POLYTOPES IN THE 0/1-CUBE WITH BOUNDED CHVÁTAL-GOMORY RANK

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Cutting-plane proofs and Chvátal-Gomory closures
Given a polytope $R = \{ x \in \mathbb{R}^n : Ax \leq b \}$ with $S = R \cap \mathbb{Z}^n$, how can we prove that a certain linear inequality is valid for $S$?

**Cutting-plane proof**

Start with $Ax \leq b$, and iteratively

- add a conic combination of previous inequalities, and
- possibly round down its right-hand side if the left-hand side has only integer coefficients.

**Gomory 1958**

Every linear inequality that is valid for $S$ can be obtained after a finite number of iterations.
Example

\[ x_1 + x_2 \leq 1, \; x_2 + x_3 \leq 1, \; x_3 + x_4 \leq 1, \; x_4 + x_5 \leq 1, \; x_1 + x_5 \leq 1 \]
\[ \Rightarrow 2x_1 + \cdots + 2x_5 \leq 5 \]
\[ \Rightarrow x_1 + \cdots + x_5 \leq 2.5 \]
\[ \Rightarrow x_1 + \cdots + x_5 \leq \lceil 2.5 \rceil = 2 \]
### Definition

Given a polytope $P \subseteq \mathbb{R}^n$, the first Chvátal-Gomory (CG) closure of $P$ is

$$P' := \{ x \in \mathbb{R}^n : c^T x \geq \left\lceil \min_{y \in P} c^T y \right\rceil \forall c \in \mathbb{Z}^n \}$$

$P^{(0)} := P$, $P^{(t)} := (P^{(t-1)})'$ is the $t$-th CG closure of $P$.

### Definition

The smallest $t$ such that $P^{(t)} = \text{conv}(P \cap \mathbb{Z}^n)$ is the CG-rank of $P$.

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**Chvátal 1973**

The CG-rank of every polytope is finite.
Fact

Let $R \subseteq \mathbb{R}^n$ be a polytope $R$ with CG-rank $k$. Then every linear inequality that is valid for $S := R \cap \mathbb{Z}^n$ has a cutting-plane proof of length at most

$$\left( n^{k+1} - 1 \right) / (n - 1).$$

Fact

Even if we fix $S = \{(0, 0), (0, 1)\} \subseteq \mathbb{R}^2$, the CG-rank of $R$ can be arbitrarily large.

... but not if $R \subseteq [0, 1]^n$!
Reverse CG-rank

**Definition**

For $S \subseteq \{0, 1\}^n$ let $\text{cgr}(S)$ denote the largest CG-rank of a polytope $R \subseteq [0, 1]^n$ with $R \cap \mathbb{Z}^n = S$.

Similar to Conforti, Del Pia, Di Summa, Faenza, Grappe (SIAM J. Discrete Math, 2015), but here we restrict to polytopes in $[0, 1]^n$.

**Eisenbrand, Schulz 2003**

Let $S \subseteq \{0, 1\}^n$. Then $\text{cgr}(S) \leq O(n^2 \log n)$.

**Rothvoß, Sanità 2013**

There exist $S \subseteq \{0, 1\}^n$ with $\text{cgr}(S) \geq \Omega(n^2)$. 
Today: What properties of $S \subseteq \{0, 1\}^n$ ensure that $\text{cgr}(S)$ is bounded by a constant?
Previous work

- \( \tilde{S} := \{0, 1\}^n \setminus S \)
- \( H[\tilde{S}] := \) undirected graph with vertices \( \tilde{S} \), two vertices are adjacent iff they differ in one coordinate

Easy

If \( H[\tilde{S}] \) is a stable set, then \( cgr(S) \leq 1 \).

Cornuéjols, Lee 2016

If \( H[\tilde{S}] \) is a forest, then \( cgr(S) \leq 3 \).

Cornuéjols, Lee 2016

If the treewidth of \( H[\tilde{S}] \) is at most 2, then \( cgr(S) \leq 4 \).
What makes cgr($S$) large?
A large pitch!

**Definition**

The pitch of $S \subseteq \{0, 1\}^n$ is the smallest number $p \in \mathbb{Z}_{\geq 0}$ such that every $p$-dimensional face of $[0, 1]^n$ intersects $S$.

(If the pitch is $p$, there is a $p - 1$-dimensional face of $[0, 1]^n$ disjoint from $S$)

**Fact**

If $S \subseteq \{0, 1\}^n$ with pitch $p$, then $\text{cgr}(S) \geq p - 1$. 
Large coefficients!

