1 INTRODUCTION

Assignment models are one of the fundamental instruments for the analysis and the design of a transportation system, in particular they are used to simulate the flows and the performances resulting from the user choice behaviour. The most common models are based on an equilibrium approach where the relevant state of the system is defined by the mutual consistency of transportation flows and costs, where flows depend on costs, through the users’ choices, and costs depend on flows, through congestion. A more general approach is based on dynamic process models, which allow the analysis of the evolution of the system in time and the convergence toward different types of attractors (a more detailed discussion in Cantarella and Cascetta, 1995).

This paper proposes an analysis of the stability of the fixed-point states, coinciding with the equilibrium states, and the possible bifurcations for a simple three-link network, through a dynamic process model.

2 ASSIGNMENT MODELS

The structure of the assignment models includes:

– the supply model, simulating how the results of user choice behaviour affect the level of service of the transportation system;
– the demand model, simulating how the supplied level of service affects the user choice behaviour;
– the supply-demand interaction model, simulating the interaction between the supply and the demand of transportation; as already mentioned in the introduction, this model can be based on equilibrium or dynamic process approaches.
2.1 Equilibrium approach: a short review

The SUPPLY MODEL for a transportation system is specified through a network (whose topology is defined by a graph made up of nodes and links) described by mathematical relations defining the consistency between the path cost vector, \( g \), and the link costs, \( c \), and between the link flow vector, \( f \), and the path flow one, \( h \):

\[
\begin{align*}
g &= \Delta^T c \quad (1) \\
f &= \Delta h \quad (2)
\end{align*}
\]

where \( \Delta \) is the link-path incidence matrix.

The network is congested if the link cost vector depends on the link flow vector, according to the cost function:

\[
c = c(f)
\]

Under the hypothesis of continuity of the first-order partial derivatives (hence of differentiability and continuity of the cost function) it is possible to define the Jacobian matrix, \( \text{Jac}[c(f)] \). If such a matrix is positive definite, the cost function is monotone strictly increasing: \([c(f') - c(f'')]^{\top} [f' - f''] > 0 \ \forall \ f' \neq f''\).

Considering, for simplicity’s sake, that the user choice dimensions concern only the path choice, let \( d_i \) be the demand value between the \( i \)th O-D pair. Moreover, it is assumed that the user behaviour is simulated by a path choice model derived from the random utility theory expressed, for the \( i \)th OD pair, by the relation, \( p_i = p(g_i) \), between the path choice probability vector, \( p_i \), and the cost one, \( g_i \). For the \( i \)th O-D pair, the DEMAND MODEL expresses the relation between the path flow vector, \( h_i \), and the cost one, \( g_i \):

\[
h_i = d_i \ p(g_i) \quad (3)
\]

Combining the expressions (1), (2) e (3) the network loading function is obtained:

\[
f = \sum_i d_i \ \Delta_i \ p(\Delta_i^T c)
\]

This function, under the mild assumptions usually adopted for path choice model (a review in Cantarella and Cascetta, 2001) is (continuous and) differentiable and monotone non-increasing: \([f(e') - f(e'')]^{\top} [e' - e''] \leq 0 \ \forall \ e', e''\); moreover, it has a symmetrical Jacobian matrix (besides being semi-defined negative), \( \text{Jac}[f(e)] = \sum_i d_i \ \Delta_i \ \text{Jac}[p(g_i = \Delta_i^T c)] \Delta_i^T \).

The equilibrium model is expressed by the relations:

\[
\begin{align*}
e^* &= c(f^*) \quad (4) \\
f^* &= f(e^*) \quad (5)
\end{align*}
\]

or by the equivalent fixed-point model: \( f^* = f(c(f^*)) \). Sufficient conditions for the existence of the fixed-point state are obtained through the Brouwer theorem requiring the continuity of the cost functions and of the network loading function. Moreover, assuming that the loading function is monotone non-increasing with respect to the link costs, if the cost function is strictly increasing a sufficient condition for the uniqueness of the fixed-point state is obtained.
2.2 Dynamic process approach: the model proposed

As regards the supply, in the dynamic process approach there are not substantial differences in the formulation of the model in comparison with the equilibrium one. It is necessary only to relate path and link costs and flows to the period \( t \) (for example the day) according to which the system is analysed in its evolution over time. The demand model needs the explicit specification of models of:

- the influence of the experience of the previous day (days) and of possible information systems on the determination of the costs of the present day (cost updating model);
- the influence of the choices made on the previous day (days) on those made on the present day (choice updating model).

