1 INTRODUCTION

The dynamic analysis of transportation networks gathered increasing attention through the past years. The static models, traditionally used in this field, are in fact unable to represent relevant phenomena, such as demand variations over time and temporary over saturation of network elements. On the other hand, dynamic models are more complex than static ones: on the supply side, they require to ensure temporal consistency, besides spatial consistency, among system variables (for instance, see Cascetta, 2001); on the demand side, the departure time choice has to be modelled explicitly in order to achieve a correct representation of users’ travel behaviour (Mahmassani and Chang, 1986; Van Vuren, Daly and Hyman, 1998; Mahmassani and Liu, 1999).

Different approaches to the latter problem can be found in the literature. One of them is to regard departure time choice as a discrete choice among temporal intervals and use static assignment to characterize the utility of each interval (Daly et al, 1990); models based on this approach yield a rough representation of travel demand during day time as a sequence of static equilibria. Another approach is based on the hypothesis that departure time and path choices are made jointly, so that, for each origin-destination pair, a discrete choice set is defined who’s finite number of alternatives is equal to the number of departure intervals multiplied by the number of paths (Arnott, De Palma and Lindsey, 1990; Cascetta, Nuzzolo and Biggierio, 1992); models based on this approach require explicit path enumeration and to introduce a diachronic graph (Van der Zijpp and Lindveld, 2000), so they can be hardly applied to congested urban networks.

In this paper a Nested Logit demand model with five choice levels is presented. The model is conceived to extend to the case of elastic demand the multimodal within-day Dynamic Traffic Assignment (DTA) model proposed in Gentile, Meschini and Papola (2001). The above model doesn’t require to introduce a diachronic graph and allows implicit path enumeration. With reference to departure time choice, the proposed demand model adopts a continuous approach, thus not requiring to enumerate explicitly the desired departure time intervals. The resulting within-day DTA model is then capable of representing both supply and demand dynamic phenomena concerning congested multimodal urban networks, and leads to a fixed point formulation that can be solved by an efficient MSA algorithm applicable to real networks.
2 EQUILIBRIUM MODEL

In analogy with the static case, the within-day DTA, regarded as a dynamic user equilibrium, can be consistently formalized through a fixed point problem expressed in terms of arc flow temporal profiles, by combining the arc performance function with the Network Loading Map (NLM) and thus avoiding to introduce the Dynamic Network Loading (DNL) problem (Bellei, Gentile and Papola, 2000). To this purpose, the NLM is here extended to the case of elastic demand with departure time choice, as depicted in Figure 1.

![Figure 1]

3 CHOICE AND DEMAND MODEL

In modelling travel demand we follow the behavioural approach based on random utility theory, where it is assumed that each user is a rational decision-maker who, when making his travel choice: a) considers a positive, finite number of mutually exclusive travel alternatives constituting his choice set $J$; b) associates to each travel alternative $j$ of his choice set a perceived utility, not known with certainty, and thus regarded by the analyst as a random variable $U_j$; and c) selects the travel alternative that maximises his utility. With these hypotheses the probability of alternative $j$ is formally expressed as:
\[ P_j = \text{Prob}\left[ \cap_{k \in J} \varepsilon_k \geq V_j - V_k + \varepsilon_j \right] , \]  

where \( V_j \) and \( \varepsilon_j \) are respectively the systematic utility and the random residual of the generic alternative \( j \). The expected value of the maximum perceived utility is called satisfaction:

\[ W = E\left[ \max_{j \in J} \{ V_j + \varepsilon_j \} \right] \]  

As usual, we assume that it is possible to divide the choice process into a hierarchic sequence of decisions; at each level the user has a specific choice set, dependent on the choices made at previous levels. The utility associated to each alternative available at a given level is the sum of a specific term and of the satisfaction that takes into account the alternatives available at the successive levels.

The proposed demand model is a Nested Logit with five choice levels, namely: generation, distribution, modal split, departure time choice, and path choice.

In the case of path choice, the utility of the alternatives, beside varying over time, are related to values of arc performances taken at different instants. Then, in order to adopt an implicit path enumeration approach, we need to assume that the distribution of random residuals is constant over time; this hypothesis is essential in order to define arc conditional probabilities consistent with the path choice model (see Bellei, Gentile and Papola, 2000).

In the following sections the models related to each choice level will be shortly examined.