Definition

The gap of \( S \subseteq \{0, 1\}^n \) is the smallest number \( \Delta \in \mathbb{Z}_{\geq 0} \) such that \( \text{conv}(S) \) can be described by inequalities of the form

\[
\sum_{i \in I} c_i x_i + \sum_{j \in J} c_j (1 - x_j) \geq \delta
\]

with \( I, J \subseteq [n] \) disjoint, \( \delta, c_1, \ldots, c_n \in \mathbb{Z}_{\geq 0} \) with \( \delta \leq \Delta \).

Fact

If \( S \subseteq \{0, 1\}^n \) with gap \( \Delta \), then \( \text{cgr}(S) \geq \frac{\log \Delta}{\log n} - 1 \).
Proof ingredients:

**Easy**

For every $S \subseteq \{0, 1\}^n$, there exists a polytope $R \subseteq [0, 1]^n$ with $R \cap \mathbb{Z}^n = S$ such that $R$ can be described by linear inequalities with coefficients in $\{-1, 0, 1\}$.

**Lemma**

Let $P = \{x \in \mathbb{R}^n \mid Ax \geq b\}$, where $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$. Letting $P'$ denote the first CG-closure of $P$, there is a description $P' = \{x \in \mathbb{R}^n \mid Bx \geq c\}$ with $B$ and $c$ integer such that $\|B\|_\infty \leq n\|A\|_\infty$. 
Our main result

**Theorem**

If $S \subseteq \{0, 1\}^n$ with pitch $p$ and gap $\Delta$, then

$$\text{cgr}(S) \leq p + \Delta - 1.$$ 

**Corollary**

Let $S \subseteq \{0, 1\}^n$ and let $t$ be the treewidth of $H[\overline{S}]$. Then

$$\text{cgr}(S) \leq t + 2t^{t/2}.$$
Comparing to treewidth

Bounded treewidth implies bounded pitch and gap:

**Proposition**

Let $S \subseteq \{0, 1\}^n$ with pitch $p$ and gap $\Delta$. If $t$ is the treewidth of $H[\bar{S}]$, then we have $p \leq t + 1$ and $\Delta \leq 2^t t^{t/2}$.

\[ \text{contracting edges} \]

clique of size 6
Proof idea of main theorem

- induction on the rhs of the inequality to obtain

- every inequality of the form $\sum_{i \in I} x_i \geq 1$ can be obtained after $n + 1 - |I|$ rounds of CG.

- note that $n + 1 - |I| \leq p$

- $\rightsquigarrow$ all inequalities with rhs $1$ can be obtained after $p$ rounds.

- for inequalities with larger rhs, proof by example
Proof idea (2)

- suppose that $7x_1 + 3x_2 + 2x_3 \geq 5$ is valid for $S$, then also
  
  $$(7 - 1)x_1 + 3x_2 + 2x_3 \geq 4$$
  $$7x_1 + (3 - 1)x_2 + 2x_3 \geq 4$$
  $$7x_1 + 3x_2 + (2 - 1)x_3 \geq 4$$

  are valid for $S$

- induction: all obtained after $p + 4 - 1$ rounds

- thus, we also already have obtained
  $$(7 - \varepsilon)x_1 + (3 - \varepsilon)x_2 + (2 - \varepsilon)x_3 \geq 4$$

- and therefore also $7x_1 + 3x_2 + 2x_3 \geq 4 + \varepsilon'$

- rounding up the rhs, we obtain the desired inequality
Further properties of sets with bounded pitch
Proposition

For every $S \subseteq \{0, 1\}^n$ with pitch $p$ and every $c \in \mathbb{R}^n$, the problem $\min \{c^T s : s \in S\}$ can be solved using $O(n^p)$ oracle calls to $S$.

Proof:

· may assume that $c_1, \ldots, c_n \geq 0$
· note: optimal solution over $\{0, 1\}^n$ would be $\emptyset$
· claim: only need to check all vectors with support at most $p$
Bounded pitch allows for fast approximation:

**Corollary**

Let $S \subseteq \{0, 1\}^n$ with pitch $p$ and let $R$ be any relaxation of $S$. Let $\varepsilon \in (0, 1)$ with $p \varepsilon^{-1} \in \mathbb{Z}$. If

$$\sum_{i \in I} c_i x_i + \sum_{j \in J} c_j (1 - x_j) \geq \delta$$

with $\delta \geq c_1, \ldots, c_n \geq 0$ is valid for $S$, then the inequality

$$\sum_{i \in I} c_i x_i + \sum_{j \in J} c_j (1 - x_j) \geq (1 - \varepsilon)\delta$$

is valid for $R^{(p \varepsilon^{-1} - 1)}$. 