjecture).
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Agenda

1. Quadratic Combinatorial Optimization

2. Results
   - Models & Monotonizations
   - Complete Description & Validity

3. Proofs & Ideas
   - Strategy
   - Downward Monotonization
   - Upward Monotonization
For the remainder of the talk:

- $K_{m,n} = (A \cup B, E)$ is the complete bipartite graph with node partition $A \cup B$ and edges $E$.
- Define $m := |A|$, $n := |B|$ and assume $m, n \geq 2$.
- Two fixed edges $e_1 = \{a_1, b_1\}$ and $e_2 = \{a_2, b_2\}$. 
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Polytopes of interest:
The convex hulls of vectors $(\chi(M), y) \in \{0, 1\}^E \times \{0, 1\}$ for which $M$ is a matching in $K_{m,n}$ and:

- $P_{\text{match}}^{1Q}$: $y = 1$ if and only if $e_1, e_2 \in M$. 
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- $P_{\text{match}}^{1Q}$: $y = 1$ if and only if $e_1, e_2 \in M$.
- $P_{\text{match}}^{1Q\downarrow}$: $y = 1$ implies $e_1, e_2 \in M$. “downward monotonization w.r.t. $y$”
- $P_{\text{match}}^{1Q\uparrow}$: $y = 0$ implies $e_1 \not\in M$ or $e_2 \not\in M$. “upward monotonization w.r.t. $y$”
Intersecting Monotonizations

Easy to see:

\[ P_{\text{match}}^{1Q} = \text{conv}(P_{\text{match}}^{1Q\downarrow} \cap P_{\text{match}}^{1Q\uparrow} \cap \mathbb{Z}^{|E|+1}) \]
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Interesting:

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$$P^{1Q}_{\text{match}} = \text{conv}(P^{1Q}_{\downarrow \text{match}} \cap P^{1Q}_{\uparrow \text{match}} \cap \mathbb{Z}^{|E|+1})$$

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$$P^{1Q}_{\text{match}} = P^{1Q}_{\downarrow \text{match}} \cap P^{1Q}_{\uparrow \text{match}}$$

Reason (works for arbitrary convex sets):

- “$$\subseteq$$” is easy.
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- “\(\subseteq\)” is easy.
- Let \((x, y) \in P_{\text{match}}^{1Q\downarrow} \cap P_{\text{match}}^{1Q\uparrow}\).
- \((x, y) \in P_{\text{match}}^{1Q\downarrow}\): There exists \(\lambda_1 \in [0, 1]\) such that \((x, y + \lambda_1) \in P_{\text{match}}^{1Q}\).
- \((x, y) \in P_{\text{match}}^{1Q\uparrow}\): There exists \(\lambda_2 \in [0, 1]\) such that \((x, y - \lambda_2) \in P_{\text{match}}^{1Q}\).
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- \((x, y) \in P_{\text{match}}^{1Q\uparrow}\): There exists \(\lambda_2 \in [0, 1]\) such that \((x, y - \lambda_2) \in P_{\text{match}}^{1Q}\).
- Suitable convex combination of the two yields \((x, y) \in P_{\text{match}}^{1Q}\).
Q: But $P_{\text{match}}^{1Q} = P_{\text{match}}^{1Q\uparrow} \cap P_{\text{match}}^{1Q\downarrow}$ is useless if the two have more complex descriptions?
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Every facet-defining inequality of \( P_{\text{match}}^{1Q} \) is facet-defining for \( P_{\text{match}}^{1Q\downarrow} \) or for \( P_{\text{match}}^{1Q\uparrow} \).
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- The inequalities of the form $a^T x + y \leq \beta$ “come from” $P_{\text{match}}^{1Q\downarrow}$.
- The inequalities of the form $a^T x - y \leq \beta$ “come from” $P_{\text{match}}^{1Q\uparrow}$.
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This approach works for one-term-linearizations for arbitrary polytopes!
Theorem (W. ’16+):