Hereinafter, a dynamic process model based on exponential filters is considered (Cantarella e Cascetta, 1995, Cantarella e Velonà, 2000):

\[
\begin{align*}
\mathbf{x}_t &= \beta \mathbf{c}(\mathbf{y}_{t-1}) + (1 - \beta) \mathbf{x}_{t-1} \\
\mathbf{y}_t &= \alpha \mathbf{f}(\mathbf{x}_t) + (1 - \alpha) \mathbf{y}_{t-1}
\end{align*}
\]

(6) (7)

where

\( \mathbf{x}_t \) is the vector of the expected cost for the day \( t \);
\( \mathbf{y}_t \) is the link flow vector on the day \( t \);
\( \mathbf{c}(\cdot) \) is the cost function previously introduced, thus \( \mathbf{c}(\mathbf{y}_{t-1}) \) is the vector of the actual link costs for day \( t-1 \);
\( \mathbf{f}(\cdot) \) is the network loading function previously introduced, thus \( \mathbf{f}(\mathbf{x}_t) \) is the link flow vector for day \( t \) if all the users reconsider the choice of the previous day;
\( \beta \in (0, 1] \) is the weight attributed to yesterday’s costs;
\( \alpha \in (0, 1] \) is the probability to reconsider yesterday’s choice.

The variables defining the state of the system on the day \( t \) are \( \mathbf{x}_t \) and \( \mathbf{y}_t \). The transition function connects the state variables of the day \( t \) with the ones of the day \( t-1 \): \( (\mathbf{x}_t, \mathbf{y}_t) = \psi(\mathbf{x}_{t-1}, \mathbf{y}_{t-1}) \).

According to the non-linear dynamic system theory, the evolution of the system may convergence towards various types of attractors (brief reviews in Cantarella e Cascetta, 1995, Cantarella e Velonà, 2000):

- Fixed-point attractors: the system always takes up the same point;
- \( K \)-periodic attractors: the system takes up \( K \) points periodically;
- Quasi-periodic attractors: the system moves on a toroidal surface;
- \( A \)-periodic attractors: the system moves in a fractal set.

As already noticed in previous papers, the fixed-point states of the system (6)-(7) coincide with the equilibrium states represented by the relations (4) and (5). The study of their stability is carried out by considering the eigenvalues of the Jacobian of the transition function and, by verifying that each of them has a modulus less than one. The Jacobian matrix of the transition function \( \psi \) at the point \((\mathbf{x}, \mathbf{y})\) for the system (6)-(7) is given by:
\[
\text{Jac}[\psi(x, y)] = \begin{bmatrix}
(1 - \beta) I & \beta \text{Jac}[c(y)] \\
(1 - \alpha) \text{Jac}[f(x)] & (1 - \alpha) I + \alpha \beta \text{Jac}[f(x)] \text{Jac}[c(y)]
\end{bmatrix}
\]

It can be observed that two eigenvalues \( \lambda_k' = \lambda_k e^{\lambda_k''} = \lambda_{m+k} \) of the matrix \( \text{Jac}[\psi(x, y)] \), functions of the parameters \( \alpha \) and \( \beta \), can be defined for each of the \( n \) eigenvalues \( \gamma_k \) of the matrix \( \text{Jac}[f(x)] \text{Jac}[c(y)] = J_f J_c : \)

\[
\begin{align*}
\lambda_k' &= [(1 - \beta) + (1 - \alpha) + \alpha \beta \gamma_k - (\chi_k)^{1/2}] / 2; \\
\lambda_k'' &= [(1 - \beta) + (1 - \alpha) + \alpha \beta \gamma_k + (\chi_k)^{1/2}] / 2;
\end{align*}
\]
where \( \chi_k = [(1 - \beta) + (1 - \alpha) + \alpha \beta \gamma_k]^2 - 4 (1 - \beta) (1 - \alpha). \)

The condition of stability can be defined according to the eigenvalues of the matrix \( J_f J_c, \gamma_k \), and it is represented by the interior of an ellipse on the Argand plane, depending on the parameters \( \alpha \) and \( \beta \) (Cantarella e Cascetta, 1995).

### 3 APPLICATIONS

A three-link network was considered. A cost function was associated to each link; in particular, separable cost functions were associated to the first and third link and a non-separable function was associated to the second link, with \( Q \) being the link capacity:

- **link 1** \( c_1 = c_{01}[1 + k_1(f_1 / Q_1)^k] \)
- **link 2** \( c_2 = c_{02}[1 + k_3 f_2 + ((f_2 + f_1) / Q_2)^k] \)
- **link 3** \( c_3 = c_{03}[1 + k_6 (f_3 / Q_3)^k] \)

In this way (see Cantarella, 1997) it would be possible to observe periodic and quasi-periodic attractors. A Logit model was used for simulating the path choice behaviour:

\[
p_i(c) = \frac{\exp(-\theta c_i)}{\sum_j \exp(-\theta c_j)}
\]
where \( \theta > 0 \) is inversely proportional to the standard deviation \( \sigma \) of the random residual: \( \theta = \pi / (6^{1/2} \sigma) \approx 1.282 / \sigma. \)

Given the demand value \( d \), the demand level \( m \) can be introduced, equal to the ratio between the maximum of the link capacities and the demand value.

