# Symmetry Breaking Polytopes: A Framework for Symmetry Handling in Binary Programs


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joint work with Marc E. Pfetsch

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Discrete Optimization


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# Symmetry in Binary Programs



A symmetry group of a binary program

$$\max\{w^\top x \ : \ Ax \leq b, \ x \in \{0,1\}^n\}$$

is a group $\Gamma \leq \mathcal{S}_n$ such that

- $\gamma \in \Gamma$ transforms feasible solutions $x$ to feasible solutions $\gamma(x)$,
- $\gamma(x)$ and $x$ have the same objective value for each feasible $x$.

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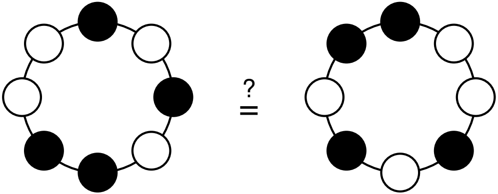

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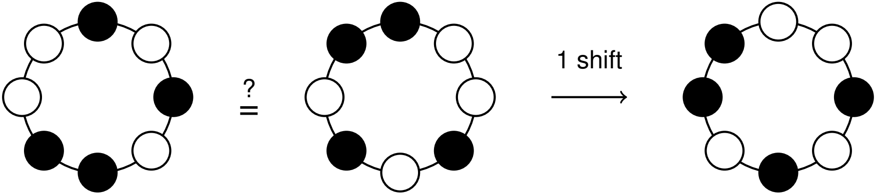

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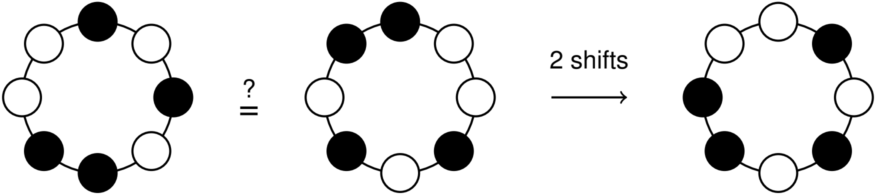




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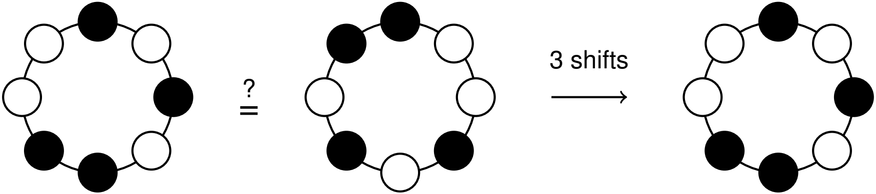

# Representative Systems

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## Idea

- Handle symmetries by computing only solution representatives.

## But

- How can we achieve this?



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# Symmetry Handling Techniques

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Examples:

- isomorphism pruning (Margot [2002])
- orbital branching (Ostrowski et al. [2011])
- symmetry handling inequalities (Liberti [2012])

# Outline

Symretopes

Symresacks

Full Orbitopes

Symmetry Handling Exploiting Problem Information
- Packing and Partitioning Constraints

Numerical Results

# Lexicographic Representative Systems

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Friedman's approach (Friedman [2007]):

- define universal ordering vector: $\bar{c} = (2^{n-1}, 2^{n-2}, \dots, 2, 1) \in \mathbb{R}^n$
- FD-inequalities: $\bar{c}^\top x \geq \bar{c}^\top \gamma(x)$
- Given permutation group $\Gamma$:
  $x \in \{0, 1\}^n$ fulfills FD-inequalities for all $\gamma \in \Gamma$ if and only if
  $x$ is lexicographically maximal in its orbit, see Friedman [2007].



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- impractical in applications
  - very large coefficients
  - separation routine unknown



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# Symretopes



**Definition**

Symmetry breaking polytope (symretope) for $\Gamma$:

$$S(\Gamma) \coloneqq \mathrm{conv}(\{x \in \{0,1\}^n \;:\; \bar{c}^\top x \geq \bar{c}^\top \gamma(x),\ \gamma \in \Gamma\}).$$

- valid inequalities $\Leftrightarrow$ symmetry handling inequalities
- facet inequalities are “strongest” symmetry handling inequalities

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Complete linear descriptions?

