Colorful complete bipartite subgraphs in generalized Kneser graphs

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Any proper 3-coloring of the Petersen graph contains a \( C_6 \) colored cyclically with the 3 colors.
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Plan

- Chen’s theorem
- Generalization of Chen’s theorem
- Proof techniques and lemmas
- Applications and open questions
Chen’s theorem
The Petersen graph is also the graph with

\[ V = \binom{[5]}{2} \]

\[ E = \left\{ XY \in \binom{V}{2} : X \cap Y = \emptyset \right\} \]
Kneser graphs

The Petersen graph is the Kneser graph $KG(5, 2)$.

$KG(n, k)$ is the Kneser graph with

\[ V = \binom{[n]}{k} \]
\[ E = \left\{ XY \in \binom{V}{2} : X \cap Y = \emptyset \right\} \]

**Theorem (Lovász 1979)**

$\chi(KG(n, k)) = n - 2k + 2.$
Chen’s theorem

Theorem (Chen 2012)

Any proper coloring of $KG(n, k)$ with a minimum number of colors contains a $K_{n-2k+2,n-2k+2}^*$ with all colors on each side.

$K_{t,t}^* = K_{t,t}$ minus a perfect matching.

Petersen graph: $K_{3,3}^* = C_6$ and there always exists a
Any proper 4-coloring of $KG(6, 2)$ contains a $K^*_4$, $K_4$ with all 4 colors on each side.
Any proper 4-coloring of $\text{KG}(6, 2)$ contains a $K_{4,4}^{*}$ with all 4 colors on each side.
Generalization of Chen’s theorem
Generalized Kneser graphs

Let $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ be a hypergraph.

$KG(\mathcal{H})$ is the generalized Kneser graph with

$$
\begin{align*}
V &= E(\mathcal{H}) \\
E &= \left\{ ef \in \binom{V}{2} : e \cap f = \emptyset \right\}
\end{align*}
$$

$KG(n, k)$ obtained with $\mathcal{H} = \text{complete } k\text{-uniform hypergraph on } n \text{ vertices.}$

Every simple graph is a generalized Kneser graph.
Dol’nikov’s theorem

Hypergraph $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$. \textbf{2-colorability defect} of $\mathcal{H}$:

$$\text{cd}_2(\mathcal{H}) = \left( \text{minimum number of vertices to remove so that the remaining hypergraph is 2-colorable} \right)$$

$$\text{cd}_2(\mathcal{H}) = \min |X| \text{ s.t. } (V(\mathcal{H}) \setminus X, \{e \in E(\mathcal{H}) : e \cap X = \emptyset \}) \text{ is 2-colorable}$$

\textbf{Theorem (Dol’nikov 1993)}

$$\chi(KG(\mathcal{H})) \geq \text{cd}_2(\mathcal{H}).$$
Examples

When $\mathcal{H}$ is the $k$-uniform complete hypergraph on $n$ vertices: $\text{cd}_2(\mathcal{H}) = n - 2k + 2$.

When $\mathcal{H}$ is a graph: $\text{cd}_2(\mathcal{H}) =$ minimum of vertices to remove so that we get a bipartite graph.

The graph $\mathcal{H}$ shown in the diagram has $\chi(\text{KG}(\mathcal{H})) = 4$ and $\text{cd}_2(\mathcal{H}) = 2$. 
Theorem (Alishahi-Hajiabolhassan-M. 2017)

Let $\mathcal{H}$ be a hypergraph with no singleton. If $\chi(KG(\mathcal{H})) = \text{cd}_2(\mathcal{H})$, then any proper coloring of $KG(\mathcal{H})$ with a minimum number of colors contains a $K^*_{\text{cd}_2(\mathcal{H}),\text{cd}_2(\mathcal{H})}$ with all colors on each side.

Example: $\chi(KG(\mathcal{H})) = \text{cd}_2(\mathcal{H}) = 4$. 
Proof techniques and lemmas
Techniques

- Combinatorics
- Topological combinatorics
Case $\text{cd}_2(\mathcal{H}) = 1$

Let $\mathcal{H}$ be a hypergraph with no singleton. If $\chi(KG(\mathcal{H})) = \text{cd}_2(\mathcal{H}) = 1$, then any proper coloring of $KG(\mathcal{H})$ with a minimum number of colors contains a monochromatic $K^*_{1,1}$.
Let $\mathcal{H}$ be a hypergraph with no singleton. If $\chi(KG(\mathcal{H})) = cd_2(\mathcal{H}) = 1$, then $KG(\mathcal{H})$ has two non-adjacent vertices.
Let $\mathcal{H}$ be a hypergraph with no singleton.
If $\chi(KG(\mathcal{H})) = cd_2(\mathcal{H}) = 1$, then $KG(\mathcal{H})$ has two non-adjacent vertices.

