# Polynomially computable sharp probability bounds

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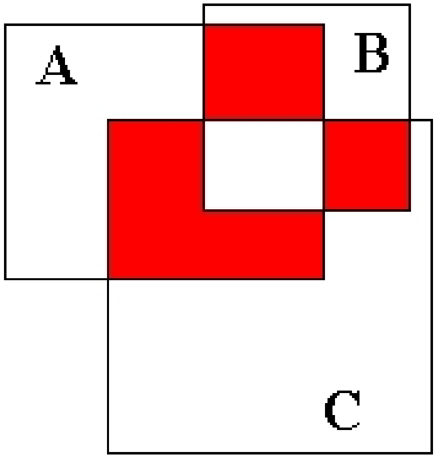
# Boole's Problem




$$\mathbf{E_1} = (\mathbf{A} \cap \mathbf{B} \cap \overline{\mathbf{C}}) \cup (\mathbf{A} \cap \overline{\mathbf{B}} \cap \mathbf{C}) \cup (\overline{\mathbf{A}} \cap \mathbf{B} \cap \mathbf{C}) \qquad Prob(\mathbf{E_1}) = \frac{1}{3}$$

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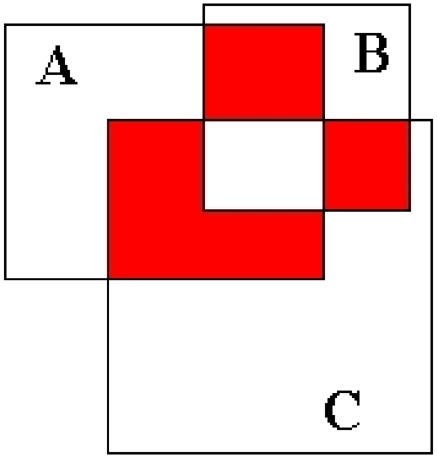


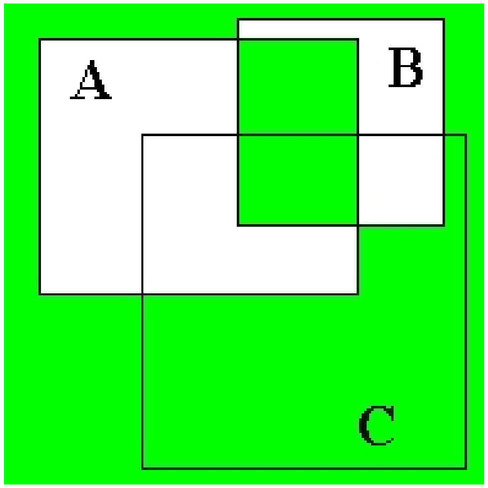


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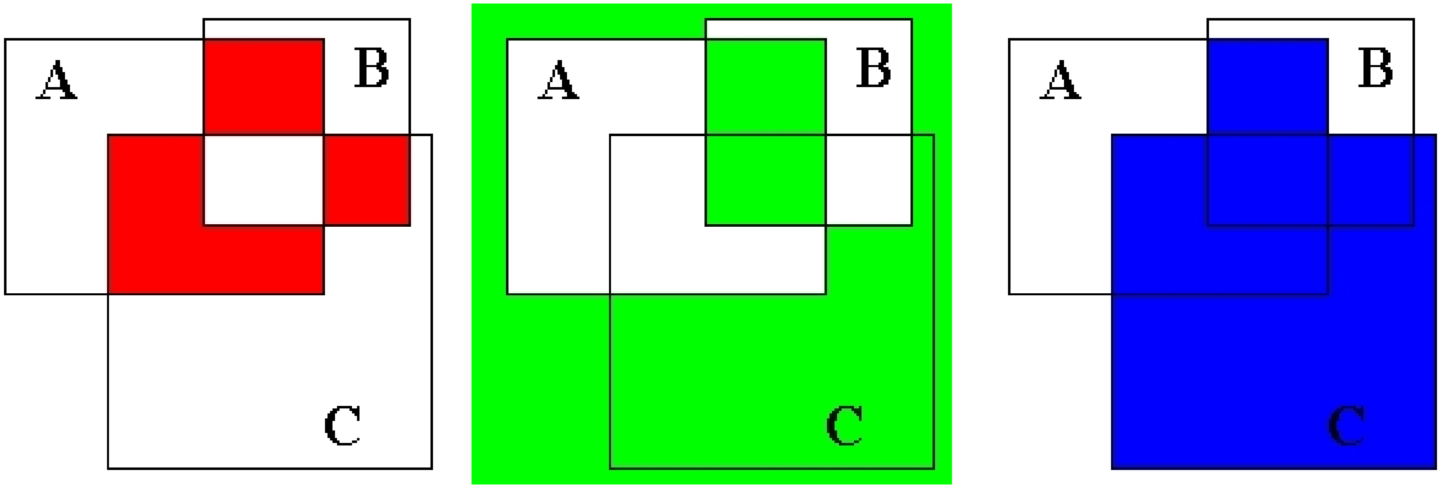