- available for $\Gamma = \mathcal{S}_n$, $\Gamma = \mathcal{A}_n$, and $\Gamma = \mathcal{S}_n \wr \mathcal{S}_m$
- bad news: NP-hard to optimize over $S(\Gamma)$
- goal: Find tractable IP formulations for $S(\Gamma)$.

# Symresacks



Problem with symretopes:

- possibly many defining inequalities,
- very complicated in general.

A bit simpler:

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A bit simpler:

**Definition**

Symresack for $\gamma \in \Gamma$: $\mathrm{P}_\gamma \coloneqq \mathrm{conv}(\{x \in \{0,1\}^n \ : \ \bar{c}^\top x \geq \bar{c}^\top \gamma(x)\})$.

Origin of name: Complementing binary variables yields a 0/1 knapsack problem:

$$\bar{c}^\top x \geq \bar{c}^\top \gamma(x) \Leftrightarrow \sum_{i=1}^{n} (\bar{c}_i x_{\gamma^{-1}(i)} - \bar{c}_i x_i) \leq 0 \Leftrightarrow \sum_{i=1}^{n} (2^{n-\gamma(i)} - 2^{n-i}) x_i \leq 0.$$

# Results on Symresacks

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**Theorem, H., Pfetsch [2017]**

The linear optimization problem over symresacks $P_\gamma$ can be solved in $O(n\,\alpha(n))$ time, where $\alpha$ is the inverse Ackermann function.

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Consequences:

- intersecting IP formulation of $P_\gamma$ for all $\gamma \in \Gamma$ and
- separating valid inequalities for $P_\gamma$ yields
- IP formulation for $S(\Gamma)$.



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# Cover Inequalities



Still one problem: The coefficients of the defining inequality for $\mathrm{P}_\gamma$ can be exponentially large $\to$ numerical problems.

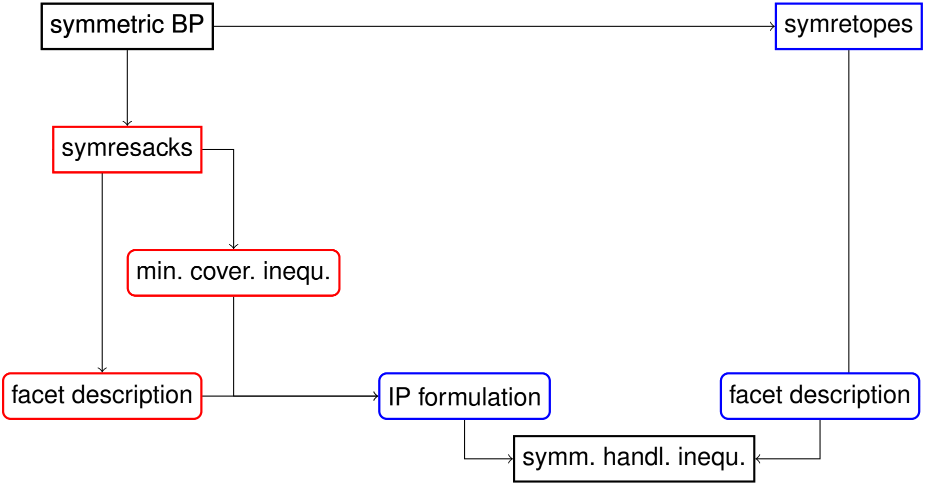
Possible remedy:

**Theorem, H., Pfetsch [2017]**

The separation problem of minimal cover inequalities for $\mathrm{P}_\gamma$ can be solved in $O(n\,\alpha(n))$ time.

- IP formulation of $\mathrm{P}_\gamma$ with coefficients in $\{0, \pm 1\}$
- Corresponding IP formulation for $\mathrm{S}(\Gamma)$ can be separated in $O(|\Gamma|\, n\,\alpha(n))$ time.