Two sets of parameters of the cost functions were considered, both of them assuring one of the conditions of uniqueness of the fixed-point states, that is, the Jacobian of the cost functions is defined positive (the other condition, that the Jacobian of the loading function is semi-defined negative, is verified because the path choice model is a Logit one).

### Setting 1

Table 1 shows the specification of the parameters and of the attributes used in this first simulation.
Tab. 1 – Values of the parameters and of the attributes

<table>
<thead>
<tr>
<th>Link</th>
<th>$c_0$</th>
<th>$k_1$</th>
<th>$k_2$</th>
<th>$k_3$</th>
<th>$k_4$</th>
<th>$k_5$</th>
<th>$k_6$</th>
<th>$k_7$</th>
<th>$Q$ (vehic/h)</th>
</tr>
</thead>
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<tr>
<td>1</td>
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<td>1.5</td>
<td>1</td>
<td></td>
<td></td>
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<td>25</td>
<td></td>
<td>0.001</td>
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<td></td>
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<td>2000</td>
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<td></td>
<td>1</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td>1200</td>
</tr>
</tbody>
</table>

(m = 0.4, d = 800 vehicles)

For each value of the Logit parameter $\theta$, the couple of eigenvalues of the corresponding matrix $J_f J_c$ can be represented on the complex plane (Fig. 1). With parameters of table 1, when the two eigenvalues form a complex conjugate pair (with imaginary part different from zero) they are represented by two points always in the interior of the ellipse of stability.

Moreover, when the parameter $\theta$ value becomes greater than the value $\theta = 0.185$, the two eigenvalues of $J_f J_c$ become real and one of them tends to leave the ellipse of stability, whereas the other tends to the origin. This means that the passage from stability to instability occurs through a flip bifurcation (Cantarella, 1997) and, therefore, through the appearance of a periodic attractor (Fig. 2).

Figure 1. Representation of the eigenvalues of the matrix $J_f J_c$ in relation to the ellipse of stability.

Figure 2. Bifurcation for the variable $x_1$ (flow on the link 1), with respect to parameter $\theta$ (left); representation of the attractor on the plane of the states $(x_1, x_2)$ for $\theta = 0.2$ (right).

**Setting 2**

Table 2 shows the specification of the parameters and the attributes used in this second simulation.
Tab.2 – Values of the parameters and of the attributes

<table>
<thead>
<tr>
<th>Link</th>
<th>$c_0$</th>
<th>$k_1$</th>
<th>$k_2$</th>
<th>$k_3$</th>
<th>$k_4$</th>
<th>$k_5$</th>
<th>$k_6$</th>
<th>$k_7$</th>
<th>$Q$ (vehic/h)</th>
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<tbody>
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<td>2</td>
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<td>1200</td>
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<td></td>
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</tr>
</tbody>
</table>

(m = 0.9, d = 800 vehicles)

By carrying out the analysis on the complex plane (Fig. 3), with parameters of table 2 the eigenvalues with an imaginary part different from zero are not always in the interior of the ellipse of stability.

![Fig. 3. Representation of the eigenvalues of the matrix $J_f J_c$ in relation to the ellipse of stability.](image)

When the parameter $\theta$ value increases, the eigenvalues of $J_f J_c$ tend to leave the ellipse of stability ($\theta = 0.012$), still with the imaginary part different from zero. In this case, a Neimark bifurcation can be observed (Cantarella, 1997, Cantarella e Velonà, 2000) and, therefore, the instability of the fixed-point states leads to the convergence toward a quasi-periodic attractor (Fig. 4).

![Fig. 4. Bifurcation for the variable $x_1$ (flow on the link 1), with respect to parameter $\theta$ (left); representation of the attractor on the plane of the states ($x_1, x_2$) for $\theta = 0.015$ (right).](image)
4 CONCLUSIONS

In this paper a dynamic process model is applied to a three-link network in order to discuss the fixed-point state stability with respect to the eigenvalues of the product of the cost and network loading function Jacobian matrices. The theoretical remarks proposed in previous papers (Cantarella e Cascetta, 1995, Cantarella, 1997) about the quality of the bifurcations, concerning the type of eigenvalues through which a system leaves the stability domain, are confirmed by the application carried out.

QUOTED REFERENCES


OTHER REFERENCES


