Solving Bulk-Robust Assignment Problems to Optimality

Matthias Walter (RWTH Aachen)

Joint work with

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Viktor Bindewald & Dennis Michaels (TU Dortmund)

Aussois Combinatorial Optimization Workshop, 2018
Assignment Problem:

- Input: Bipartite graph $G = (V, E)$ with $V = A \cup B$, edge costs $c \in \mathbb{R}^E$
- Feasible sets: Perfect matchings $M \subseteq E$ (assuming $|A| = |B|$)
- Goal: Minimize cost $c(M) := \sum_{e \in M} c_e$
Bulk-Robustness for Assignment Problems

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Bulk-Robustness:
- Possible (or likely) failure scenarios are given (explicitly or implicitly).
- Goal: Buy edges such that for every scenario, there still exists a perfect matching using the (bought) edges that survived.
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Literature:
- Concept formally introduced by Adjiashvili, Stiller & Zenklusen (MPA 2015)
- Classical related problems: $k$-edge connected spanning subgraph problem robustifies spanning-tree problem against failure of any $(k-1)$-edge set.
- LP-based $\mathcal{O}(\log(|V|))$-approximation algorithm by Adjiashvili, Bindewald & Michaels (ICALP 2016)
Bulk-Robust Assignments with Edge Failures

**Input:**

- Bipartite graph $G = (V, E)$ with $V = A \cup B$
- Failure scenarios $\mathcal{F} = \{\{f_1\}, \ldots, \{f_\ell\}\}$ with $f_i \in E$
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Example:

Hardness:
- SetCover reduces to the problem.
- For any $d < 1$, it admits no $(d \log |V|)$-approximation, unless $\text{NP} \subseteq \text{DTIME}(|V|^{\log \log |V|})$. 
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  /                   /
 O------------------O
  |                   |
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Robust Assignments

Models

CG Cuts

Bulk-Robust Assignments with Edge Failures

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Bulk-Robust Assignments with Node Failures

Input:

- Bipartite graph $G = (V, E)$ with $V = A \cup B$
- Failure scenarios $\mathcal{F} = \{\delta(b_1), \ldots, \delta(b_\ell)\}$ with $b_i \in B$.
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Related Problem:
- Related version where nodes from $B$ are bought (in contrast to edges) has approximation algorithm by Adjiashvili, Bindewald & Michaels (2017).
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General Case

Input:

• Bipartite graph \( G = (V, E) \) with \( V = A \cup B \)
• Failure scenarios \( \mathcal{F} = \{F_1, \ldots, F_\ell\} \) with \( F_i \subseteq E \)
  with cardinalities \( k(F) \) for all \( F \in \mathcal{F} \)
• Edge costs \( c \in \mathbb{R}^E \)

Goal:

• Find \( X \subseteq E \) with minimum \( c(X) \) such that
  for all \( F \in \mathcal{F} \), the subgraph \( (V, X \setminus F) \) contains a matching of size \( k(F) \).
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- Find $X \subseteq E$ with minimum $c(X)$ such that for all $F \in \mathcal{F}$, the subgraph $(V, X \setminus F)$ contains a matching of size $k(F)$.

**Special Cases:**
- Edge failures: Set $k(F_i) := |A| = |B|$ and $F_i := \{f_i\}$ for all $i \in \llbracket \ell \rrbracket$.
- Node failures: Set $k(F_i) := |A|$ and $F_i := \delta(b_i)$ for all $i \in \llbracket \ell \rrbracket$. 
Integer Programming Models

Straight-forward model (see Adjiashvili et al., ICALP 2016):

\[
\begin{align*}
\text{min} & \quad c^T x \\
\text{s.t.} & \quad x \geq y^{(F)} \quad \text{for all } F \in \mathcal{F} \\
& \quad y^{(F)} \in P_{k(F)}\text{-match}(G - F) \quad \text{for all } F \in \mathcal{F} \\
& \quad x_e \in \mathbb{Z}_+ \quad \text{for all } e \in E
\end{align*}
\]

- Has $\mathcal{O}(|\mathcal{F}| \cdot |E|)$ variables and constraints.
**Integer Programming Models**

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- Has \( \mathcal{O}(|\mathcal{F}| \cdot |E|) \) variables and constraints.