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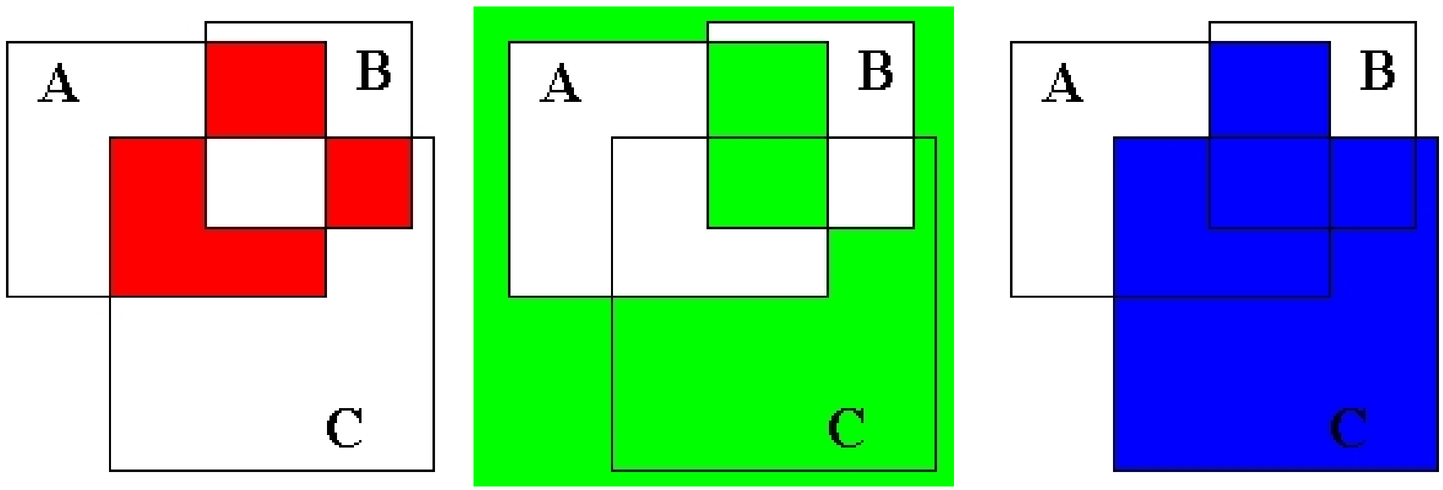


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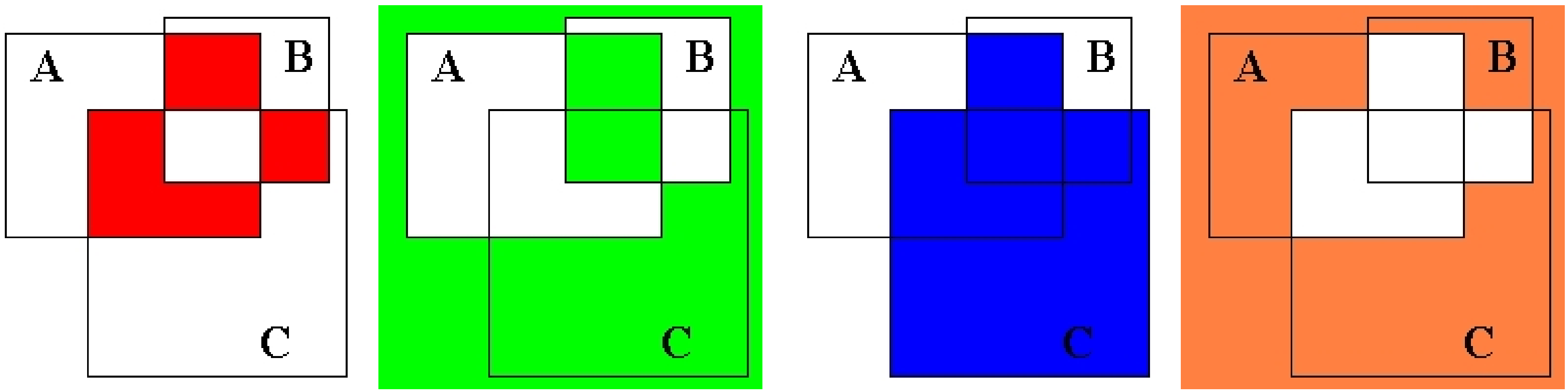
**Is this possible?**

