POLYTOPES IN THE 0/1-CUBE WITH BOUNDED CHVÁTAL-GOMORY RANK

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Cutting-plane proofs and Chvátal-Gomory closures

Cutting-plane proofs

Given a polytope $R = \{x \in \mathbb{R}^n : Ax \leq b\}$ with $S = R \cap \mathbb{Z}^n$, how can we prove that a certain linear inequality is valid for S?

Cutting-plane proof

Start with $Ax \leq b$, and iteratively

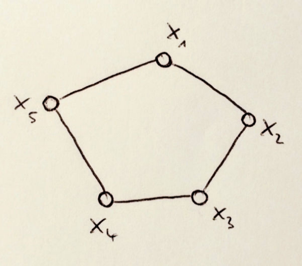
- add a conic combination of previous inequalitites, and
- possibly round down its right-hand side if the left-hand side has only integer coefficients.

Gomory 1958

Every linear inequality that is valid for S can be obtained after a finite number of iterations.

Cutting-plane proofs (2)

Example



$x_1 + x_2 \leq 1$, $x_2 + x3 \leq 1$, $x_3 + x_4 \leq 1$, $x_4 + x_5 \leq 1$, $x_1 + x_5 \leq 1$

$\Rightarrow 2x_1 + \cdots + 2x_5 \leq 5$

$\Rightarrow x_1 + \cdots + x_5 \leq 2.5$

$\Rightarrow x_1 + \cdots + x_5 \leq \lfloor 2.5 \rfloor = 2$

Chvátal-Gomory

Definition

Given a polytope $P \subseteq \mathbb{R}^n$, the first Chvátal-Gomory (CG) closure of P is

$$P' := \{x \in \mathbb{R}^n : c^\intercal x \geq \lceil \min_{y \in P} c^\intercal y \rceil \; \forall\, c \in \mathbb{Z}^n\}$$

$P^{(0)} := P$, $P^{(t)} := (P^{(t-1)})'$ is the t-th CG closure of P.

Definition

The smallest t such that $P^{(t)} = \mathrm{conv}(P \cap \mathbb{Z}^n)$ is the CG-rank of P.

Chvátal 1973

The CG-rank of every polytope is finite.

Chvátal-Gomory (2)

Fact

Let $R \subseteq \mathbb{R}^n$ be a polytope R with CG-rank k. Then every linear inequality that is valid for $S := R \cap \mathbb{Z}^n$ has a cutting-plane proof of length at most

$$(n^{k+1} - 1)/(n - 1).$$

Fact

Even if we fix $S = \{(0,0),(0,1)\} \subseteq \mathbb{R}^2$, the CG-rank of R can be arbitarily large.

... but not if $R \subseteq [0,1]^n$!

Reverse CG-rank

Definition

For $S \subseteq \{0,1\}^n$ let $\mathrm{cgr}(S)$ denote the largest CG-rank of a polytope $R \subseteq [0,1]^n$ with $R \cap \mathbb{Z}^n = S$.

Similar to *Conforti, Del Pia, Di Summa, Faenza, Grappe (SIAM J. Discrete Math, 2015)*, but here we restrict to polytopes in $[0,1]^n$

Eisenbrand, Schulz 2003

Let $S \subseteq \{0,1\}^n$. Then $\mathrm{cgr}(S) \leq \mathcal{O}(n^2 \log n)$.


Rothvoß, Sanità 2013

There exist $S \subseteq \{0,1\}^n$ with $\mathrm{cgr}(S) \geq \Omega(n^2)$.


Today: What properties of $S \subseteq \{0,1\}^n$ ensure that $\mathrm{cgr}(S)$ is bounded by a constant?

Previous work

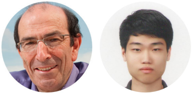
- $\bar{S} := \{0,1\}^n \setminus S$
- $H[\bar{S}] :=$ undirected graph with vertices $\bar{S}$, two vertices are adjacent iff they differ in one coordinate

Easy

If $H[\bar{S}]$ is a stable set, then $\text{cgr}(S) \leq 1$.

Cornuéjols, Lee 2016

If $H[\bar{S}]$ is a forest, then $\text{cgr}(S) \leq 3$.



Cornuéjols, Lee 2016

If the treewidth of $H[\bar{S}]$ is at most 2, then $\text{cgr}(S) \leq 4$.

What makes $\mathrm{cgr}(S)$ large?

First parameter

A large pitch!

Definition

The pitch of $S \subseteq \{0,1\}^n$ is the smallest number $p \in \mathbb{Z}_{\geq 0}$ such that every p-dimensional face of $[0,1]^n$ intersects S.

(If the pitch is p, there is a $p-1$-dimensional face of $[0,1]^n$ disjoint from S)

Fact

If $S \subseteq \{0,1\}^n$ with pitch p, then $\mathrm{cgr}(S) \geq p - 1$.

Second parameter

Large coefficients!