# The Symretope Framework



# Outline

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# Full Orbitopes

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**Definition**

Full Orbitope $O_{m,n}(\Gamma)$ for $\Gamma \leq \mathcal{S}_n$:
convex hull of matrices $X \in \{0, 1\}^{m \times n}$ whose columns are sorted lex. max. w.r.t. permutations from $\Gamma$.

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Examples:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \xrightarrow[\text{group}]{\text{cyclic}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \xrightarrow[\text{group}]{\text{symmetric}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

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Applications:

- cyclic group: periodic timetabling
- symmetric group: graph coloring, assignment problems



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# Full Orbitope $O_{m,n}(\mathcal{S}_n)$ – Properties

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- linear objective can be maximized in $O(m^2 n)$ time, Kaibel and Pfetsch [2008]
- there exists extended formulation of polynomial size, Kaibel and Loos [2010]
- complete linear description
  - open for $n \geq 3$
  - available for $n = 2 \rightsquigarrow$ orbisack $O_m \coloneqq O_{m,2}(\mathcal{S}_2)$

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**Theorem, Kaibel and Loos [2011]**

The orbisack $O_m$ is described completely by $\Theta(3^m)$ inequalities whose largest coefficient is in $\Theta(2^m)$.

# Orbisacks and Orbitopes

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## Orbisack $\mathrm{O}_m$

- special case of symresack
- $X = (X^1, X^2) \in \mathrm{O}_m$ iff $X^1 \succeq X^2$

## Orbitope $\mathrm{O}_{m,n}(\mathcal{S}_n)$

- special case of symretope
- $X = (X^1, \dots, X^n) \in \mathrm{O}_{m,n}(\mathcal{S}_n)$ iff $X^1 \succeq X^2 \succeq \cdots \succeq X^{n-1} \succeq X^n$

**Theorem, H., Pfetsch [2017]**

Let $\mathrm{O}_m^j$ be the orbisack for $(X^j, X^{j+1})$ and let $S^j$ be an IP formulation of $\mathrm{O}_m^j$. Then,

- $\bigcap_{j=1}^{n-1} S^j$ is an IP formulation of $\mathrm{O}_{m,n}(\mathcal{S}_n)$.
- there exists an IP formulation with $\{0, \pm 1\}$-coefficients of $\mathrm{O}_{m,n}(\mathcal{S}_n)$ that can be separated in $O(mn)$ time.
- every non-trivial facet of $\mathrm{O}_m^j$ can be trivially lifted to a facet of $\mathrm{O}_{m,n}(\mathcal{S}_n)$.

# Packing and Partitioning Orbitopes

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## Packing and Partitioning Orbitope

Convex hull of $X \in \mathrm{O}_{m,n}(\mathcal{S}_n)$ with

$$\sum_{j=1}^{n} X_{ij} \begin{cases} \leq 1, \\ = 1. \end{cases}$$

- incorporate structural properties into orbitopes
- facet description available, Kaibel and Pfetsch [2008]
- $\rightsquigarrow$ “strongest” symmetry handling inequalities using additional structure are known

# Outline

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# Constrained Symresacks

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Let $\zeta_1 \circ \cdots \circ \zeta_m$ be the disjoint cycle decomposition of $\gamma$ and let $Z_\ell = \operatorname{supp}(\zeta_\ell)$.

**Definition Kaibel and Pfetsch [2008]**

Packing Symresack:

$$\mathrm{P}_\gamma^{\leq} \coloneqq \operatorname{conv}\left(\left\{x \in \mathrm{P}_\gamma \cap \{0,1\}^n \;:\; \sum_{i \in Z_\ell} x_i \leq 1,\ \ell \in [m]\right\}\right)$$

Partitioning Symresack:

$$\mathrm{P}_\gamma^{=} \coloneqq \operatorname{conv}\left(\left\{x \in \mathrm{P}_\gamma \cap \{0,1\}^n \;:\; \sum_{i \in Z_\ell} x_i = 1,\ \ell \in [m]\right\}\right)$$

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Good news: Optimization problem over $\mathrm{P}_\gamma^{\leq}$ and $\mathrm{P}_\gamma^{=}$ can be solved in $O(n^2)$ time.