$P_{match}^{1Q}$ is completely described by

1. $x_e \geq 0$ for all $e \in E$
2. $x(\delta(v)) \leq 1$ for all $v \in A \cup B$
3. $y \leq x_{e_i}$ for all $v \in A \cup B$
4. $x(E[S]) + y \leq \frac{|S| - 1}{2}$ for all $S \in S^\downarrow$
5. $x(E[S]) + x_{e_1} + x_{e_2} - y \leq \frac{|S|}{2}$ for all $S \in S^\uparrow$

with index sets:

$S^\downarrow := \{ S \subseteq A \cup B \mid |S| \text{ odd and } S \cap \{a_1, a_2, b_1, b_2\} \text{ is equal to } \{a_1, a_2\} \text{ or } \{b_1, b_2\} \}$

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Validity

\[ x(E[S]) + y \leq \frac{|S| - 1}{2} \quad \text{for all } S \in S^\downarrow \]
\[ x(E[S]) + x_{e_1} + x_{e_2} - y \leq \frac{|S|}{2} \quad \text{for all } S \in S^\uparrow \]
Proof Strategy

\((\hat{x}, \hat{y})\) satisfies a certain inequality with equality.
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Graph $K_{m,n}$  |  Related graph $\tilde{G}$

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$(\hat{x}, \hat{y})$ satisfies a certain inequality with equality. | construct | Vector $\bar{x}$ in certain face (♦) of $\tilde{G}$'s matching polytope.
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$(\hat{x}, \hat{y})$ satisfies a certain inequality with equality. \[ \text{construct} \]

Vector $\bar{x}$ in certain face (♦) of $\tilde{G}$'s matching polytope. \[ \text{exists} \]

Matchings $\tilde{M}_1, \ldots, \tilde{M}_k$ in $\tilde{G}$. 

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---|---
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Convex combination with special property (★) exists

Matchings $\tilde{M}_1, \ldots, \tilde{M}_k$ in $\tilde{G}$. 
Proof Strategy

\begin{itemize}
\item Graph $K_{m,n}$
\item Related graph $\tilde{G}$
\item $(\hat{x}, \hat{y})$ satisfies a certain inequality with equality. 
\item Construct Vector $\bar{x}$ in certain face ($\blacklozenge$) of $\tilde{G}$'s matching polytope.
\item Convex combination with special property ($\star$) exists
\item Matchings $\tilde{M}_1, \ldots, \tilde{M}_k$ in $\tilde{G}$.
\item $\tilde{M}_j$ have structure due to ($\blacklozenge$) and ($\star$).
\end{itemize}
Proof Strategy

Graph $K_{m,n}$

- $(\hat{x}, \hat{y})$ satisfies a certain inequality with equality.

- Matchings $\hat{M}_1, \ldots, \hat{M}_k$ in $K_{m,n}$.

Related graph $\tilde{G}$

- Construct vector $\tilde{x}$ in certain face ($\bullet$) of $\tilde{G}$'s matching polytope.

- Convex combination with special property ($\star$) exists.

- Matchings $\tilde{M}_1, \ldots, \tilde{M}_k$ in $\tilde{G}$.

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Graph $K_{m,n}$ | Related graph $\tilde{G}$

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construct | exists

Matchings $\hat{M}_1, \ldots, \hat{M}_k$ in $K_{m,n}$. | Matchings $\tilde{M}_1, \ldots, \tilde{M}_k$ in $\tilde{G}$.

$\hat{y}k$ of them contain the edges $e_1$ and $e_2$. | $\tilde{M}_j$ have structure due to ($\blacklozenge$) and ($\blackstar$).

Convex combination with special property ($\blackstar$)
Proof Strategy

Graph $K_{m,n}$  |  Related graph $\tilde{G}$

- $(\hat{x}, \hat{y})$ satisfies a certain inequality with equality.

**Construct**
- Vector $\bar{x}$ in certain face ($\circ$) of $\tilde{G}$'s matching polytope.
- Convex combination with special property ($\star$) exists.