### 3.1 Implicit path choice model and network flow propagation model

The multimodal network is defined on a graph \( G(N, A) \), where \( N \) is the node set and \( A \) is the arc set. At any instant \( \tau' \), each mode \( m \) is characterized by specific arc flows and performances:

- \( f_a^m(\tau') \): entering flow of arc \( a \),
- \( c_a^m(\tau') \): generalized costs of arc \( a \),
- \( t_a^m(\tau') \): exit time of arc \( a \).

Referring to users travelling towards destination \( d \) on mode \( m \), the implicit path choice and the network flow propagation models require to define the following variables for every instant \( \tau' \):

- \( w_x^md(\tau') \): satisfaction of node \( x \),
- \( p_a^md(\tau') \): conditional probability of arc \( a \),
- \( d_a^md(\tau') \): actual demand flow from origin \( o \),
- \( f_a^md(\tau') \): flow on arc \( a \).

With reference to the Logit case, we have (Bellei, Gentile and Papola, 2000):

\[ w_x^md(\tau') = \theta_R \cdot \ln(\sum_{a \in FSE_{x,a}(\tau')} \exp((-c_a^m(\tau') + w_{HD(a)}^md(t_a^m(\tau'))) / \theta_R)) \]  

\[ p_a^md(\tau') = \exp((-c_a^m(\tau') + w_{HD(a)}^md(t_a^m(\tau'))) - w_{TL(a)}^md(t_a^m(\tau'))) / \theta_R) \]  

\[ f_a^md(\tau') = p_a^md(\tau') \cdot [d_{TL(a)}^md(\tau') + \sum_{b \in BSE_{a,a}(TL(a))} [f_b^md(t_b^m(\tau')) \cdot \partial t_b^m(\tau') / \partial \tau]] \]  

\[ f_a^m(\tau') = \sum_{d \in N} f_a^md(\tau') \]
where \( FSE_{md}(x) \) and \( BSE_{md}(x) \) are respectively the efficient forward and backward stars of node \( x \), while \( TL(a) \) and \( HD(a) \) are respectively the initial and final node of arc \( a \).

### 3.2 Departure time choice and actual demand models

The role of the actual demand model is to transform the desired demand temporal profile of every \( o-d \) pair and mode \( m \) into the corresponding actual demand temporal profile, on the basis of the departure time probabilities produced by the departure time choice model.

With reference to the latter model, we assume that the choice set is an appropriate neighbourhood of the desired departure instant. Note that, by considering a continuous choice set, we release one of the main hypotheses of discrete choice models (here we have an infinite number of alternatives). Moreover, we assume that the random residuals associated to each infinitesimal alternative are independently and identically distributed (i.i.d.) Gumble variables; so that the resulting choice model belongs to the Logit family. In the following we refer to users travelling from origin \( o \) to destination \( d \) by mode \( m \), then the corresponding indices will be dropped from the notation in order to improve the readability.

With reference to a desired departure instant \( \tau' \), let’s define:

\[
\begin{align*}
q(\tau') & \quad \text{desired demand flow} \\
p(\sigma)(\tau') & \quad \text{probability density function of leaving at time } \sigma \\
V(\sigma)(\tau') & \quad \text{systematic utility of leaving at time } \sigma \\
T(\tau') & \quad \text{choice set}
\end{align*}
\]

By thinking the continuous choice set \( T(\tau') \) as discretized into an infinite number of infinitesimal departure intervals, the Logit formula can be still applied to calculate the probability of choosing the generic departure interval \([\sigma-d\sigma/2, \sigma+d\sigma/2]\) as follows:

\[
\begin{align*}
p(\sigma)(\tau') \cdot d\sigma &= \frac{\exp\left(\frac{V(\sigma)(\tau')}{\theta_{DT}}\right)}{\int_{T(\tau')} \exp\left(\frac{V(x)(\tau')}{\theta_{DT}}\right) \cdot dx} \cdot d\sigma, \\
\end{align*}
\]

where, clearly, the summation is replaced by an integration. In analogy to the discrete case, the denominator in equation (7) is directly related to the departure time satisfaction:

\[
W(\tau') = \theta_{DT} \cdot \ln \left[ \int_{T(\tau')} \exp\left(\frac{V(x)(\tau')}{\theta_{DT}}\right) \cdot dx \right]
\]

We assume that the continuous choice set is the following interval:

\[
T(\tau') = [\tau' -ADV, \tau' +DEL]
\]

where \( ADV \) and \( DEL \) are respectively the maximum advance and delay accepted. Moreover, we assume that the systematic utility is given by the difference between the satisfaction of leaving at time \( \tau' \) (expressed through the satisfaction of the origin yielded by the path choice model) and a disutility term proportional to the departure advance or delay:
\( V(\sigma)(\tau') = w(\sigma) - \max\{b_{ADV}(\tau' - \sigma), b_{DEL}(\sigma - \tau')\} \)  

