OPTIMAL PRICING AND DESIGN OF A NEW PUBLIC
TRANSPORT SERVICE

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1. INTRODUCTION

In this work, we are concerned with modeling and solving general network
pricing problems, and will focus on the particular application of evaluating
innovative public transport services. The problem is viewed from the point
of view of the service provider; the provider controls the parameters of the
service, including, in this application, prices and service frequency.

Based on the prices that the service provider sets, the users of the net-
work will choose one or another transport mode, and along with the mode,
an itinerary will be obtained. In the case of existing bus lines, the itinerary
of the trip is given, while it is determined simultaneously with the choice of
mode for the car-based trips. This process of determining transport modes
and itineraries, where applicable, is accomplished by calculating an equilib-
rium point of the user decisions on the network. The underlying principle
of the equilibrium, due to Wardrop (1952) [6], states that, at equilibrium, the
cost on all used routes serving each origin-destination pair is equal,
and the unused routes are at higher cost.

It is well known that this definition of equilibrium, when there is only
one class of users, and furthermore the user's cost on a link depends only on
the number of users on that link, can be expressed as a convex optimization
problem.

In this setting, two differences exist in the definition of the equilibrium
that we require. The first is that the cost that each user experiences on
a link is not simply a function of the number of users on the link, giving
way to a nonlinear (and convex) delay function, but also function of the
price and frequency variables. This implies that the equilibrium problem is
parametric. The second difference is that we require multiple user classes,
as well as multiple transport modes. Consequently, there is no longer an
equivalent convex optimization problem to describe the model.

The service provider, on the other hand, seeks to optimize a profit function
that involves his two sets of control variables, prices and frequency, and
depends as well on the outcome of the parametric equilibrium problem.
In this paper, we formulate the model of the network revenue management problem as applied to the planning of an innovative public transport service. We study numerically the resulting model on a real-life application, and propose a remarkably effective method for solving it. We can demonstrate experimentally the closeness of the solutions we obtain to the global optimum of this highly nonconvex and nonlinear problem.

Both the model and the solution method used here extend readily to the optimal pricing and network design of toll roads, telecommunication networks, and internet service networks.

2. MATHEMATICAL MODEL OF THE MULTIMODAL OPTIMAL TRANSIT NETWORK DESIGN PROBLEM

We define an underlying graph, \( G = (\mathcal{N}, \mathcal{A}) \) where \( \mathcal{N} \) is the set of all nodes and \( \mathcal{A} \) the set of all arcs.

Definition and notations

- \( K \): is the set of origin-destination (OD) pairs.
- \( K^n \): is the subset of the OD pairs linked by the new bus line
- \( m \): represents the three different modes: \( b \) existing bus lines, \( n \) new bus line and \( c \) car
- \( \mathcal{R}_m^k \): is the set of routes joining OD pair \( k \) using mode \( m \). \( \mathcal{R}^k = \bigcup_m \mathcal{R}_m^k \)
- \( d_k^k \): gives the demand for each OD pair \( k \in K \)
- \( \nu_j \): discrete value of time (VOT), \( j = 1,..,V \)
- \( d_{k,j}^j \): is the portion of the demand for commodity \( k \) with VOT \( \nu_j \),
- \( h_{m,r}^k \): is the flow for commodity \( k \), VOT \( \nu_j \) and route \( r \) of mode \( m \). We consider in the case of the new bus line that there is a unique route for each \( k \in K^n \) so we write \( h_{n,r}^k \)
- \( p_b \): vector of prices for the existing bus lines
- \( p_c \): cost of using a car
- \( p_m \): vector of prices for the new bus line.
- \( t_{m,a} \): travel time on link \( a \) of mode \( m \),
- \( c_{m,r}^k \): generalized cost for commodity \( k \), VOT \( \nu_j \) and route \( r \) of mode \( m \),
- \( \pi_{m,r}^k \): minimum generalized cost for commodity \( k \) and VOT \( j \)
- \( f \): frequency for the new bus line,
- \( c_{f} \): fixed cost of putting in place another vehicle for the new bus line.

