Admissibility of solution estimators to stochastic optimization problems

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Joint Work with Tu Nguyen and Ao Sun

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A general stochastic optimization problem

$F : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}$

$X \subseteq \mathbb{R}^d$

ξ is a random variable taking values in $\mathbb{R}^m$

$$\min_{x \in X} \ \mathbb{E}_\xi[\ F(x, \xi)\ ]$$

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Example:

Supervised Machine Learning: One sees samples $(z, y) \in \mathbb{R}^n \times \mathbb{R}$ of labeled data from some distribution, and one aims to find a function $f \in \mathcal{F}$ in a finitely parameterized hypothesis class $f \in \mathcal{F}$ that minimizes the expected loss $\mathbb{E}_{(z,y)}[\ell(f(z), y)]$, where $\ell : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ is some loss function. Then d is the number of parameters, X is a description of $\mathcal{F}$ via the parameters, $m = n + 1$, $\xi = (z, y)$, and

$$F(f, (z, y)) = \ell(f(z), y).$$

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(News) Vendor Problem: (News) Vendor buys some units of a product (newspapers) from supplier at cost of $c > 0$ dollars/unit; at most u units available. Stochastic demand for product. Product sold at price $p > c$ dollars/unit. End of day, vendor can return unsold product to supplier at $r < c$ dollars/unit. Find number of units to buy to maximize (minimize) the expected profit (loss).

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$d = m = 1$, $X = [0, u]$,

$$F(x,\xi) = cx - p \min\{x,\xi\} - r \max\{x - \xi, 0\}.$$

Solving the problem

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Natural idea: Given samples $\xi^1, \ldots, \xi^n \in \mathbb{R}^d$, solve the deterministic problem

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Stochastic optimizers call this sample average approximation (SAA); machine learners call this empirical risk minimization.

Concrete Problem

$d = m.$ $\quad F(x, \xi) = \xi^T x$

$X \subseteq \mathbb{R}^d$ is a compact set (e.g., polytope, integer points in a polytope).

$\xi \sim N(\mu, \sigma I)$.

$$\min_{x \in X} \ \mathbb{E}_\xi[\, F(x, \xi) \,] = \min_{x \in X} \ \mathbb{E}_\xi[\, \xi^T x \,] = \min_{x \in X} \ \mu^T x$$

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Sample Average Approximation (SAA):

$$\min_{x \in X} \; \frac{1}{n} \sum_{i=1}^{n} F(x, \xi^i) \;=\; \min_{x \in X} \; \overline{\xi}^T x$$

where $\overline{\xi} := \frac{1}{n} \sum_{i=1}^{n} \xi^i$.

A quick tour of Statistical Decision Theory

Set of states of nature, modeled by a set Θ.

Set of possible actions to take, modeled by $\mathcal{A}$.

In a particular state of nature $\theta \in \Theta$, the performance of any action $a \in \mathcal{A}$, is evaluated by a loss function $\mathcal{L}(\theta, a)$. Goal: choose action to minimize loss.

(Partial/Incomplete) Information about θ is obtained through a random variable y taking values in a sample space χ. The distribution of y depends on the particular state of nature θ, denoted by P_θ.

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Decision Rule: Takes $y \in \chi$ as input and reports an action $a \in \mathcal{A}$. Denote by $\delta : \chi \to \mathcal{A}$.

Our problem cast as statistical decision problem

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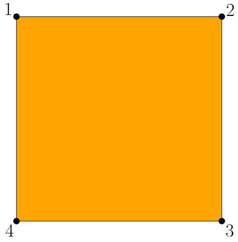
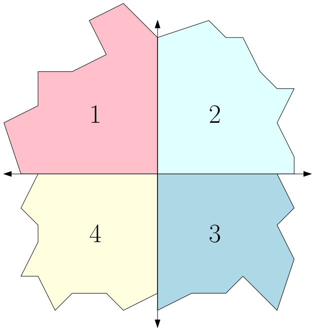
$$\text{SAA:} \quad \delta(\xi^1, \ldots, \xi^n) \in \arg\max\{\bar{\xi}^T x : x \in X\}$$

Decision Rules





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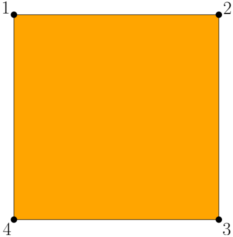

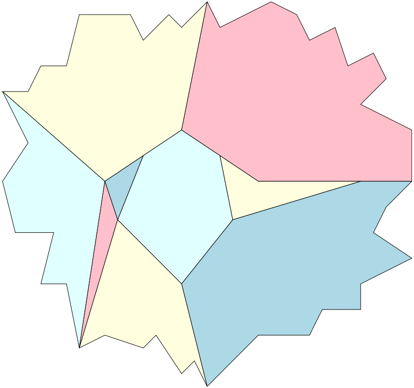

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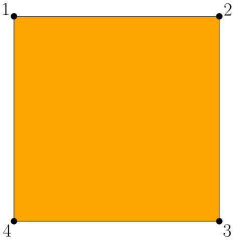

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If a decision rule δ is not dominated by any other decision rule, we say that δ is admissible. Otherwise, it is inadmissible.