With these hypotheses we have:

\[
\exp \left( \frac{W(\tau')}{\theta_{DF}} \right) = \int_{x=\cdot-ADV}^{\cdot+ADV} \exp \left( w(x) - b_{ADV}(\tau' - x) \right) \cdot dx + \int_{x=\cdot}^{\cdot-DEL} \exp \left( w(x) - b_{DEL}(x - \tau') \right) \cdot dx \quad (11)
\]

We now address the problem of transforming the desired demand temporal profile into the actual demand temporal profile. The number of trips started within the infinitesimal departure interval \([\sigma - d\sigma/2, \sigma + d\sigma/2]\) is given by the integral, for each desired departure time \(\tau\) such that \(\sigma \in T(\tau)\), of the number of desired trips within the infinitesimal interval \([\tau - d\tau/2, \tau + d\tau/2]\), multiplied by the probability that the departure occurs within the interval \([\sigma - d\sigma/2, \sigma + d\sigma/2]\):

\[
d(\sigma) \cdot d\sigma = \int_{\tau=\cdot-DEL}^{\cdot+DEL} q(\tau) \cdot d\tau \cdot p(\sigma) \cdot d\sigma \quad (12)
\]

On the basis of (7), (8), (9) and (10), equation (12) becomes:

\[
d(\sigma) = \int_{\tau=\cdot-DEL}^{\cdot+ADV} q(\tau) \cdot \exp \left( w(\tau) - b_{ADV}(\tau - W(\tau)) \right) \cdot d\tau + \int_{\tau=\cdot}^{\cdot-DEL} q(\tau) \cdot \exp \left( w(\tau) - b_{DEL}(\tau - W(\tau)) \right) \cdot d\tau \quad (13)
\]

As in general the profiles \(q(\tau), W(\tau)\) and \(w(\tau)\) can take any form, the integrals in equations (11) and (13) cannot be solved in closed form; then they are to be calculated numerically. An ad-hoc procedure based on the assumption that the profile \(q(\tau)\) is piece-wise constant, while the profiles \(W(\tau)\) and \(w(\tau)\) are piece-wise linear, is presented in section 4.1.

### 3.3 Modal split model

Referring to users travelling from origin \(o\) toward destination \(d\) and to a desired departure instant \(\tau'\), let’s define:

- \(q_{od}(\tau')\) desired demand flow
- \(V_{m,od}(\tau')\) specific utility of modal alternative \(m\)
- \(P_{m,od}(\tau')\) choice probability of modal alternative \(m\)
- \(W_{od}(\tau')\) modal split satisfaction

In the Logit case, we have:

\[
W_{od}(\tau') = \theta_M \cdot \ln \left[ \sum_{m \in M} \exp \left( \frac{V_{m,od}(\tau')}{\theta_M} \right) \right], \quad P_{m,od}(\tau') = \exp \left( \frac{V_{m,od}(\tau') - W_{od}(\tau')}{\theta_M} \right) \quad (14)
\]

As usual, we assume:

\[
V_{m,od}(\tau') = \beta_M^T \cdot X_{m,od} + W_{odm}(\tau'), \quad (15)
\]

where \(X_{m,od}\) is a vector of attributes and \(\beta_M\) is the corresponding vector of coefficients, while \(W_{odm}(\tau')\) is the departure time satisfaction given by (8).

Thus, the desired demand flow on mode \(m\) at time \(\tau'\) is:
\[ q^{odm}(\tau') = q^{od}(\tau') \cdot P_{m}^{od}(\tau') \] (16)

### 3.4 Distribution model

Referring to users departing from origin \( o \) and to a desired departure instant \( \tau' \), let’s define:

- \( q^{o}(\tau') \) desired demand flow
- \( V_{d}^{o}(\tau') \) specific utility of destination alternative \( d \)
- \( P_{d}^{o}(\tau') \) choice probability of destination alternative \( d \)
- \( W^{o}(\tau') \) distribution satisfaction

In the Logit case, we have:

\[ W^{o}(\tau') = \theta_{D} \cdot \ln \left( \sum_{d \in N} \exp \left( \frac{V_{d}^{o}(\tau')}{\theta_{D}} \right) \right), \quad P_{d}^{o}(\tau') = \exp \left( \frac{V_{d}^{o}(\tau') - W^{o}(\tau')}{\theta_{D}} \right) \] (17)

We assume:

\[ V_{d}^{o}(\tau') = \beta_{D}^{T} \cdot X_{d}^{o} + W^{od}(\tau') \] , (18)

where \( X_{d}^{o} \) is a vector of attributes and \( \beta_{D} \) is the corresponding vector of coefficients, while \( W^{od}(\tau') \) is the modal split satisfaction given by (14).