- Barycenter of all $\chi(\hat{M}_j)$ is equal to $\hat{x}$.
- $\chi$ exists

**Prove**
- Matchings $\hat{M}_1, \ldots, \hat{M}_k$ in $K_{m,n}$.
- $\hat{y}$ of them contain the edges $e_1$ and $e_2$.

- Matchings $\tilde{M}_1, \ldots, \tilde{M}_k$ in $\tilde{G}$.
- $\tilde{M}_j$ have structure due to ($\circ$) and ($\star$).
Downward Monotonization

\[ y \leq x_{e_i} \text{ for } i = 1, 2 \quad \text{and} \quad x(E[S]) + y \leq \frac{|S| - 1}{2} \text{ for all } S \in S^{\downarrow} \]

Define graph \( \tilde{G} = (\tilde{V}, \tilde{E}) \):
- \( \tilde{V} := A \cup B \).
- \( e_A := \{a_1, a_2\}, e_B := \{b_1, b_2\} \).
- \( \tilde{E} := E \cup \{e_A, e_B\} \).

Define vector \( \tilde{x} \in \mathbb{R}^\tilde{E} \):
- \( \tilde{x}_e := \hat{x}_e \text{ for all } e \in E \setminus \{e_1, e_2\} \).
- \( \tilde{x}_{e_i} := \hat{x}_{e_i} - \hat{y} \text{ for } i = 1, 2 \).
- \( \tilde{x}_{e_a} := \tilde{x}_{e_b} := \hat{y} \).
Schrijverization

**Lemma (Schrijver ’83):**
Let $\bar{x} \in \mathbb{Q}^\bar{E}$ be in $\bar{G}$’s matching polytope. Then there exist (potentially with duplicates) matchings $\bar{M}_1, \ldots, \bar{M}_k$ in $\bar{G}$ that satisfy

$$\bar{x} = \frac{1}{k} \sum_{j=1}^{k} \chi(\bar{M}_j).$$
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$$\bar{x} = \frac{1}{k} \sum_{j=1}^{k} \chi(\bar{M}_j).$$

**Special property (★):** Number $\ell$ of $\bar{M}_j$ with $|\bar{M}_j \cap \{e_a, e_b\}| = 1$ is minimum.
Lemma (Schrijver ’83):
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Special property (★): Number $\ell$ of $\bar{M}_j$ with $|\bar{M}_j \cap \{e_a, e_b\}| = 1$ is minimum.
Claim: $\ell = 0$. 
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Claim: $\ell = 0$.

Proof by contradiction:
- Let $M_a$ be one of the matchings such that $e_a \in M_a$, but $e_b \notin M_a$.
- Let $M_b$ be one of the matchings such that $e_b \in M_b$, but $e_a \notin M_b$.
- Let $C$ be component of $M_a \Delta M_b$ that contains $e_a$. 
Schrijverization

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$$
\bar{x} = \frac{1}{k} \sum_{j=1}^{k} \chi(\bar{M}_j).
$$

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Claim: $\ell = 0$.

Proof by contradiction:

- Let $M_a$ be one of the matchings such that $e_a \in M_a$, but $e_b \notin M_a$.
- Let $M_b$ be one of the matchings such that $e_b \in M_b$, but $e_a \notin M_b$.
- Let $C$ be component of $M_a \Delta M_b$ that contains $e_a$.
- $C$ does not contain $e_b$ since remaining graph is bipartite.
- Replace $M_a$ and $M_b$ in combination by $M_a \Delta C$ and $M_b \Delta C$.
Downward Monotonization

\[ \hat{x}_{e_1} - \hat{y} \]

\[ \hat{x}_{e_2} - \hat{y} \]
Downward Monotonization

- \( \hat{y} \) of the \( \bar{M}_j \) contain both edges \( e_a \) and \( e_b \): \( \hat{M}_j := \bar{M}_j \Delta \{ e_a, e_b, e_1, e_2 \} \). These matchings contain both edges \( e_1 \) and \( e_2 \)!
Downward Monotonization

