Generalizing the Kawaguchi-Kyan Bound to Stochastic Parallel Machine Scheduling

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Problem $P\|\sum w_j C_j$

Given: Weights $w_j \geq 0$ and processing times $p_j \geq 0$ of jobs $j = 1, \ldots, n$ and number $m$ of machines.

Task: Process each job nonpreemptively for $p_j$ time units on one of the $m$ machines such that the total weighted completion time $\sum_{j=1}^n w_j C_j$ is minimized.
The WSPT Rule

**WSPT rule**

Whenever a machine becomes idle, start the available job with largest ratio $w_j/p_j$ on it.

The WSPT rule is optimal for a single machine (Smith (1956)) and for unit weights (Conway, Maxwell, Miller (1967)).
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Theorem (Kawaguchi, Kyan (1986))
The WSPT rule is a $\frac{1}{2}(1 + \sqrt{2})$-approximation, and this bound is tight.
Problem $P|p_j \sim \text{stoch}| \mathbb{E}\left[\sum w_j C_j\right]$ 

Given: Weights $w_j \geq 0$ and distributions of independent random processing times $p_j \geq 0$ of jobs $j = 1, \ldots, n$ and number $m$ of machines.

Task: Find a nonpreemptive scheduling policy $\Pi$ for $m$ identical parallel machines such that the expected weighted sum of completion times is minimized.
Problem \( P|p_j \sim \text{stoch}| E[\sum w_j C_j] \)

Given: Weights \( w_j \geq 0 \) and distributions of independent random processing times \( p_j \geq 0 \) of jobs \( j = 1, \ldots, n \) and number \( m \) of machines.

Task: Find a nonpreemptive scheduling policy \( \Pi \) for \( m \) identical parallel machines such that the expected weighted sum of completion times is minimized.

A policy must be nonanticipative, i.e. a decision made at time \( t \) may only depend on the information known at time \( t \).
The WSEPT Rule

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Whenever a machine becomes idle, start the available job with largest ratio $w_j / E[p_j]$ on it.
Known Results

- WSEPT has no constant performance guarantee (even for unit weights). (Cheung et al. (2014), Im, Moseley, Pruhs (2015))
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+ WSEPT is optimal if
  - there is only one machine (Rothkopf (1966)),
  - all jobs have unit weight and processing times are pairwise stochastically comparable (Weber, Varaiya, Walrand (1986)).
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  - all jobs have unit weight and processing times are pairwise stochastically comparable (Weber, Varaiya, Walrand (1986)).

+ If \( \frac{\text{Var}[p_j]}{E[p_j]^2} \leq \Delta \) for all \( j \), then WSEPT has performance guarantee

\[
1 + \frac{(m - 1)}{2m} \cdot (1 + \Delta) \leq 1 + \frac{1}{2} \cdot (1 + \Delta).
\]

(Möhring, Schulz, Uetz (1999))
Performance Guarantees

\[ \frac{1}{2} (1 + \sqrt{2}) = \frac{7}{6} \]

\[ 1 + \frac{1}{2} (1 + \Delta) \]
Performance Guarantees

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Performance Guarantees

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\[ \frac{1}{2}(1 + \sqrt{2}) \]

this talk: \[ 1 + \frac{1}{2}(\sqrt{2} - 1)(1 + \Delta) \]
Auxiliary Objective Function

Given: Smith ratios $\rho_j$ and distributions of independent random processing times $p_j \geq 0$ of jobs $j = 1, \ldots, n$ and number $m$ of machines.