**Polyhedral combinatorics helps:**

- What does this mean for \( x \)?

\[
\exists y : x \geq y, \ y \in P_{k(F)}\text{-match}(G')
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• Projection onto \( x \) is the dominant of the \( k(F)\)-matching polytope.
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- Has $O(|\mathcal{F}| \cdot |E|)$ variables and constraints.

Polyhedral combinatorics helps:

- What does this mean for $x$?

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\exists y : x \geq y, \quad y \in P_{k(F)}\text{-match}(G')
\]

- Projection onto $x$ is the dominant of the $k(F)$-matching polytope.

- Inequalities known (Fulkerson 1970):

\[
\sum_{e \in E[S]} x_e \geq |S| - |V| + k(F) \quad \text{for all } S \subseteq V
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\] (1)

\(\Rightarrow\) Has \(\mathcal{O}(|\mathcal{F}| \cdot |E|)\) variables and constraints.

Equivalent (derived from dominant):

\[
\begin{align*}
\min & \quad c^T x \\
\text{s.t.} & \quad \sum_{e \in E[S] \setminus F} x_e \geq |S| - |V| + k(F) \quad \text{for all } S \subseteq V \text{ for all } F \in \mathcal{F} \\
& \quad x_e \in \mathbb{Z}_+ \quad \text{for all } e \in E
\end{align*}
\] (4)

\(\Rightarrow\) Has \(\mathcal{O}(|E|)\) variables and \(\mathcal{O}(|\mathcal{F}| \cdot 2^{|V|})\) constraints.

\(\Rightarrow\) For every \(F \in \mathcal{F}\), separation problem reduces to a minimum \(s-t\)-cut problem.
Models in Practice: LP Relaxation

Setup:
- All experiments done with SCIP 5.0.0 (recently released).
- Complete bipartite graphs with $|A| = |B| = n$
- Uniform failures $\mathcal{F} = \{\{e\} \mid e \in E\}$, unit costs $c = 1$
- Time limit 600 s, no heuristics, no general purpose cuts

Running times for LP relaxation
Models in Practice: LP Relaxation

Setup:

- Erdős-Rényi graphs with $|A| = |B| = n$, $p = 0.5$
- Uniform failures $\mathcal{F} = \{\{e\} \mid e \in E\}$, unit costs $c = 1$
- Time limit 600 s, no heuristics, no general purpose cuts

Running times for LP relaxation

![Graph showing running times for LP relaxation with compact and dominant methods. The x-axis represents $n = |A| = |B|$ and the y-axis represents running time in seconds. The graph illustrates the increase in running time as $n$ increases.](image-url)
Setup:

- Complete bipartite graphs with $|A| = |B| = n$
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Results for compact vs. dominant model (IP)

<table>
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<th>n</th>
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<th>Final bnd</th>
<th>Time</th>
<th>Root time</th>
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<td>11.1</td>
<td>600.0</td>
<td>0.0</td>
<td>18</td>
<td>0.4</td>
</tr>
<tr>
<td>10</td>
<td>20</td>
<td>14.8</td>
<td>11.4</td>
<td>600.0</td>
<td>0.0</td>
<td>20</td>
<td>243.2</td>
</tr>
</tbody>
</table>
Strengthening the Model

Chvátal-Gomory cuts:

- Consider $F_1, \ldots, F_\ell$ with constant $k(F_i) = k$ for all $i \in [\ell]$ ($\ell \geq 2$).
- Sum up all inequalities for fixed $S$ with $|S| - |V| + k \geq 1$. 
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$$
\sum_{e \in E[S]} \left| \{ i \in [\ell] : e \in E \setminus F_i \} \right| x_e \geq \ell(|S| - |V| + k)
$$
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$$\sum_{e \in E[S]} \{ i \in [\ell] \mid e \in E \setminus F_i \} x_e \geq \ell (|S| - |V| + k)$$

- Scale it by $1/(\ell - 1)$.
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$$\sum_{e \in E[S]} \frac{| \{ i \in [\ell] \mid e \in E \setminus F_i \} |}{\ell - 1} x_e \geq \frac{\ell}{\ell - 1}(|S| - |V| + k)$$

Weakened for coefficients with $e$ in no $F_i$.