Question: Can we find a better IP formulation than the formulation via minimal cover inequalities?

# Vertices of $P_\gamma^{\leq}$ and $P_\gamma^{=}$

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consider $\gamma = (1, 7, 3, 6)(2, 9, 5)(4, 8)$

| $x$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| $\gamma(x)$ | 6 | 5 | 7 | 8 | 9 | 3 | 1 | 4 | 2 |

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# Vertices of $P^{\leq}_{\gamma}$ and $P^{=}_{\gamma}$

consider $\gamma = (1, 7, 3, 6)(2, 9, 5)(4, 8)$

| $x$ | 1 | 2 | (3) | 4 | (5) | 6 | 7 | (8) | 9 |
|---|---|---|---|---|---|---|---|---|---|
| $\gamma(x)$ | 6 | (5) | 7 | (8) | 9 | (3) | 1 | 4 | 2 |

descent points: $\mathcal{D} \coloneqq \{i \in [n] \,:\, \gamma(i) < i\}$, ascent points: $\mathcal{A} \coloneqq \{i \in [n] \,:\, \gamma(i) \geq i\}$

Reason for infeasibility:

- there exists descent point $j$ with $x_j = 1$ and
- there is no ascent point $i < \gamma(j)$ with $x_i = 1$.

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Reason for infeasibility:

- there exists descent point $j$ with $x_j = 1$ and
- there is no ascent point $i < \gamma(j)$ with $x_i = 1$.

$$\rightsquigarrow x_j \leq \sum_{i \in \mathcal{A}\,:\, i < \gamma(j)} x_i, \qquad j \in \mathcal{D}$$



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# IP formulation for $P^{\leq}_{\gamma}$ and $P^{=}_{\gamma}$

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**IP formulation for $P^{\leq}_{\gamma}$ and $P^{=}_{\gamma}$**

- packing/partitioning constraints
- ordering constraints: $x_j \leq \sum_{i \in \mathcal{A}\colon i < \gamma(j)} x_i$, $j \in \mathcal{D}$,
- non-negativity constraints

Consequences:

- There exists IP formulation with coefficients in $\{0, \pm 1\}$.
- In contrast to minimal cover inequalities: IP formulation has linear size.

Open question: complete linear description?

# Monotone Permutations

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**Definition**

- monotone cycle $\zeta$: contains exactly one descent point
- monotone permutation $\gamma = \zeta_1 \circ \cdots \circ \zeta_m$: all cycles are monotone

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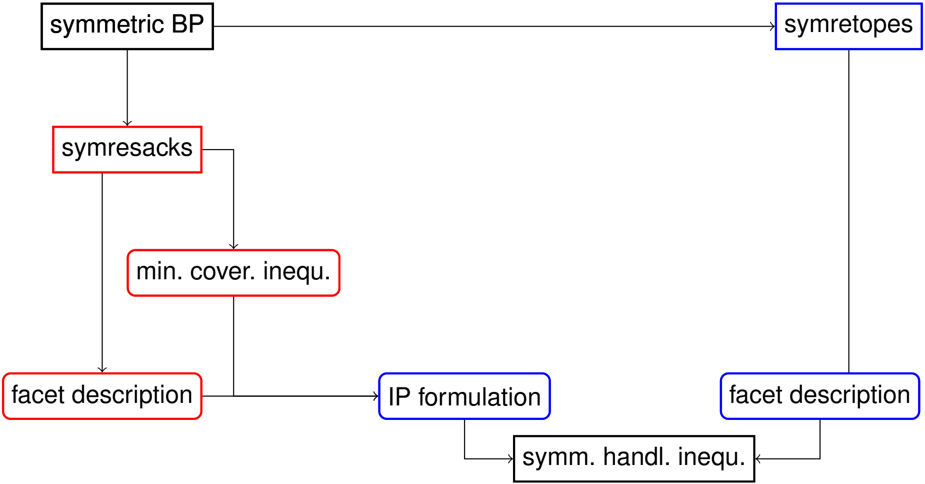
**Theorem, H. [2017]**

Complete linear description of $P^{\leq}_{\gamma}$ and $P^{=}_{\gamma}$ for monotone permutations by packing/partitioning constraints, ordering constraints, and non-negativity constraints.