Thus, the desired demand flow travelling toward destination \( d \) at time \( \tau' \) is:

\[ q^{od}(\tau') = q^{o}(\tau') \cdot P_{d}^{o}(\tau') \] (19)

### 3.5 Emission model

Referring to users potentially departing from origin \( o \) and to a desired departure instant \( \tau' \), let’s define:

- \( N^{o}(\tau) \) flow of users potentially travelling (assumed to be known)
- \( V^{o}(\tau) \) specific utility of travelling
- \( P^{o}(\tau) \) choice probability of travelling
- \( S^{o}(\tau) \) total satisfaction

In the Logit case, we have:

\[ S^{o}(\tau') = \theta_{E} \cdot \ln \left( 1 + \exp \left( \frac{V^{o}(\tau')}{\theta_{E}} \right) \right), \quad P^{o}(\tau') = \exp \left( \frac{V^{o}(\tau') - S^{o}(\tau')}{\theta_{E}} \right) \] (20)

We assume:

\[ V^{o}(\tau') = \beta_{E}^{T} \cdot X^{o} + W^{o}(\tau') \] , (21)

where \( X^{o} \) is a vector of attributes and \( \beta_{E} \) is the corresponding vector of coefficients, while \( W^{o}(\tau') \) is the distribution satisfaction given by (17). Thus, the desired demand flow at instant \( \tau' \) is:

\[ q^{o}(\tau') = N^{o}(\tau') \cdot P^{o}(\tau') \] (22)
In order to implement the proposed DTA model with departure time choice and elastic demand, it is necessary to divide the period of analysis into a sequence of temporal intervals \((\tau_{i-1}, \tau_i], i = 1, \ldots, I\). Then, each temporal profile is approximated through a function either piece-wise constant, or piece-wise linear, consistent with the above intervals, so that it is represented numerically by means of a \((I+1 \times 1)\) vector. Specifically, the temporal profiles of the flows are assumed to be piece-wise constant, while the temporal profiles of performances, satisfactions and choice probabilities are assumed to be piece-wise linear. The resulting fixed point problem formalizing the DTA is solved through the Method of Successive Averages (MSA) as outlined in the following:

\[
\begin{align*}
0) & \quad k = 0, \quad f^{k+1} = \{0 | f^\text{iniz}\} \quad \text{initialization} \\
1) & \quad k = k + 1 \quad \text{update the iteration counter} \\
2) & \quad t^k = t(f^k), \quad c^k = c(f^k, t^k) \quad \text{calculate the arc performances} \\
3) & \quad w^k = w(c^k, t^k), \quad p^k = p(w^k, c^k, t^k) \quad \text{implicit path choice model} \\
4) & \quad W_{DT}^k = W_{DT}(w^k) \quad \text{departure time choice satisfactions} \\
5) & \quad W_M^k = W_M(W_{DT}^k), \quad P_M^k = P_M(W_M^k, W_{DT}^k) \quad \text{modal split model} \\
6) & \quad W_D^k = W_D(W_M^k), \quad P_D^k = P_D(W_D^k, W_M^k) \quad \text{distribution model} \\
7) & \quad S^k = S(W_D^k), \quad P_G^k = P_G(S^k, W_D^k) \quad \text{emission model} \\
8) & \quad q^k = q(N, P_G^k, P_D^k, P_M^k) \quad \text{calculate the desired demand flows} \\
9) & \quad d^k = d(q^k, W_{DT}^k, w^k) \quad \text{calculate the actual demand flows} \\
10) & \quad y^k = \omega(p^k, t^k, d^k) \quad \text{network flow propagation model} \\
11) & \quad f^{k+1} = f^k + 1/k \cdot (y^k - f^k) \quad \text{updates the arc flows} \\
12) & \quad \text{if } \max_{a \in A} \max_{i \in I} |v_{a,i} - f_{a,i}^k| > \varepsilon \text{ and } k < k_{\text{max}} \text{ then goto 2 \ stop criterion}
\end{align*}
\]