Task: Find a nonpreemptive scheduling policy for $m$ identical parallel machines such that the expected weighted sum of completion times is minimized, where each job is weighted with its Smith ratio times its actual processing time.
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- The weight of a job is a random variable $w_j = \rho_j p_j$.
- The Smith ratio $\rho_j$ of a job is deterministic.
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Remark
List scheduling the jobs in nonincreasing order of their Smith ratios $\rho_j$ is a $\frac{1}{2}(1 + \sqrt{2})$-approximation for the auxiliary objective function.
Proof of WSEPT’s Performance Guarantee

Claim

The WSEPT rule is a $1 + \frac{1}{2}(\sqrt{2} - 1) \cdot (1 + \Delta)$-approximation for $P|p_j \sim \text{stoch}| E[\sum w_j C_j]$. 
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Consider auxiliary objective function with weight factors $\rho_j := w_j / E[p_j]$.

Then, for every policy $\Pi$:

$$\text{Obj}(\Pi) = \sum_{j=1}^{n} \rho_j E[p_j] E[C_{j}^{\Pi}]$$

original objective function value
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$$E[p_j C_j^\Pi] = E[p_j (S_j^\Pi + p_j)] = E[p_j S_j^\Pi] + E[p_j^2]$$
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nonanticipativity
Proof of WSEPT’s Performance Guarantee

Hence,

\[ \text{Obj}^{'}(\Pi) = \text{Obj}(\Pi) + \sum_{j=1}^{n} \rho_j \text{Var}[p_j] \]

\[ \leq: c \]
Proof of WSEPT’s Performance Guarantee

Hence,

$$\text{Obj}'(\Pi) = \text{Obj}(\Pi) + \sum_{j=1}^{n} \rho_j \text{Var}[p_j] \leq \text{Obj}(\Pi) + \sum_{j=1}^{n} \Delta w_j \text{E}[p_j].$$

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\]

\[
\text{OPT} \quad \text{WSEPT} \quad \text{OPT}' \quad \text{WSEPT}'
\]

\[
WSEPT \quad c \quad c 
\]

\[
\leq \frac{1}{2} (1 + \sqrt{2})
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\text{WSEPT} = \text{WSEPT}' - c
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\[ WSEPT = WSEPT' - c \leq \frac{1}{2} (1 + \sqrt{2}) \text{OPT}' - c \]

\[ = \frac{1}{2} (1 + \sqrt{2}) (\text{OPT} + c) - c \]
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Hence,

\[ \text{Obj}'(\Pi) = \text{Obj}(\Pi) + \sum_{j=1}^{n} \rho_j \text{Var}[p_j] \leq \text{Obj}(\Pi) + \sum_{j=1}^{n} \Delta w_j \text{E}[p_j] \leq c \leq \text{Obj}'(\Pi) + \sum_{j=1}^{n} \Delta w_j \text{E}[p_j] \leq \Delta \text{OPT} \]

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Hence,

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\[
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\text{OPT} \quad \text{WSEPT} \quad \text{OPT}' \quad \text{WSEPT}'
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\]

\[
c \leq \Delta \text{OPT} \leq (1 + \frac{1}{2}(\sqrt{2} - 1)(1 + \Delta)) \text{OPT}
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Remarks

- Considering $\alpha$-points instead of completion times reduces the constant $c$, and thus yields the better performance guarantee.
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- The derived performance guarantee is the best known performance ratio of any algorithm for $P|p_j \sim \text{stoch}| \sum w_j \mathbf{C}_j$.

For exponentially distributed processing times, WSEPT’s approximation ratio lies in $[1, \frac{4}{3}]$ (lower bound by Jagtenberg, Schwiegelshohn, Uetz (2013)). Even in this special case no better approximation is known.

The performance guarantee can be refined for fixed numbers of machines.
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- For exponentially distributed processing times, WSEPT’s approximation ratio lies in $[1.243, 4/3]$ (lower bound by Jagtenberg, Schwiegelshohn, Uetz (2013)). Even in this special case no better approximation is known.
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Thank you!
Literature

- W. C. Cheung, F. Fischer, J. Matuschke, and N. Megow: *A \(\Omega(\Delta^{1/2})\) gap example for the WSEPT policy*, cited as personal communication on an exercise sheet by Marc Uetz from the MDS Autumn School 2014