# Boole's Problem

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$$\mathbf{E}_4 = (\overline{\mathbf{A}} \cap \overline{\mathbf{B}}) \cup (\overline{\mathbf{A}} \cap \overline{\mathbf{C}}) \cup (\overline{\mathbf{B}} \cap \overline{\mathbf{C}}) \qquad Prob(\mathbf{E}_4) = ?$$

**How large (small) can $Prob(\mathbf{E}_4)$ be?**

# Brief History

- Boole's Problem (Boole 1854, 1868 (1850))

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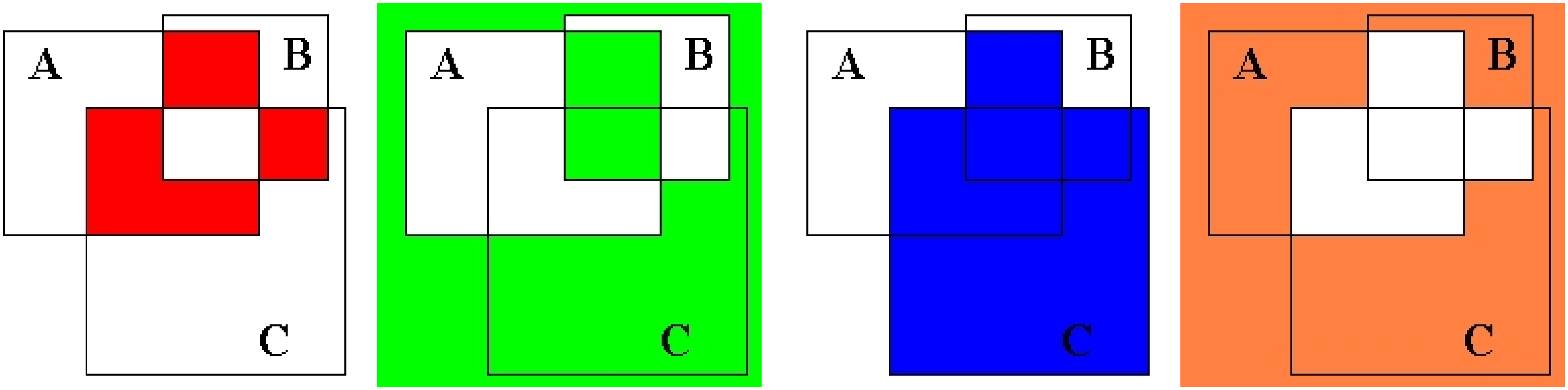
- Linear programming formulation (Hailperin 1965)

# Linear Programming Formulation




$$\mathbf{x}_0 = Prob(\overline{\mathbf{A}} \cap \overline{\mathbf{B}} \cap \overline{\mathbf{C}}), \qquad \mathbf{x}_1 = Prob(\mathbf{A} \cap \overline{\mathbf{B}} \cap \overline{\mathbf{C}}), \qquad \cdots, \qquad \mathbf{x}_7 = Prob(\mathbf{A} \cap \mathbf{B} \cap \mathbf{C})$$

$$\begin{array}{ccccccccccc}
\emptyset & \{\mathbf{A}\} & \{\mathbf{B}\} & \{\mathbf{C}\} & \{\mathbf{A},\mathbf{B}\} & \{\mathbf{A},\mathbf{C}\} & \{\mathbf{B},\mathbf{C}\} & \{\mathbf{A},\mathbf{B},\mathbf{C}\} & & & \\
\mathbf{x}_0 & +\mathbf{x}_1 & +\mathbf{x}_2 & +\mathbf{x}_3 & & & & & \longrightarrow & \left\{ \begin{array}{c} \text{max} \\ \text{min} \end{array} \right\} & \\
\mathbf{x}_0 & +\mathbf{x}_1 & +\mathbf{x}_2 & +\mathbf{x}_3 & +\mathbf{x}_4 & +\mathbf{x}_5 & +\mathbf{x}_6 & +\mathbf{x}_7 & = & 1 & \\
 & & & & +\mathbf{x}_4 & +\mathbf{x}_5 & \mathbf{x}_6 & & = & \frac{1}{3} & \\
\mathbf{x}_0 & & & +\mathbf{x}_3 & +\mathbf{x}_4 & & & +\mathbf{x}_7 & = & \frac{1}{2} & \\
 & & & +\mathbf{x}_3 & +\mathbf{x}_4 & +\mathbf{x}_5 & +\mathbf{x}_6 & +\mathbf{x}_7 & = & \frac{5}{6} & \\
 & & & & & & & \mathbf{x}_j & \geq & 0, \ j = 0, ..., 8 &
\end{array}$$

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- Probabilistic logic (Nilsson 1986)

- Probabilistic satisfiability (PSAT)

  (Georgakopoulos, Kavvadias and Papadimitriou 1988)

  Feasibility is **NP-hard!**

  What about optimization with feasible input?