- \( \hat{y}k \) of the \( \tilde{M}_j \) contain both edges \( e_a \) and \( e_b \): \( \hat{M}_j := \tilde{M}_j \triangle \{ e_a, e_b, e_1, e_2 \} \). These matchings contain both edges \( e_1 \) and \( e_2 \)!
- The remaining \( \tilde{M}_j \) contain neither of the edges: \( \hat{M}_j := \tilde{M}_j \).
  It remains to prove that these matchings do not contain both, \( e_1 \) and \( e_2 \).
Downward Monotonization

- \( \hat{y} k \) of the \( \bar{M}_j \) contain both edges \( e_a \) and \( e_b \): \( \hat{M}_j := \bar{M}_j \Delta \{ e_a, e_b, e_1, e_2 \} \). These matchings contain both edges \( e_1 \) and \( e_2 \)!
- The remaining \( \bar{M}_j \) contain neither of the edges: \( \hat{M}_j := \bar{M}_j \).
  It remains to prove that these matchings do not contain both, \( e_1 \) and \( e_2 \).

**Case 1: Some \( y \leq x_{e_i} \)-inequality is tight.**
- Then \( \bar{x}_{e_i} = 0 \), i.e., no \( \bar{M}_j \) contains edge \( e_i \).
Downward Monotonization

- \( \hat{y}k \) of the \( \bar{M}_j \) contain both edges \( e_a \) and \( e_b \): \( \hat{M}_j := \bar{M}_j \Delta \{ e_a, e_b, e_1, e_2 \} \).
  These matchings contain both edges \( e_1 \) and \( e_2 \! 
- The remaining \( \bar{M}_j \) contain neither of the edges: \( \hat{M}_j := \bar{M}_j \).
  It remains to prove that these matchings do not contain both, \( e_1 \) and \( e_2 \).

Case 1: Some \( y \leq x_{e_i} \)-inequality is tight.
- Then \( \bar{x}_{e_i} = 0 \), i.e., no \( \bar{M}_j \) contains edge \( e_i \).

Case 2: Some \( x(E[S]) + y \leq \frac{1}{2}(|S| - 1) \)-inequality is tight.
- Then blossom inequality \( \bar{x}(E[S]) \leq \frac{1}{2}(|S| - 1) \) is tight: (•)
- At most one of the cut-edges \( e_1, e_2 \) can be in each matching!
Upward Monotonization

\[ x(E[S]) + x_{e_1} + x_{e_2} - y \leq \frac{|S|}{2} \quad \text{for all } S \in \mathcal{S}^\uparrow \]

Diagram: Graph with vertices labeled as follows:
- \(a_1\) and \(b_1\)
- \(a_2\) and \(b_2\)
- \(u\) and \(v\)

Edges and labels:
- \(\hat{x}_{e_1} - \frac{1}{2} \hat{y}\) from \(a_1\) to \(u\)
- \(1 - \hat{x}_{e_1} - \hat{x}_{e_2} + \hat{y}\) from \(a_1\) to \(u\)
- \(\hat{x}_{e_2} - \frac{1}{2} \hat{y}\) from \(a_2\) to \(v\)
- \(\frac{1}{2} \hat{y}\) from \(u\) to \(v\)
- \(\frac{1}{2} \hat{y}\) from \(v\) to \(b_2\)
- \(\frac{1}{2} \hat{y}\) from \(u\) to \(b_2\)
Upward Monotonization

\[ x(E[S]) + x_{e_1} + x_{e_2} - y \leq \frac{|S|}{2} \quad \text{for all } S \in S^\uparrow \]
Summary:

- A strengthening for linearized quadratic objective terms.
- A proof strategy that it is more geometric than by dual solutions!

Open questions:

- What happens for two or more monomials (description / proof technique)?
- Bipartite matching is a matroid intersection problem, so what about others, e.g., arborescence polytopes?
- Your questions?