Strengthened for coefficients with $e$ in all $F_i$.

Stronger right-hand side.
Strengthening the Model

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- $x$ is integer and nonnegative, so round up coefficients and right-hand side.
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1. Consider $F_1, \ldots, F_\ell$ with constant $k(F_i) = k$ for all $i \in [\ell]$ ($\ell \geq 2$).
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4. $x$ is integer and nonnegative, so round up coefficients and right-hand side.

$$\sum_{e \in E[S]} \left\{ \begin{array}{ll} 2 & \text{if } e \text{ in no } F_i \\ 0 & \text{if } e \text{ in all } F_i \\ 1 & \text{otherwise} \end{array} \right\} x_e \geq |S| - |V| + k + 1$$
Strengthening the Model

Chvátal-Gomory cuts:

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- **Strengthened** for coefficients with $e$ in all $F_i$.
- **Stronger** right-hand side.
Separation Problem

Input:
- Bipartite graph $G = (V, E)$ with bipartition $V = A \cup B$.
- Edge weights $w \in \mathbb{R}_+^E$.
- Parameter $k$.

Goal:
- Find $S \subseteq V$ with $|S| \geq |V| - k + 1$ minimizing $w(E[S]) - |S| + |V| - k$
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IP Model:
- Variables $y$ and $z$ with
  - $y_v = 1 \iff v \in S$
  - $z_e = 1 \iff e \in E[S]$

$$
\begin{align*}
\min & \quad - \sum_{v \in V} y_v + \sum_{e \in E} w_e z_e \\
\text{s.t.} & \quad -y_a - y_b + z_{a,b} \geq -1 \quad \text{for all } \{a, b\} \in E \\
& \quad y(A) + y(B) \geq |V| - k + 1 \\
& \quad y, z \text{ binary}
\end{align*}
$$

Observe: TU system plus a single inequality.
Bad News: NP-hardness

Separation problem:
- Input: bipartite graph $G = (V, E)$, a nonnegative vector $w \in \mathbb{Q}_+^E$ and a number $\ell \in \mathbb{N}$.
- Goal: find a set $S \subseteq V$ with $|S| \geq \ell$ that minimizes $w(E[S]) - |S|$.

Some NP-hard problem:
- Input: bipartite Graph $G = (V, E)$, numbers $m, n \in \mathbb{N}$.
- Goal: is there a set of at most $n$ nodes that cover at least $m$ of $G$’s edges?
- Hardness: Apollonio & Simeone (2014)
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Reduction idea:
- Node complementing ($\ell := |V| - n$) and proper scaling ($w := (|V| + 1)1_E$)
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Reduction idea:
- Node complementing ($\ell := |V| - n$) and proper scaling ($w := (|V| + 1)\mathbb{1}_E$)
- Existence of $S$ with $|S| \leq n$ and $|\{e \in E \mid e \cap S \neq \emptyset\}| \geq m$ is equivalent to existence of $\bar{S}$ with $|\bar{S}| \geq \ell$ and
  \[
  |E \setminus E[\bar{S}]| \geq m \iff |E[\bar{S}]| \leq (|E| - m)
  \iff (|V| + 1)|E[\bar{S}]| \leq (|V| + 1)(|E| - m)
  \iff (|V| + 1)|E[\bar{S}]| - |\bar{S}| \leq (|V| + 1)(|E| - m)
  \iff w(E[\bar{S}]) - |\bar{S}| \leq (|V| + 1)(|E| - m).
\]

