LYAPUNOV STABILITY OF DISTRIBUTED QUEUE EQUILIBRIA IN STOCHASTIC SIDE CONSTRAINED ASSIGNMENT MODELS

Leonid Engelson
Centre for Traffic Simulation Research
Royal Institute of Technology
SE-100 44 Stockholm, Sweden
E-mail: lee@infra.kth.se

1 INTRODUCTION

Traditional static network equilibrium models suggest that delay caused by travelling along a road link only depends on the traffic volume on the link and that the volume-delay function is increasing, convex and defined for all non-negative volumes. Hence the link capacity restrictions and queuing delays are usually ignored. One of the ways of improving the quality of a traffic assignment model is to include explicit restrictions on link flows.

The concept of Queue Equilibrium (QE) for capacitated traffic networks was introduced by Thompson and Payne (1975) and further exploited by Smith (1987). This concept introduces fixed link flow restrictions and queue delays in the well known Wardrop principle of user equilibrium. Bell (1995) considered stochastic user equilibrium for capacitated networks and showed that link delays can be obtained as Lagrange multipliers for the capacity restrictions at equilibrium both under deterministic and under logit based stochastic assignment.

Larsson and Patriksson (1999) generalised capacitated network equilibrium by introducing side constrains into optimisation problem that corresponds to the Wardrop conditions. A side constraint may involve several link flows and can be used to describe e.g. interaction of vehicles in a junction or joint capacity of a two-way street. Larsson and Patriksson (1999) showed that solution to the problem with side constraints can also be characterised as queue equilibrium with properly chosen link capacities.

In the real congested transportation systems, queues are never constant. Therefore it is important to investigate evolutions of queues, in particular Lyapunov stability of QE. Only stable equilibria are interesting from the practical point of view because unstable ones are not probable to be observed. Stability of QE deserves specific interest in the context of real time driver information since the information provides a system feedback from queue length to path flows.
Engelson (2002) studied dynamic properties of QE under assumption that the drivers adjust their paths en route using information about instantaneous queuing delays. He proved existence of a queue equilibrium under natural assumptions about drivers’ route choice and proposed a method for investigating Lyapunov stability of queue equilibria based on the theory of Projected Dynamical Systems (see Nagurney and Zhang, 1996) transferred to retarded equations (see e.g. Hale and Lunel, 1993).

In this paper, some results by Engelson (2002) are extended from link capacity constraints to the side constraints. The paper introduces a dynamical system in the space of distributed queuing delays corresponding to the side constraints. The feedback is provided by path flows that vary due to drivers’ response to the information about queues. Steady states of the system correspond to distributed queue equilibria. Sufficient condition for existence of an equilibrium is presented. Although the dynamic system involves non-continuous feedback due to hard constraints, stability of the equilibria can be investigated via stability of lower dimension dynamical system obtained by projection of the original system onto minimal face of the feasible set containing the equilibrium.

2 THE STATIC MODEL

Let the network be represented by nodes $n \in N$ and links $a \in A$. The car travel demand is represented by a fixed vector of origin-destination (OD) flows $d \in \mathbb{N}_+^{|W|}$ where $W \subset N \times N$ is the set of OD pairs. Denote $R_w$ the set of paths connecting OD pair $w \in W$ and $h^w_r$ the flow along path $r \in R_w$. Let the travel cost of the link $a$ consist of the constant undelayed travel time $v_a$ and the queuing delay $q_a$. Hence the travel cost of the path is

$$C^w_r = \sum_{a \in A} (v_a + q_a) \delta^w_{ra} \quad \forall r \in R_w, \forall w \in W$$

where $\delta^w_{ra}$ is equal to 1 if link $a$ lies on path $r$ and 0 otherwise.

Assume that the allocation of trips between alternative paths conforms to the following logit model

$$h^w_r = d_w \frac{\exp(-\alpha C^w_r)}{\sum_{s \in R_w} \exp(-\alpha C^w_s)} \quad \forall r \in R_w, \forall w \in W$$

where $\alpha > 0$ is a given parameter. Link flows resulting from this assignment are
Consider a set of linear side constraints
\[ \langle G^k, f \rangle \leq u_k, \quad k \in K \]  (4)
where \( G^k = \{g^k_a\} \in \mathbb{R}_+^{|A|} \) are fixed vectors, \( u_k \geq 0 \) are fixed numbers and \( f \in \mathbb{R}_+^{|A|} \) is a vector of link flows. Each constraint may give rise to a distributed queuing delay \( y_k \geq 0 \) whereby the link queuing delays are calculated as
\[ q_a = \sum_{k \in K} \frac{g^k_a}{\sum_{b \in A} g^k_b} \quad \forall a \in A. \]  (5)

**Definition** (adapted from Larsson and Patriksson, 1999). The vector \( y \in \mathbb{R}_+^{|K|} \) is said to be a distributed queue equilibrium (DQE) if the link flows resulting from (5), (1), (2), (3) satisfy constraints (4) and \( y_k = 0 \) for those constraints \( k \) that are satisfied strictly.