Is the Sample Average Approximation (SAA) rule admissible?

Admissibility in stochastic optimization

Stochastic optimization setup:

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Want to solve with access to n i.i.d. samples of ξ.

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Statistical decision theory view:

$\xi \sim N(\mu, I)$; states of nature $\Theta = \mathbb{R}^m = \{\text{all possible } \mu \in \mathbb{R}^m\}$.

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Given a decision rule $\delta : \chi \to X$, the risk of δ

$$R_\delta(\mu) := \mathbb{E}_{\xi^1, \ldots, \xi^n}[\, \mathcal{L}(\mu, \delta(\xi^1, \ldots, \xi^n)) \,]$$

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Sample Average Approximation (SAA) can be inadmissible!!

Inadmissibility of SAA: Stein's Paradox

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$$\min_{x \in \mathbb{R}^d}\ \mathbb{E}_{\xi}[\ F(x, \xi)\ ]\ =\ \min_{x \in \mathbb{R}^d}\ \mathbb{E}_{\xi}[\ \|x - \xi\|^2\ ]$$

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Generalized to arbitrary convex quadratic function with uncertain linear term in Davarnia and Cornuéjols 2018. Follow-up work from a Bayesian perspective in Davarnia, Kocuk and Cornuéjols 2018.

A class of problems with no Stein's paradox

THEOREM Basu-Nguyen-Sun 2018

Consider the problem of optimizing an uncertain linear objective $\xi \sim N(\mu, I)$ over a fixed compact set $X \subseteq \mathbb{R}^d$:

$$\min_{x \in X} \; \mathbb{E}_{\xi \sim N(\mu, I)}[\, \xi^T x \,]$$

The Sample Average Approximation (SAA) rule is admissible.

Main technical ideas/tools

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Sufficient Statistic: $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$ family of distributions for r.v. y in sample space χ. Sufficient statistic for this family is a function $T : \chi \to \tau$ such that the conditional probability $P(y|T = t)$ does not depend on θ.

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FACT:

$$\chi = \underbrace{\mathbb{R}^d \times \ldots \times \mathbb{R}^d}_{n \text{ times}}, \ \mathcal{P} = \{\underbrace{N(\mu, I) \times \ldots \times N(\mu, I)}_{n \text{ times}} : \mu \in \mathbb{R}^d\},$$

i.e., $(\xi^1, \ldots, \xi^n) \in \chi$ are i.i.d samples from the normal distribution $N(\mu, I)$. Then $T(\xi^1 \ldots, \xi^n) = \overline{\xi} := \frac{1}{n}\sum_{i=1}^n \xi^i$ is a sufficient statistic for $\mathcal{P}$.

Main technical ideas/tools

For any decision rule δ, define the function

$$F(\mu) = R_{\delta}(\mu) - R_{\delta_{SAA}}(\mu).$$

Suffices to show that there exists $\hat{\mu} \in \mathbb{R}^d$ such that $F(\hat{\mu}) > 0$.

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First observe: $F(0) = 0$.

Main technical ideas/tools

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$$F(\mu) = R_{\delta}(\mu) - R_{\delta_{SAA}}(\mu).$$

Suffices to show that there exists $\hat{\mu} \in \mathbb{R}^d$ such that $F(\hat{\mu}) > 0$.

First observe: $F(0) = 0$.

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Use a fact from probability theory that for any Lebesgue integrable function $f : \mathbb{R}^n \to \mathbb{R}^d$, the map

$$\mu \mapsto \mathbb{E}_{y \in N(\mu,\Sigma)} \left[f(y) \right] := \int_{\mathbb{R}^d} f(y) \exp\left(-\frac{1}{2}(y-\mu)^T \Sigma^{-1} (y-\mu) \right) dy$$

has derivatives of all orders and these can be computed by taking the derivative under the integral sign.

Open Questions

- What about nonlinear objectives over compact feasible regions? For example, what if $F(x, \xi) = x^T Q x + \xi^T x$ for some fixed PSD matrix Q, and X is a compact (convex) set?

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- METATHEOREM (from Gérard Cornuéjols): Admissible if and only if feasible region is bounded !?

THANK YOU !

Questions/Comments ?