# A Special Case: Union of Events

**Events:** $\mathbf{A}_i \subseteq \Omega$, $i \in \mathbf{V} = \{1, 2, ..., n\}$

**Input:** $p_I = Prob\left(\bigcap_{i \in I} \mathbf{A}_i\right)$ for $I \subseteq \mathbf{V}$, $|I| \leq m$, $(p_\emptyset = 1)$

**Problem:** Find lower and upper bounds for the probability of the union of these $n$ events:

$$LB\,(p_I \mid I \subseteq \mathbf{V}, |I| \leq m) \leq Prob\left(\bigcup_{i \in \mathbf{V}} \mathbf{A}_i\right) \leq UB\,(p_I \mid I \subseteq \mathbf{V}, |I| \leq m)$$

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**Remark** [Fréchet 1935]: For $m = 1$ the bounds

$$\max_{1 \leq i \leq n} Prob(\mathbf{A}_i) \quad \leq Prob\left(\bigcup_{i \in \mathbf{V}} \mathbf{A}_i\right) \leq \quad \min\left\{1, \sum_{i=1}^{n} Prob(\mathbf{A}_i)\right\}$$

are **sharp**.

# Linear Programming Formulation

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**Variables:** $\mathbf{x}_J = Prob\left(\left(\bigcap_{i \in J} \mathbf{A}_i\right) \cap \left(\bigcap_{i \notin J} \overline{\mathbf{A}}_i\right)\right)$ for $J \subseteq \mathbf{V}$

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and

$$p_I = \sum_{\mathbf{V} \supseteq J \supseteq I} \mathbf{x}_J \qquad \text{for all } I \subseteq \mathbf{V}, |I| \leq m$$

# Linear Programming Formulation

Let us define

$$LB_m^* = \min \sum_{\emptyset \neq J \subseteq \mathbf{V}} \mathbf{x}_J \quad \text{and} \quad UB_m^* = \max \sum_{\emptyset \neq J \subseteq \mathbf{V}} \mathbf{x}_J$$

subject to the constraints

$$\sum_{\mathbf{V} \supseteq J \supseteq I} \mathbf{x}_J = p_I \quad \text{for all } I \subseteq \mathbf{V}, |I| \leq m$$

$$\mathbf{x}_J \geq 0 \quad \text{for all } J \subseteq \mathbf{V}$$

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**Claim**: Bounds for the probability of any other event defined in terms of $\mathbf{A}_1$, $\mathbf{A}_2$, ..., $\mathbf{A}_n$ can be computed from a similar LP formulation, in which only the objective function will be different.

E.g. "at least $r$ out of these $n$ events occur", "at most $q$ out of these $n$ events occur", etc.

# Linear Programming Formulation

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**Claim** [Hailperin 1965]: The bounds $LB^*_m = LB\,(p_I \mid I \subseteq \mathbf{V}, |I| \leq m)$ and $UB^*_m = UB\,(p_I \mid I \subseteq \mathbf{V}, |I| \leq m)$ are **sharp**.

From the optimal solutions of these linear programs one can construct examples for which $Prob(\bigcup_{i \in \mathbf{V}} \mathbf{A}_i)$ attains these bounds. $(\sum_{J \subseteq \mathbf{V}} \mathbf{x}_J = p_\emptyset = 1)$

# Linear Programming Formulation

**Claim**: Computing $LB_m^*$ and $UB_m^*$ **maybe hard**!

- Feasibility is **NP-hard**.
  (Georgakopoulos, Kavvadias and Papadimitriou 1988)

- Exponentially many variables in LP formulation!

- Column generation (row generation in dual LP) is an **NP-hard** sub-problem (even for $m = 2$).
  (Jaumard, Hansen and Poggi de Aragão 1991)

# Relaxation I: Aggregation

Summing up equations for $I \subseteq \mathbf{V}$, $|I| = k$ (for $k = 0, 1, ..., m$), and introducing new variables $y_j = \sum_{J \subseteq \mathbf{V},\ |J|=j} \mathbf{x}_J$ (for $j = 0, 1, ..., n$) yields a **relaxation**, the so called Binomial Moment Problem (Prékopa 1988):

$$\widetilde{LB}_m = \text{min} \sum_{j \geq 1} y_j \quad \text{and} \quad \widetilde{UB}_m = \text{max} \sum_{j \geq 1} y_j$$

subject to the constraints

$$\sum_{j=0}^{n} \binom{j}{k} y_j = \mathbf{S}_k \qquad \text{for } k = 0, 1, ..., m$$

$$y_j \geq 0 \qquad \text{for } j = 0, 1, ..., n$$

$\mathbf{S}_k = \sum_{\substack{I \subseteq \mathbf{V} \\ |I|=k}} p_I$ is called the $k$-th binomial moment of $\mathbf{A}_1$, $\mathbf{A}_2$, ..., $\mathbf{A}_n$.