# The Symretope Framework





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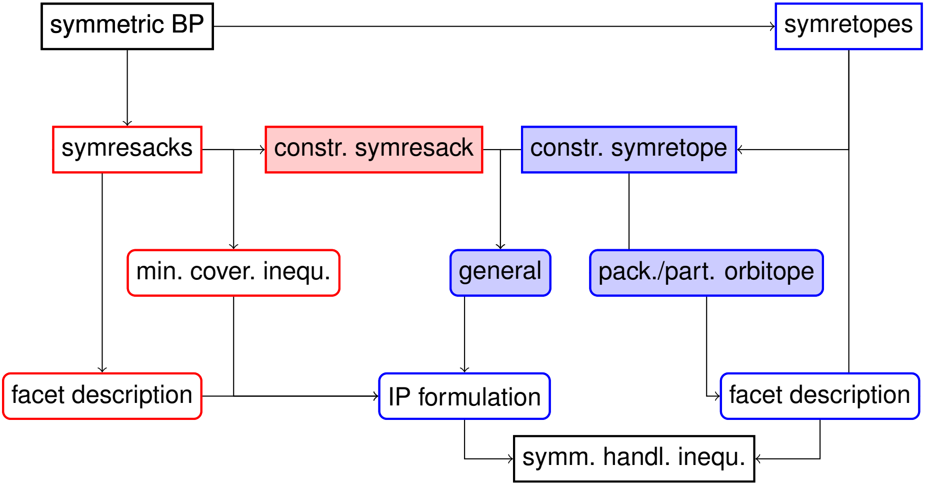



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Full Orbitopes

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# Implementation and Setup

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- implemented plug-ins for SCIP 5.0
  - symresack
    - minimal cover inequalities
    - propagation
  - orbisack
    - minimal cover inequalities and facet inequalities
    - propagation
- use CPLEX 12.7.1 as LP solver
- time limit 3600s

# Results Benchmark Instances

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| Setting | time | speed-up | #opt |
|---|---|---|---|
| `M2010-bench` (87): | | | |
| default | 341.02 | 1.00 | 75 |
| **orbital fixing** | 296.72 | **0.87** | **77** |
| symresack | 307.14 | 0.90 | 75 |
| pp-symresack | 309.86 | 0.91 | 75 |
| `Margot` (15): | | | |
| default | 1085.46 | 1.00 | 6 |
| orbital fixing | 330.53 | 0.30 | 10 |
| symresack | 91.59 | 0.08 | **13** |
| **pp-symresack** | **80.12** | **0.07** | **13** |



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# Results Highly Symmetric Instances

| Setting | time | speed-up | #opt |
|---|---|---|---|
| `Tennis` (10): | | | |
| default | 34.9 | 1.00 | 9 |
| **orbital fixing** | **7.7** | **0.22** | **10** |
| symresack | 17.7 | 0.51 | **10** |
| pp-symresack | 16.8 | 0.48 | **10** |
| `WB` (120): | | | |
| default | 1687.59 | 1.00 | 30 |
| orbital fixing | 523.23 | 0.31 | 61 |
| **symresack** | **12.83** | **0.01** | **120** |
| pp-symresack | 13.01 | 0.01 | **120** |
| `Graph Coloring` (55): | | | |
| default | 574.26 | 1.00 | 20 |
| symresack | 540.26 | 0.94 | 21 |
| **pp-symresack** | **467.03** | **0.81** | **24** |

Experiments for graph coloring use a specialized graph coloring code.

# Conclusion and Outlook

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## Conclusion

- analysis of symretopes allows to derive symmetry handling inequalities
- symmetry handling is possible with inequalities with small coefficients
- incorporating problem information leads to smaller and tighter IP formulations

## Future Work

- complete linear descriptions of general (packing/partitioning) symresacks?
- incorporation of different problem information, e.g., covering constraints

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Thank you for your attention!

# Literature

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