The Network GEV model

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Discrete choice models play a major role in transportation demand analysis. Their nice and strong theoretical properties, and their flexibility to capture various situations, provide a vast topic of interest for both researchers and practitioners, that has (by far) not been totally exploited yet. The theory on Generalized Extreme Value (GEV) models has been introduced by McFadden (1978). It provides a tremendous potential, as it defines a whole family of models, consistent with random utility theory. It appears that only a few members of this family have been exploited so far, the Multinomial Logit model and the Nested Logit model being the most popular (Ben-Akiva and Lerman, 1985). Recent research on the Cross-Nested logit model (Small, 1987, Vovsha, 1997, Vovsha and Bekhor, 1998, Ben-Akiva and Bierlaire, 1999, Papola, 2000, Bierlaire, 2001a, Wen and Koppelman, 2001, Swait, 2001) has slightly extended the number of GEV models used in practice.

The most general GEV model published thus far is probably the Recursive Nested Extreme Value (RNEV) model, proposed by Daly (2001). It is an elegant generalization of the Cross-Nested logit model, where multiple layers of nests are allowed. RNEL is designed to be easily estimated, as it requires moderate extensions to nested logit estimation packages like ALOGIT (Daly, 1987) or HieLoW (Bierlaire, 1995, Bierlaire and Vandevyvere, 1995).

In this paper, we propose a new modeling approach, providing an intuitive way of generating a wide class of concrete GEV models. The idea is an extension of the use of trees to represent Nested Logit models (Ben-Akiva and Lerman, 1985, Daly, 1987). Here, we base the model definition on a network representation. The advantages of this approach are the following.

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We formally prove that any model based on a network representation complying with some simple properties is indeed a GEV model. Therefore, consistency with random utility theory is guaranteed.

A network representation allows to intuitively capture complex correlation structures of actual modeling situations. This feature, intensively exploited with trees for the Nested Logit models in the literature, can now be extended to a wide class of GEV models.

The recursive definition of the model, based on the network structure, greatly simplifies its formulation.

The main objective of the paper is to provide a general theoretical result, such that the development of new GEV models will be easier in the future. Indeed, in addition to the intuitive approach due to the network structure, any instance of the Network GEV model is proven to be a GEV model and therefore, no more theoretical justification is required for such models.

1 The GEV model

The Generalized Extreme Value (GEV) model has been derived from the random utility model by McFadden (1978). This general model consists of a large family of models that include the Multinomial Logit, the Nested Logit and the Cross-Nested Logit models. The probability of choosing alternative \(i\) within the choice set \(C\) of a given choice maker is

\[
P(i|C) = \frac{y_i \frac{\partial G}{\partial y_i}(y_1, \ldots, y_J)}{\mu G(y_1, \ldots, y_J)}
\]

where \(J\) is the number of available alternatives, \(y_i = e^{V_i}\), \(V_i\) is the deterministic part of the utility function associated to alternative \(i\), and \(G\) is a non-negative differentiable function defined on \(\mathbb{R}_+^J\) with the following properties:

1. \(G\) is homogeneous of degree \(\mu > 0\), that is \(G(\alpha y) = \alpha^\mu G(y)\),

2. \(\lim_{y_i \to +\infty} G(y_1, \ldots, y_i, \ldots, y_J) = +\infty\), for each \(i = 1, \ldots, J\),
3. the $k$th partial derivative with respect to $k$ distinct $y_i$ is non-negative if $k$ is odd and non-positive if $k$ is even that is, for any distinct indices $i_1, \ldots, i_k \in \{1, \ldots, J\}$, we have

$$(-1)^k \frac{\partial^k G}{\partial x_{i_1} \cdots \partial x_{i_k}}(x) \leq 0, \ \forall x \in \mathbb{R}^J_+.$$  \hspace{1cm} (2)

Note that the homogeneity of $G$ and Euler’s theorem give

$$P(i|C) = \frac{e^{V_i + \ln G_i(\ldots)}}{\sum_{j=1}^{J} e^{V_j + \ln G_j(\ldots)}},$$  \hspace{1cm} (3)

where $G_i = \frac{\partial G}{\partial y_i}$.

## 2 The Network GEV model

Let $(V, E)$ be a directed graph, where $V$ is the set of vertices and $E$ the set of edges. Each edge $(i, j)$ is associated with a non-negative parameter $\alpha_{(i,j)} \geq 0$, so that the directed graph is a network. The network has the following properties:

1. It does not contain any circuit.
2. It has one special node with no predecessor, called the root, and denoted by $v_0$.
3. It has $J$ special nodes with no successor, called the alternatives, and denoted by $v_1, \ldots, v_J$.
4. For each node $v_i$ in the network, there exists at least a path $(v_{i_0}, v_{i_1}), (v_{i_1}, v_{i_2}), \ldots, (v_{i_{P-1}}, v_{i_p})$ connecting $v_0 = v_{i_0}$ and $v_i = v_{i_p}$ such that

$$\prod_{k=1}^{P} \alpha_{(i_{k-1}, i_k)} > 0,$$  \hspace{1cm} (4)

that is all parameters on the path are non-zero.

We associate with each node $v_i$ of the network

- a set $I_i \subseteq \{1, \ldots, J\}$ of $J_i$ relevant alternatives,
• a homogeneous function $G^i : \mathbb{R}^J \rightarrow \mathbb{R}$, and
• an homogeneity parameter $\mu_i$.

We define $I_i = \{i\}$ for the nodes representing the alternatives, that is for $i = 1, \ldots, J$, and $I_i = \bigcup_{j \in \text{succ}(i)} I_j$ for all others. Note that we can deduce from network property 4 that $I_0 = \{1, \ldots, J\}$. The homogeneous functions are defined as follows.

$$G^i : \mathbb{R} \rightarrow \mathbb{R} : G^i(x_i) = x_i^{\mu_i} \quad i = 1, \ldots, J, \quad (5)$$

and

$$G^i : \mathbb{R}^J \rightarrow \mathbb{R} : G^i(x) = \sum_{j \in \text{succ}(i)} \alpha_{i,j} G^j(x)^{\frac{\mu_i}{\mu_j}}. \quad (6)$$

If $\mu_i \leq \mu_j$ for each edge $(i, j)$ such that $\alpha_{i,j} \neq 0$, the function $G^0$ associated with the root node, and entirely defined by the network structure, is a GEV generating function. Indeed, $G^1, \ldots, G^J$ defined by (5) trivially verify the GEV conditions. The $G^i$ functions (6) associated with all other nodes, including the root, also verify the GEV conditions, as proven by Theorem 1.

**Theorem 1** If $G^i : \mathbb{R}^J \rightarrow \mathbb{R}, i = 1, \ldots, p$ are GEV generating functions with homogeneity factor $\mu_i$, than the function $G : \mathbb{R}^J \rightarrow \mathbb{R}$, defined by

$$G(x) = \sum_{i=1}^{p} \alpha_i G^i(x)^{\frac{\mu_i}{\mu}} \quad (7)$$

is also a GEV generating function with homogeneity factor $\mu$ if the following conditions are verified:

1. $\alpha_i \geq 0, i = 1, \ldots, p$,
2. $\sum_{i=1}^{p} \alpha_i > 0$
3. $\mu > 0$,
4. $\mu_i > 0, i = 1, \ldots, p$,
5. $\mu \leq \mu_i, i = 1, \ldots, p$.

A formal proof of the theorem and discussions about practical issues will be discussed in the full paper.
References


URL: http://www.strc.ch

URL: http://rosowww.epfl.ch/mbi/biogeme