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If $\xi$ is the random variable denoting the number of events occurring from $\{\mathbf{A}_1, \mathbf{A}_2, ..., \mathbf{A}_n\}$, then

$$\mathbf{S}_k = Exp\left[\binom{\xi}{k}\right].$$

# Binomial Moment Problem

- $\widetilde{LB}_m \leq LB^*_m$ and $\widetilde{UB}_m \geq UB^*_m$ are polynomially computable sharp bounds (sharp in terms of $\{\mathbf{S}_k \mid k \leq m\}$, but may not be sharp in terms of $\{p_I \mid I \subseteq \mathbf{V},\ |I| \leq m\}$).

- Dual feasible basic solutions are characterized (Prékopa 1988) $\rightsquigarrow$ closed form optimal solutions for $m \leq 4$ (Prékopa 1988; B and Prékopa 1989)

- Several closed form bounds in the literature are of this type, or generalized by these type of bounds.

# Binomial Moments Based Bounds

- $\mathbf{S}_1 - \mathbf{S}_2 + \cdots - \mathbf{S}_{2s} \leq \widetilde{LB}_{2s}$ and $\widetilde{UB}_{2s+1} \leq \mathbf{S}_1 - \mathbf{S}_2 + \cdots + \mathbf{S}_{2s+1}$

  (Bonferroni 1937)

- $\widetilde{LB}_2 \quad \geq \quad \frac{\mathbf{S}_1^2}{\mathbf{S}_1 + 2\mathbf{S}_2}$ (Chung and Erdös 1952)

- $\widetilde{LB}_2 = \frac{2}{i+1}\mathbf{S}_1 - \frac{2}{i(i+1)}\mathbf{S}_2, \qquad i = 1 + \lfloor \frac{2\mathbf{S}_2}{\mathbf{S}_1} \rfloor$

  (Dawson and Sankoff 1967; Kwerel 1975; Galambos 1977)

- $\widetilde{UB}_2 = \mathbf{S}_1 - \frac{2}{n}\mathbf{S}_2$

  (Kwerel 1975; Sathe, Pradhan and Shah 1980; Platz 1985)

# Binomial Moments Based Bounds

- $\widetilde{LB}_3 = \frac{i+2n-1}{(i+1)n}\mathbf{S}_1 - \frac{2(2i+n-2)}{i(i+1)n}\mathbf{S}_2 + \frac{6}{i(i+1)n}\mathbf{S}_3$,

  where $i = 1 + \lfloor \frac{2(n-2)\mathbf{S}_2 - 6\mathbf{S}_3}{(n-1)\mathbf{S}_1 - 2\mathbf{S}_2} \rfloor$, and

- $\widetilde{UB}_3 = \mathbf{S}_1 - \frac{2(2i-1)}{i(i+1)}\mathbf{S}_2 + \frac{6}{i(i+1)}\mathbf{S}_3$,

  where $i = 1 + \lfloor \frac{3\mathbf{S}_3}{\mathbf{S}_2} \rfloor$

(Kwerel 1975; B and Prékopa 1989)

- $\widetilde{UB}_4 = \mathbf{S}_1 - \frac{2((i-1)(i-2)+(2i-1)n)}{i(i+1)n}\mathbf{S}_2 + \frac{6(2i+n-4)}{i(i+1)n}\mathbf{S}_3 - \frac{24}{i(i+1)n}\mathbf{S}_4$,

  where $i = 1 + \lfloor \frac{2(n-2)\mathbf{S}_2 + 3(n-5)\mathbf{S}_3 - 12\mathbf{S}_4}{(n-2)\mathbf{S}_2 - 3\mathbf{S}_3} \rfloor$

(B and Prékopa 1989)

# Stronger Lower Bounds

- $LB^*_{m=2} \geq \sum_{i=1}^{n} \alpha_i p_i$, where $p_i = P(\mathbf{A}_i)$, $p_{i,j} = P(\mathbf{A}_i \cap \mathbf{A}_j)$

  and $\sum_{j \neq i} \alpha_j p_{i,j} = (1 - \alpha_i) p_i$ for $i = 1, ..., n$.