- $\chi(KG(\mathcal{H})) = 1$ means that any two edges of $\mathcal{H}$ intersect.
- $cd_2(\mathcal{H}) = 1$ implies that there are at least two edges.
Case $\text{cd}_2(\mathcal{H}) = 2$

Let $\mathcal{H}$ be a hypergraph with no singleton.

If $\chi(\text{KG}(\mathcal{H})) = \text{cd}_2(\mathcal{H}) = 2$, then any proper coloring of $\text{KG}(\mathcal{H})$ with a minimum number of colors contains a monochromatic $K_{2,2}$.
Case $cd_2(H) = 2$

Let $H$ be a hypergraph with no singleton. If $\chi(KG(H)) = cd_2(H) = 2$, then $KG(H)$ has two disjoint edges.
Case \( \text{cd}_2(\mathcal{H}) = 2 \)

Let \( \mathcal{H} \) be a hypergraph with no singleton. If \( \chi(KG(\mathcal{H})) = \text{cd}_2(\mathcal{H}) = 2 \), then \( KG(\mathcal{H}) \) has two disjoint edges.
The topological method

The topological method in a nutshell

\[ \exists \text{ proper coloring } c \text{ of } G = (V, E) \text{ with } t \text{ colors} \]
\[ \implies \exists \ Z_2\text{-complex } L(G) \text{ and } Z_2\text{-equivariant map } \phi : L(G) \to S^{f(t)}. \]

Obstruction (e.g., the Borsuk-Ulam theorem) ⇒ lower bound on \( t \).

Proof (Ziegler 2001, Matoušek 2003) of Dol’nikov’s theorem \( \chi(KG(\mathcal{H})) \geq \text{cd}_2(\mathcal{H}) \):

\[ \exists \text{ simplicial } Z_2\text{-map } \phi : \text{sd } Z_2^n \to Z_2^{*(n-\text{cd}_2(\mathcal{H})+t)} \]

where \( n = |V(\mathcal{H})| \), conclude with Tucker’s lemma:

\[ n \leq n - \text{cd}_2(\mathcal{H}) + t. \]
\[ x \in \{+, -, 0\}^n \setminus \{0\} \mapsto \phi(x) \in \{ \pm 1, \pm 2, \ldots, \pm (n - \text{cd}_2(\mathcal{H}) + t)\} \]

\[ x^+ = \{i \in [n]: x_i = +\} \quad \text{and} \quad x^- = \{i \in [n]: x_i = -\} \]

\[ \phi(x) = \begin{cases} 
\pm (n - \text{cd}_2(\mathcal{H}) + \max c(S)) & \text{for } S \in E(\mathcal{H}) \text{ and } (S \subseteq x^+ \text{ or } S \subseteq x^-) \\
\pm (|x^+| + |x^-|) & \text{if such } S \text{ does not exist.}
\end{cases} \]
Fan’s lemma

Replace Tucker’s lemma by

Lemma (Fan’s lemma)

Let $T$ be a centrally symmetric triangulation of a $d$-sphere. For every simplicial $\mathbb{Z}_2$-map $\phi: T \to \mathbb{Z}_2^{*-\infty}$, there exists an alternating $d$-simplex.

An alternating simplex has an ordering of its vertices $v_0, \ldots, v_d$ s.t.

$0 < +\phi(v_0) < -\phi(v_1) < +\phi(v_2) < \cdots < (-1)^d \phi(v_d)$.

Theorem (Fan 1982, Simonyi-Tardos 2006)

There exists a colorful bipartite complete subgraph $K_{\lfloor \text{cd}_2(\mathcal{H})/2 \rfloor, \lceil \text{cd}_2(\mathcal{H})/2 \rceil}$ in any proper coloring of $KG(\mathcal{H})$.