(note that $0 \leq |\bar{S}| < |V| + 1$)
Good News: Nice Heuristic Approach

Main idea:

- Let’s move $y(A) + y(B) \geq |V| - k + 1$ into the objective function!
- Lagrange multiplier is one-dimensional: (binary) search for good values.
- Subproblem again reduces to minimum $s-t$-cut problem.
- If it returns a set $S$ then we have a most-violated inequality among all inequalities with this $|S|$. 
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Desirable side-effect:

$$\sum_{e \in E[S]} \{0, 1, 2\} x_e \geq |S| - |V| + k + 1$$

- Chvátal-Gomory strengthening is stronger for small right-hand sides.
- We can control $|S|$ via Lagrange multipliers to get a small right-hand side.
- Experimentally best strategy: aim for violated cuts with minimum $|S|$.
Models in Practice: CG Cuts

Setup:

- Complete bipartite graphs with $|A| = |B| = n$
- Uniform failures $\mathcal{F} = \{\{e\} \mid e \in E\}$, unit costs $c = 1$
- Time limit 600 s, no general purpose cuts

Running times for IP

![Graph showing running times for CG and CG+degree models.](image-url)

$n = |A| = |B|$
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Running times for IP

- Note that we are solving the IP and not just the relaxation!
Models in Practice: CG Cuts

Setup:
- Complete bipartite graphs with $|A| = |B| = n$
- Uniform failures $\mathcal{F} = \{\{e\} | e \in E\}$, unit costs $c = 1$
- Time limit 600 s, no general purpose cuts
- Special case of CG cuts are strengthened degree inequalities $x(\delta(v)) \geq 2$.

Running times for IP

- Note that we are solving the IP and not just the relaxation!
Setup:

- Erdős-Rényi graphs with $|A| = |B| = n$, $p = 0.5$
- Uniform failures $\mathcal{F} = \{\{e\} \mid e \in E\}$, unit costs $c = 1$
- Time limit 600 s, no general purpose cuts

Running times for IP

![Graph showing running times for CG and CG+degree algorithms]

- $n = |A| = |B|$
Models in Practice: CG Cuts

Setup:

- Complete bipartite graphs with $|A| = |B| = n$
- Uniform failures $\mathcal{F} = \{\{e\} \mid e \in E\}$
- Random costs $c_e \in \{1, \ldots, 2\}$ for all $e \in E$ independently.
- Time limit 600 s, no general purpose cuts

Running times for IP

![Running time graph](image-url)
Models in Practice: CG Cuts

Setup:

- Complete bipartite graphs with $|A| = |B| = n$
- Uniform failures $\mathcal{F} = \{\{e\} \mid e \in E\}$
- Random costs $c_e \in \{1, \ldots, 4\}$ for all $e \in E$ independently.
- Time limit 600 s, no general purpose cuts

Running times for IP

![Graph showing running times for IP with CG and CG+degree]
Models in Practice: IP Bounds with CG Cuts

Setup:

- Complete bipartite graphs with $|A| = n$ and $|B| = \lceil 1.5n \rceil$
- Node failures $\mathcal{F} = \{\delta(b) \mid b \in B\}$, unit costs $c = 1$
- Time limit 600 s, no general purpose cuts

Remark: Problem is on primal side, i.e., finding an optimal solution!
Thanks!

Things you’ve seen:

- Speed-up of dominant formulation vs. compact one.
- Derivation of Chvátal-Gomory (CG) cuts.
- Fast heuristic separation with Lagrange multiplier.
- Strength of CG cuts, in particular strengthened degree.
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- Speed-up of dominant formulation vs. compact one.
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- Fast heuristic separation with Lagrange multiplier.
- Strength of CG cuts, in particular strengthened degree.

Things you might see in the future:
- Structured instances:
  - ... obtained from the SetCover reduction
  - ... obtained from other sources (QAPLIB?)
  - ... yours?
- Implementation of / comparison with approximation algorithm