  (Gallot 1966; Kounias 1968)

- $LB^*_2 \geq \sum_{i \in \mathbf{V}} \frac{p_i^2}{p_i + \sum_{j \neq i} p_{i,j}} \geq \widetilde{LB}_2$, (de Caen 1997)

- $LB^*_2 \geq \sum_{i \in \mathbf{V}} \left( \frac{\theta_i p_i^2}{(2 - \theta_i) p_i + \sum_{j \neq i} p_{i,j}} + \frac{(1 - \theta_i) p_i^2}{(1 - \theta_i) p_i + \sum_{j \neq i} p_{i,j}} \right) \geq \widetilde{LB}_2$,

  where $\theta_i = \frac{\sum_{j \neq i} p_{i,j}}{p_i} - \lfloor \frac{\sum_{j \neq i} p_{i,j}}{p_i} \rfloor$

  (Kuai, Alajai and Takahara 2000)

# Stronger Bounds by Graph Structures

- $UB_2^* \leq \mathbf{S}_1 - \sum_{i \neq k} p_{i,k}$ ($k$ is fixed) (Kounias 1968)

- $UB_2^* \leq \mathbf{S}_1 - \sum_{(i,j) \in T} p_{i,j} \leq \widetilde{UB}_2$,

  where $T$ is a spanning tree (Hunter 1976; Worsley 1982)

- $UB_3^* \leq \mathbf{S}_1 - \sum_{(i,j) \in \mathcal{E}} p_{i,j} + \sum_{(i,j,k) \in \mathcal{C}} p_{\{i,j,k\}} \leq \widetilde{UB}_3$,

  where $(\mathcal{E}, \mathcal{C})$ is a cherry tree (Bukszár and Prékopa 2001)

# Aggregations and Graph Structures

- Several stronger lower and upper bounds, generalizing the previous ones, were derived recently via **partial aggregation**: considering linear combinations instead of the original equations, and introducing new variables, which are linear functions of the original variables in order to obtain a **polynomially sized relaxation**.
(Prékopa, Vizvári, Regős and Gao 2001; Prékopa and Gao 2001)

- Improved Bonferroni inequalities via binomially bounded functions.
(Dohmen and Tittmann, 2007)

- Chordal graph bound ($m = 3$, (Veneziani, 2002)) and chordal graph sieve ($m = \chi(G)$, (Dohmen, 2002)).

- Upper bounds for $m = 3$ via graph structures (positive $p_{i,j}$ effect)
(Veneziani, 2002, 2008)

# Tightening the Dual

- **Relaxing** of an LP has the same effect on its optimum value as **tightening** of its dual.

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- **Relaxing** of an LP has the same effect on its optimum value as **tightening** of its dual.

- Replace **polynomial resizing** by **efficient tightening** of the dual:

  Try to tighten the dual so that row generation (separation) becomes polynomially solvable. (Size of the formulation may not decrease!)

# Simplify LP

Eliminate $x_\emptyset$ and the normalization $\sum_{j\subseteq\mathbf{V}} x_J = 1$, and then dualize.

$$\begin{aligned}\sum_{\emptyset\neq J\subseteq\mathbf{V}} \mathbf{x}_J &\to \left\{\begin{array}{c}\text{max}\\ \text{min}\end{array}\right\}\\ \sum_{J\supseteq I} \mathbf{x}_J &= p_I,\ \ I\subseteq\mathbf{V}, |I|\leq m\\ \mathbf{x}_J &\geq 0,\ \ J\subseteq\mathbf{V}\end{aligned} \quad \equiv \quad \begin{aligned}\sum_{\emptyset\neq J\subseteq\mathbf{V}} \mathbf{x}_J &\to \left\{\begin{array}{c}\text{max}\\ \text{min}\end{array}\right\}\\ \sum_{\emptyset\neq J\supseteq I} \mathbf{x}_J &= p_I,\ \ \emptyset\neq I\subseteq\mathbf{V}, |I|\leq m\\ \mathbf{x}_J &\geq 0,\ \ \emptyset\neq J\subseteq\mathbf{V}\end{aligned}$$

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$$
\begin{aligned}
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\sum_{J\supseteq I} \mathbf{x}_J &= p_I, \quad I \subseteq \mathbf{V}, |I| \leq m \\
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\quad \equiv \quad
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\quad \equiv \quad
\begin{aligned}
\left\{ \begin{matrix} \text{min} \\ \text{max} \end{matrix} \right\} \quad &\leftarrow \quad \sum_{\substack{I\subseteq \mathbf{V} \\ 1\leq |I|\leq m}} p_I w_I \\
w(S) \quad \left\{ \begin{matrix} \geq \\ \leq \end{matrix} \right\} \quad & 1, \quad \emptyset \neq S \subseteq \mathbf{V}
\end{aligned}
$$

Here $w(S) = \sum_{I\subseteq S} w_I$ and $w = (w_I \mid 1 \leq |I| \leq m)$.

# Tighten-up the Dual

Recall that $w = (w_I \mid 1 \leq |I| \leq m)$ and $w(S) = \sum_{I \subseteq S} w_I$ for all subsets $S \subseteq \mathbf{V}$.

$$\left\{ \begin{array}{c} \min \\ \max \end{array} \right\} \sum_{\substack{I \subseteq \mathbf{V} \\ 1 \leq |I| \leq m}} p_I w_I \quad = \quad \left\{ \begin{array}{c} UB_m(\mathcal{F}) \\ LB_m(\mathcal{F}) \end{array} \right\}$$

$$w(S) \quad \left\{ \begin{array}{c} \geq \\ \leq \end{array} \right\} \quad 1 \quad \emptyset \neq S \subseteq \mathbf{V}$$

$$w \quad \in \quad \mathcal{F},$$

where $\mathcal{F}$ is a polyhedral set.