Strengthening for graphs with $\chi(KG(\mathcal{H})) = \text{cd}_2(\mathcal{H})$ (Spencer-Su 2005, Simonyi-Tardos 2007).
Chen’s lemma

Replace Fan’s lemma by

Lemma (Chen 2012)

Consider an order-preserving \( \mathbb{Z}_2 \)-map \( \phi : \{+, -, 0\}^n \setminus \{0\} \rightarrow \{\pm 1, \ldots, \pm n\} \). Suppose moreover that there is a \( \gamma \in [n] \) such that when \( x \prec y \), at most one of \( |\phi(x)| \) and \( |\phi(y)| \) is equal to \( \gamma \). Then there are two chains

\[
x_1 \preceq \cdots \preceq x_n \quad \text{and} \quad y_1 \preceq \cdots \preceq y_n
\]

such that

\[
\phi(x_i) = (-1)^i i \quad \text{for all } i \quad \text{and} \quad \phi(y_i) = (-1)^i i \quad \text{for } i \neq \gamma
\]

and such that \( x_\gamma = -y_\gamma \).

Proved with the help of Fan’s lemma.
Applications and open questions
Circular chromatic number

Graph $G = (V, E)$

$(p, q)$-coloring: $c : V \to [p]$ such that $q \leq |c(u) - c(v)| \leq p - q$ when $uv \in E$.

Circular chromatic number: $\chi_c(G) = \inf \{p/q : \exists (p, q)\text{-coloring}\}$. 
Circular chromatic number

Graph $G = (V, E)$

$(p, q)$-coloring: $c : V \rightarrow [p]$ such that $q \leq |c(u) - c(v)| \leq p - q$ when $uv \in E$.

Circular chromatic number: $\chi_c(G) = \inf\{p/q : \exists (p, q)\text{-coloring}\}$.

Properties.

- The infimum is in fact a minimum.
- $\chi(G) = \lceil \chi_c(G) \rceil$.
- Computing $\chi_c(G)$: NP-hard.
When does $\chi_c(G) = \chi(G)$ hold?

Question that has received a considerable attention (Zhu 2001).

**Theorem (Simonyi-Tardos 2006)**

$\chi(G) = \chi_c(G)$ when $G$ is “topologically $\chi(G)$-chromatic” and $\chi(G)$ is even.

**Lemma (Folklore)**

If every proper $t$-coloring of a $t$-chromatic graph $G$ contains a $K_{t,t}^*$ with all colors on each side, then $\chi(G) = \chi_c(G)$.

**Corollary (Alishahi-Hajiabolhassan-M. 2017)**

If $\chi(KG(\mathcal{H})) = \text{cd}_2(\mathcal{H})$, then $\chi(G) = \chi_c(G)$.

Categorical product

**Theorem (Alishahi-Hajiabolhassan-M. 2017)**

Let $\mathcal{H}_1, \ldots, \mathcal{H}_s$ be hypergraphs with no singleton and such that $\chi(KG(\mathcal{H}_i)) = \text{cd}_2(\mathcal{H}_i)$ for all $i$. Let $t = \min_i \text{cd}_2(\mathcal{H}_i)$.

Then any proper coloring of $KG(\mathcal{H}_1) \times \cdots \times KG(\mathcal{H}_s)$ with $t$ colors contains a $K_{t,t}^*$ with all colors on each side.

Consequence: for such hypergraphs

$$\chi(KG(\mathcal{H}_1) \times \cdots \times KG(\mathcal{H}_s)) = \chi_c(KG(\mathcal{H}_1) \times \cdots \times KG(\mathcal{H}_s))$$

$$= \min_i \chi(KG(\mathcal{H}_i))) = \min_i \chi_c(KG(\mathcal{H}_i))) = \min_i (\text{cd}(\mathcal{H}_i)).$$

They satisfy Hedetniemi’s conjecture and Hedetniemi’s conjecture for the circular coloring (Zhu 1992).
Hypergraphs $\mathcal{H}$ with $\chi(KG(\mathcal{H})) = cd_2(\mathcal{H})$

Let $A$ and $B$ be two disjoint sets, with $|A| \geq 2k - 1$ and $|B| \geq 1$. The set system

$$\mathcal{H} = \binom{A}{k} \cup \{\{i, j\} : i \in A, j \in B\} \cup \binom{B}{k}$$

satisfies $\chi(KG(\mathcal{H})) = cd_2(\mathcal{H})$.

- Deciding $\chi(G) = \chi_c(G)$ is NP-complete.
- Computing $\chi(KG(\mathcal{H}))$ is NP-hard.
- Computing $cd_2(\mathcal{H})$ is NP-hard.

What is the complexity of deciding $\chi(KG(\mathcal{H})) = cd_2(\mathcal{H})$?
Thank you