We work within a single time period, representing travel demands over one day. The operator’s control variables, service frequency, will thus be considered over one day.

The bilevel model can be formulated as follows. The lower-level problem represents a multimodal parametric traffic assignment problem, where new bus line prices and frequency are the parameters, and \( h_m \) mode flows, are the decision variables.

The bilevel optimal service and pricing problem with fixed demand is then given by:

\[
\max_{p_n,f} \sum_{k \in K^n} \sum_{j=1,...,V} h_{n, r}^k p_n^k - cf
\]
where for each value of $f$ and $p_n$ the flow $h_{m,r}^{k,j}$ is solution of the lower-level multimodal transit assignment problem

$$
\begin{align*}
    h_{m,r}^{k,j}(c_{m,r}^{k,j} - \pi^{k,j}) &= 0 \quad \forall r \in \mathcal{R}_m^k \quad \forall k \in K \quad \forall m \quad \forall j = 1, \ldots, V \\
    (c_{m,r}^{k,j} - \pi^{k,j}) &\geq 0 \quad \forall r \in \mathcal{R}_m^k \quad \forall k \in K \quad \forall m \quad \forall j = 1, \ldots, V \\
    \sum_{m} \sum_{r \in \mathcal{R}_m^k} h_{m,r}^{k,j} &= d^{k,j} \quad \forall k \in K \quad \forall j = 1, \ldots, V \\
    h_{m,r}^{k,j} &\geq 0 \quad \forall r \in \mathcal{R}_m^k \quad \forall k \in K \quad \forall m \quad \forall j = 1, \ldots, V \\
    \pi^{k,j} &\geq 0 \quad \forall r \in \mathcal{R}_m^k \quad \forall k \in K \quad \forall m \quad \forall j = 1, \ldots, V
\end{align*}
$$

(2)

For each commodity $k$ and each mode $m$ the route generalized costs depend on the travel time over the route, given by the function $t_{m,r}^k(x)$, and on the price for this route $p_{m,r}$. We assume that the travel time on a route is defined as the sum of the costs of the links defining the route and that the travel time on a link is independent of the flows on any other link in the network. The generalized cost are given by

$$
e_{m,r}^{k,j}(x) = \nu_j \left( t_{m,r}^k(x) + \xi_m(f) \right) + p_{m,r}
$$

where

$$
t_{m,r}^k(x) = \sum_{a \in A} \delta_r a m a(x_a)
$$

$$
\delta_r a = \begin{cases} 
1 & \text{if link } a \in r \\
0 & \text{otherwise,}
\end{cases}
$$

$$
x_a = (x_m a) \quad \text{with} \quad x_m a = \sum_{k \in K} \sum_{j = 1, \ldots, V} \sum_{r \in \mathcal{R}_m^k} \delta_r a h_{m,r}^{k,j}
$$

$$
\xi_m(f) = \begin{cases} 
f^{-1} & m = n \\
0 & m \neq n
\end{cases}
$$

The particular case where the travel times over links are constant $t_{m,a}(x_a) = t_{m,a}$, is the case where the generalized costs are independent of the congestion. In this case the solution of the equilibrium problem is given by

$$
h_{m,r}^{k,j} = \begin{cases} 
d^{k,j} & \text{if } m = m^* \text{ and } r = r^* \\
0 & \text{otherwise}
\end{cases}
$$

where $m^*$ and $r^*$ belong to the set

$$
\arg \min_m \min_{r \in R_m^k} c_{m,r}^{k,j}
$$

We consider this simplified assignment problem initially in calculating the service provider’s optimal investments and then follow with a congested network assignment.

3. The Algorithm

The most widely used methods are the basic subgradient, or sensitivity analysis, approaches, and the penalty method, in which the user assignment problem optimality conditions are expressed in the service provider’s problem and penalized in the objective function. In [4], it was showed that stationary points of the bilevel revenue management problem can be very far
from optimal and to overcame this problem a new global algorithm based in a decomposition of the profit function along its axes was presented. We apply this algorithm to the current optimal pricing problem. Our results are analyzed on a network of a western Paris suburban region.

References