# Tighten-up the Dual

$$\left\{\begin{array}{c}\text{min}\\\text{max}\end{array}\right\} \sum_{\substack{I\subseteq \mathbf{V}\\ 1\leq |I|\leq m}} p_I w_I \quad = \quad \left\{\begin{array}{c}UB_m(\mathcal{F})\\ LB_m(\mathcal{F})\end{array}\right\}$$

$$w(S) \left\{\begin{array}{c}\geq\\ \leq\end{array}\right\} 1 \quad \emptyset \neq S \subseteq \mathbf{V}$$

$$w \quad \in \quad \mathcal{F}.$$

**Observation 1**. If membership in $\mathcal{F}$ can be checked and for all $w \in \mathcal{F}$ the setfunction $w(S)$ can be minimized (resp. maximized) over $S \subseteq \mathbf{V}$ in polynomial time, then $UB_m(\mathcal{F})$ (resp. $LB_m(\mathcal{F})$) can be computed in polynomial time.

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**Observation 2**. $LB_m(\mathcal{F}) \leq LB_m^* \leq Prob\left(\bigcup_{i=1}^{n} A_i\right) \leq UB_m^* \leq UB_m(\mathcal{F})$

# Submodular Bounds

Set $M = \sum_{i=1}^{m} \binom{n}{i}$, and

$$\mathcal{F}_{sub} \; = \; \{ w \in \mathbb{R}^M \mid w(S) \text{ is submodular } \}$$

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**Corollary**. For $m \le 3$ the upper bound $UB_m(\mathcal{F}_{sub})$ can be computed in polynomial time (by network flow models).

# Nonpositive Bounds

$$\mathcal{F}_N \;=\; \{w \in \mathbb{R}^M \mid w_I \leq 0 \text{ for all } I \text{ with } 1 < |I| \leq m\}$$

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$$\begin{aligned} Prob(\mathbf{A_i}) \;&=\; p_{\{i\}} \text{ for } i \in \mathbf{V} \\ Prob\left(\bigcap_{i\in I} \mathbf{A_i}\right) \;&\ge\; p_I \quad \text{for } I \subseteq \mathbf{V},\ 1 < |I| \le m. \end{aligned}$$

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**Corollary**. $UB_m(\mathcal{F}_N)$ can be computed in polynomial time for all $m \ge 1$.

# $UB_2(\mathcal{F}_N) = UB_2(\mathcal{F}_{sub})$

$$\sum_{i=1}^{n} p_i w_i^1 + \sum_{1 \le i < j \le n} p_{ij} w_{ij}^2 \;\to\; \text{min} \;=\; UB_2(\mathcal{F}_N) = UB_2(\mathcal{F}_{sub})$$

$$w^1(S) + w^2(S) \;\ge\; 1 \qquad \text{for all } S \subseteq \mathbf{V},\; S \ne \emptyset,$$

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**Theorem**. The vertices of the feasible region are of the form $(w^1, w^2) = (\mathbf{1}, X^T)$ where $T$ is a spanning tree of the complete graph on $\mathbf{V}$.

# $UB_2(\mathcal{F}_N) = UB_2(\mathcal{F}_{sub})$

$$\begin{aligned} \sum_{i=1}^{n} p_i w_i^1 + \sum_{1\leq i<j\leq n} p_{ij} w_{ij}^2 \quad &\rightarrow \quad \text{min} \quad = \quad UB_2(\mathcal{F}_N) = UB_2(\mathcal{F}_{sub}) \\ w^1(S) + w^2(S) \quad &\geq \quad 1 \qquad \text{for all } S \subseteq \mathbf{V},\ S \neq \emptyset, \\ w^2 \quad &\leq \quad 0 \end{aligned}$$

**Theorem**. The vertices of the feasible region are of the form $(w^1, w^2) = (\mathbf{1}, X^T)$ where $T$ is a spanning tree of the complete graph on $\mathbf{V}$.

**Corollary**. $UB_2(\mathcal{F}_N) = UB_2(\mathcal{F}_{sub}) = UB_2(HW)$, where $UB_2(HW)$ is the **best** Hunter-Worsley bound.

These bounds are sharper than any other known upper bound for $m = 2$.