Definition

The gap of $S \subseteq \{0,1\}^n$ is the smallest number $\Delta \in \mathbb{Z}_{\geq 0}$ such that $\operatorname{conv}(S)$ can be described by inequalities of the form

$$\sum_{i \in I} c_i x_i + \sum_{j \in J} c_j (1 - x_j) \geq \delta$$

with $I, J \subseteq [n]$ disjoint, $\delta, c_1, \ldots, c_n \in \mathbb{Z}_{\geq 0}$ with $\delta \leq \Delta$.

Fact

If $S \subseteq \{0,1\}^n$ with gap Δ, then $\operatorname{cgr}(S) \geq \frac{\log \Delta}{\log n} - 1$.

Second parameter (2)

Proof ingredients:

Easy

For every $S \subseteq \{0,1\}^n$, there exists a polytope $R \subseteq [0,1]^n$ with $R \cap \mathbb{Z}^n = S$ such that R can be described by linear inequalities with coefficients in $\{-1,0,1\}$.

Lemma

Let $P = \{x \in \mathbb{R}^n \mid Ax \geq b\}$, where $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$. Letting P' denote the first CG-closure of P, there is a description $P' = \{x \in \mathbb{R}^n \mid Bx \geq c\}$ with B and c integer such that $||B||_\infty \leq n||A||_\infty$.

Our main result

Theorem

If $S \subseteq \{0,1\}^n$ with pitch p and gap Δ, then

$$\mathrm{cgr}(S) \leq p + \Delta - 1.$$

Corollary

Let $S \subseteq \{0,1\}^n$ and let t be the treewidth of $H[\bar{S}]$. Then

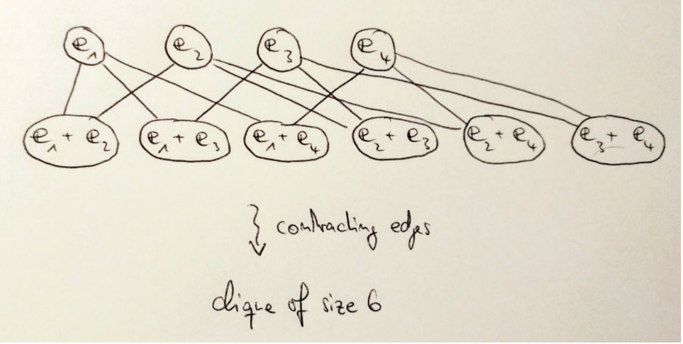
$$\mathrm{cgr}(S) \leq t + 2t^{t/2}.$$

Comparing to treewidth

Bounded treewidth implies bounded pitch and gap:

Proposition

Let $S \subseteq \{0,1\}^n$ with pitch p and gap Δ. If t is the treewidth of $H[\bar{S}]$, then we have $p \leq t+1$ and $\Delta \leq 2t^{t/2}$.



Proof idea of main theorem

- induction on the rhs of the inequality to obtain
- every inequality of the form $\sum_{i \in I} x_i \geq 1$ can be obtained after $n + 1 - |I|$ rounds of CG.
- note that $n + 1 - |I| \leq p$
- $\rightsquigarrow$ all inequalities with rhs 1 can be obtained after p rounds.
- for inequalities with larger rhs, proof by example

Proof idea (2)

- suppose that $7x_1 + 3x_2 + 2x_3 \geq 5$ is valid for S, then also

$$\begin{aligned} (7-1)x_1 + \quad 3x_2 + \quad 2x_3 &\geq 4 \\ 7x_1 + (3-1)x_2 + \quad 2x_3 &\geq 4 \\ 7x_1 + \quad 3x_2 + (2-1)x_3 &\geq 4 \end{aligned}$$

  are valid for S

- induction: all obtained after $p + 4 - 1$ rounds

- thus, we also already have obtained $(7-\varepsilon)x_1 + (3-\varepsilon)x_2 + (2-\varepsilon)x_3 \geq 4$

- and therefore also $7x_1 + 3x_2 + 2x_3 \geq 4 + \varepsilon'$

- rounding up the rhs, we obtain the desired inequality

Further properties of sets with bounded pitch

Optimizing

Proposition

For every $S \subseteq \{0,1\}^n$ with pitch p and every $c \in \mathbb{R}^n$, the problem $\min\{c^\intercal s : s \in S\}$ can be solved using $\mathcal{O}(n^p)$ oracle calls to S.

Proof:

- may assume that $c_1, \dots, c_n \geq 0$
- note: optimal solution over $\{0,1\}^n$ would be $\mathbb{O}$
- claim: only need to check all vectors with support at most p

Approximating

Bounded pitch allows for fast approximation:

Corollary

Let $S \subseteq \{0,1\}^n$ with pitch p and let R be any relaxation of S. Let $\varepsilon \in (0,1)$ with $p\varepsilon^{-1} \in \mathbb{Z}$. If

$$\sum_{i \in I} c_i x_i + \sum_{j \in J} c_j (1 - x_j) \geq \delta$$

with $\delta \geq c_1, \ldots, c_n \geq 0$ is valid for S, then the inequality

$$\sum_{i \in I} c_i x_i + \sum_{j \in J} c_j (1 - x_j) \geq (1 - \varepsilon)\delta$$

is valid for $R^{(p\varepsilon^{-1} - 1)}$.