### 4.1 Departure time choice

In this subsection, the integrals (11) and (13) are solved in closed form. Let \(\tau^{ADV}(i)\) and \(\tau^{DEL}(i)\) be the first periodization instant respectively after \(\tau^i - ADV\) and \(\tau^i + DEL\), we have:

\[
\exp\left(\frac{w'}{\theta_{\text{adv}}}\right) = \exp\left(\frac{\alpha_{\text{adv}}(i)}{\beta_{\text{adv}}(i)}\right) \left(\exp\left(\tau^{ADV}(i) \cdot \beta_{\text{adv}}(i)\right) - \exp\left((\tau^i - ADV) \cdot \beta_{\text{adv}}(i)\right)\right) + \\
\sum_{j \neq i, j < i} \exp\left(\frac{\alpha_{\text{adv}}(i)}{\beta_{\text{adv}}(i)}\right) \left(\exp\left(\tau^{ADV}(i) \cdot \beta_{\text{adv}}(i)\right) - \exp\left(\tau^{ADV}(j) \cdot \beta_{\text{adv}}(i)\right)\right) + \\
\sum_{j \neq i, j > i} \exp\left(\frac{\alpha_{\text{adv}}(i)}{\beta_{\text{adv}}(i)}\right) \left(\exp\left(\tau^{ADV}(i) \cdot \beta_{\text{adv}}(i)\right) - \exp\left(\tau^{ADV}(j) \cdot \beta_{\text{adv}}(i)\right)\right) + \\
\exp\left(\frac{\alpha_{\text{adv}}(i)}{\beta_{\text{adv}}(i)}\right) \left(\exp\left(\tau^{DEL}(i) \cdot \beta_{\text{adv}}(i)\right) - \exp\left((\tau^i + DEL) \cdot \beta_{\text{adv}}(i)\right)\right)
\]

where:

\[
\alpha_{\text{adv}} = w' - w^{i-1} \cdot \frac{\tau^i - \tau^{ADV}(i)}{\tau^i - \tau^{ADV}(i)} + \tau^{ADV}(i) \cdot b_{\text{adv}}, \quad \alpha_{\text{adv}} = w'^{-i} - w^{i-1} \cdot \frac{\tau^i - \tau^{ADV}(i)}{\tau^i - \tau^{ADV}(i)} + \tau^{ADV}(i) \cdot b_{\text{adv}}
\]

\[
\beta_{\text{adv}} = \frac{w' - w^{i-1}}{\tau^i - \tau^{ADV}(i)} \cdot b_{\text{adv}}, \quad \beta_{\text{adv}} = \frac{w'^{-i} - w^{i-1}}{\tau^i - \tau^{ADV}(i)} \cdot b_{\text{adv}}.
\]
Let $\tau^{ADV_2(i)}$ and $\tau^{DEL_2(i)}$ be the first periodization instant respectively after $\tau^{i+ADV}$ and $\tau^{i-DEL}$, we have:

$$d^t = q^{ADV_{2(i)}} \left( \exp\left( \frac{\gamma^{ADV_{2(i)}}}{\eta^{ADV_{2(i)}}} \right) \right) + \sum_{j \neq i} q^{ADV_{2(j)}} \left( \exp\left( \frac{\gamma^{ADV_{2(j)}}}{\eta^{ADV_{2(j)}}} \right) \right) + \sum_{j \neq i} q^{ADV_{2(j-1)}} \left( \exp\left( \frac{\gamma^{ADV_{2(j-1)}}}{\eta^{ADV_{2(j-1)}}} \right) \right) - \exp\left( \frac{\gamma^{ADV_{2(i+1)}}}{\eta^{ADV_{2(i+1)}}} \right) - \exp\left( \frac{\gamma^{ADV_{2(i)}}}{\eta^{ADV_{2(i)}}} \right)$$

where:

$$\gamma^{ADV_{i+1}} = \frac{w^i - w^{i+1} + \tau^i \cdot b_{ADV}}{0_{ADV}}, \quad \gamma^{ADV_{i}} = \frac{w^i - w^{i+1} - \tau^i \cdot b_{ADV}}{0_{ADV}}$$

$$\eta^{ADV} = \frac{-w^i - w^{i+1} + \tau^i \cdot b_{ADV}}{0_{ADV}}, \quad \eta^{ADV} = \frac{w^i - w^{i+1} + \tau^i \cdot b_{ADV}}{0_{ADV}}$$

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