# Upper Bounds

- $\widetilde{UB}_{2s+1} \leq \mathbf{S}_1 - \mathbf{S}_2 + \cdots + \mathbf{S}_{2s+1}$ (Bonferroni 1937)

- $UB_2^* \leq \sum_{i=1}^{n} p_i - \sum_{i \neq k} p_{i,k}$ ($k$ is fixed) (Kounias 1968)

- $\widetilde{UB}_2 = \mathbf{S}_1 - \frac{2}{n}\mathbf{S}_2$ (Kwerel 1975; Sathe, Pradhan and Shah 1980; Platz 1985)

- $\widetilde{UB}_3 = \mathbf{S}_1 - \frac{2(2i-1)}{i(i+1)}\mathbf{S}_2 + \frac{6}{i(i+1)}\mathbf{S}_3, \quad \left(i = 1 + \lfloor \frac{3\mathbf{S}_3}{\mathbf{S}_2} \rfloor\right)$
(Kwerel 1975; B and Prékopa 1989)

- $UB_2^* \leq \mathbf{S}_1 - \sum_{(i,j)\in T} p_{i,j} \leq \widetilde{UB}_2,$ where $T$ is a spanning tree
(Hunter 1976; Worsley 1982)

- $\widetilde{UB}_4 = \mathbf{S}_1 - \frac{2i^2 - i(6-4n)+4-2n}{i(i+1)n}\mathbf{S}_2 + \frac{6(2i+n-4)}{i(i+1)n}\mathbf{S}_3 - \frac{24}{i(i+1)n}\mathbf{S}_4, \quad \left(i = 1 + \lfloor \frac{2(n-2)\mathbf{S}_2 + 3(n-5)\mathbf{S}_3 - 12\mathbf{S}_4}{(n-2)\mathbf{S}_2 - 3\mathbf{S}_3} \rfloor\right)$
(B and Prékopa 1989)

- $UB_3^* \leq \mathbf{S}_1 - \sum_{(i,j)\in\mathcal{E}} p_{i,j} + \sum_{(i,j,k)\in\mathcal{C}} p_{\{i,j,k\}} \leq \widetilde{UB}_3,$ where $(\mathcal{E},\mathcal{C})$ is a cherry tree
(Bukszár and Prékopa 2001)

# $UB_2(\mathcal{F}_N) = UB_2(\mathcal{F}_{sub})$

**Observation**. $UB_2(\mathcal{F}_N) \neq UB_2^*$

There are infinitely many examples where in the optimum we have $w_i^2 > 0$ for some $i$.

# Decomposition Bounds

$$\mathcal{F}_{dec} = \left\{ w \in \mathbb{R}^M \;\middle|\; \begin{array}{c} w_I = \sum_{i \in I} u_i^{|I|} \quad \forall\ I \subseteq \mathbf{V},\ \mathbf{1} \leq |I| \leq m \\ \text{for some } u^1, \ldots, u^m \in \mathbb{R}^n \end{array} \right\}$$

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**Theorem**. For $w \in \mathcal{F}_{dec}$ the setfunction $w(S)$ can be minimized (or maximized) over $S \subseteq \mathbf{V}$ in $O(n^2 \log n)$ time.

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**Corollary**. For any $m \geq 1$ the bounds $LB_m(\mathcal{F}_{dec})$ and $UB_m(\mathcal{F}_{dec})$ can be computed in polynomial time.

# Decomposition Bounds

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$$w(S) = \sum_{k=1}^{m} \binom{|S| - 1}{k - 1} \sum_{i \in S} u_i^k$$

# Decomposition Bounds – $UB_2(\mathcal{F}_{dec})$

$$w(S) \;=\; u^1(S) + (|S|-1)u^2(S)$$

$$\sum_{i \in \mathbf{V}} \left( p_i u_i^1 + u_i^2 \sum_{j \neq i} p_{ij} \right) \;\to\; \text{min} \;= UB_2(\mathcal{F}_{dec})$$

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**Corollary**. $UB_2(\mathcal{F}_{dec}) \;=\; \sum_{i\in\mathbf{V}} p_i - \max_{i\in\mathbf{V}} \sum_{j\neq i} p_{ij} \;=\; UB_2(Ko)$, where $UB_2(Ko)$ is the **best** bound of the type introduced by Kounias (1968).

# Summary

- $LB_m(\mathcal{F}_{dec})$ dominates all known lower bounds.

- $UB_2(\mathcal{F}_N) = UB_2(HW)$ dominates all known upper bounds for $m = 2$.

- $UB_m(\mathcal{F}_{dec})$ and $UB_m(\mathcal{F}_{sub})$ are incomparable, and dominate all known polynomially computable upper bounds.
  - $UB_3(\mathcal{F}_{sub})$, $UB_3(\mathcal{F}_{dec})$ and $UB_3(V)$ are incomparable.
  - $UB_3(V)$ of Veneziani (2002, 2008) dominates $UB_3(BP)$ of Bukszár and Prékopa (2001), and $UB_3(D)$ of Dohmen (2002); none of these are known to be polynomially computable.