Consider the optimisation problem
\[
\min \{ f_a = \sum_{w \in W} \sum_{r \in R_w} \delta^w_{ra} h^w_r \quad \forall a \in A \}. \]  (3)
subject to constraints
\[ \sum_{r \in R_w} h^w_r = d_w \quad \forall w \in W, \]  (7)
the non-negativity constraints \( h^w_r \geq 0 \), and the side constraints (4), where the flow vector \( f \) is defined by formula (3).

If the feasible set is not empty, then the optimisation problem has a solution. Analysing the Kuhn-Tucker conditions, one concludes that
\[ y_k = \alpha \beta_k \sum_{a \in A} g^k_a \]  (8)
where \( \beta_k \) are Lagrange multipliers corresponding to the side constraints (4) is a DQE. This result is an extension of Proposition 3 in Bell (1995).

The problem of finding DQE can also be presented as variational inequality problem
where \( y \) is a vector of distributed delays and the vector function \( H : \mathbb{R}^{|K|}_+ \rightarrow \mathbb{R}^{|K|}_+ \) has components

\[
H_k(y) = \sum_{a \in A} f_a(y) g_a^k / u_k - 1
\]

where \( f_a(y) \) are defined via formulae (5), (1), (2) and (3).

### 3 THE DYNAMIC MODEL

Here we consider the network with variable queuing delays. The car travel demand is assumed constant. The path and flow rates vary due to drivers response to the information about path travel times.

Let us call **efficient path** a path that does not contain more than one link involved in side constraints. Denote \( E_w \) the set of efficient paths connecting OD pair \( w \in W \) and assume that \( E_w \neq \emptyset \ \forall w \in W \). Further assume that the drivers at the origins of their trips distribute themselves among the efficient paths to their destinations according to the some continuous path choice model (e.g. logit model) that determines path proportions from the path travel costs \( C_r^w(t) \) based on the instantaneous link queuing delays \( q_a(t) \). At the each node, the route choice is revised according to a similar model but the updated queuing delays are used to calculate the remaining path travel costs. The path choice sets are such that no driver can traverse more that one link involved in side constraints. It is assumed that the eventual queues on links involved in the side constraints are situated at the ends of the links.

Under these assumptions, the flow rate at time \( t \) at the end of each link can be expressed as a function of past states of the queuing delays. Moreover, as drivers only use efficient paths, the flow rate \( j_a(t) \) of vehicles joining the queues on link \( a \) depend just on queuing delays \( q_b(t - \tau_m), b \in A, m = 1, \ldots, M \), where the lags \( \tau_m \) are fixed and positive.

The side constraint (4) means that at most \( u_k / g_a^k \) vehicles can leave the distributed queue from link \( a \) per time unit. This implies that the queuing delay \( y_k \) changes with the rate

\[
\sum_{a \in A} j_a(t) g_a^k / u_k - 1 \quad \text{unless this expression is negative and } y_k = 0. \quad \text{In the latter case, } y_k \text{ remains zero.}
\]

The dynamic model can be written as a projected dynamical system (see Nagurney and Zhang, 1996)
\[ \dot{y}_k = \pi_+ \left( y_k, \sum_{a \in A} j_a g_a^k / u_k - 1 \right) \quad \forall k \in K \tag{10} \]

where the (not continuous by \( y \)) function \( \pi_+ (y, x) \) is equal to \( x \) if \( y > 0 \) or \( x \geq 0 \), and 0 otherwise.

As the flow rates \( j_a \) depend on past values of delays \( y_k \), the equation (10) is a retarded equation which is not considered by Nagurney and Zhang (1996). Engelson (2002) proved existence and uniqueness of solutions for this type of equations.

Assume that the following consistency condition is fulfilled for path choice sets concerning the vehicles travelling between each OD pair \( w \in W \). If a path is in the choice set from a node and it starts with link \( a \) then the same path (without link \( a \)) is available from the end node of link \( a \).

Moreover, assume that the initial and the intermediate path choices are governed by the logit model with the same parameter \( \alpha \). These assumptions guarantee that, under constant queuing delays, the vehicles follow the path chosen in the beginning of the trip according to the formula (2).

The steady states of system (10) are exactly the solutions to the variational inequality problem (9), i.e. the distributed queue equilibria. This allows to use the system (10) for study of dynamic properties of DQE.

Let \( Y^* \) be a regular solution to the problem (9), i.e. the components of \( Y^* \) that correspond to side constraints turning into equalities at the equilibrium are positive. Then Lyapunov stability of the system (10) can be investigated via the Minimal Face Flow (MFF) which is a dynamic system of lower dimension than (10) and is described by retarded equations with continuous right hand sides.

Having written the system of equations (10) in the vector form

\[ \dot{y}(t) = \Pi_+ (y(t), F(y(t - \tau_1), ..., y(t - \tau_M))) \tag{11} \]

one can define the corresponding MFF by the equation

\[ \dot{z}(t) = P_{S(Y^*)} (F(z(t - \tau_1) + Y^*, ..., z(t - \tau_M) + Y^*)) , \quad z \in S(Y^*) \tag{12} \]

where \( P_{S(Y^*)} \) is the orthogonal projection in \( \mathbb{R}^{|K|} \) onto subspace

\[ S(Y^*) = \{ Y = [y_1, ..., y_{|K|}] : y_k = 0 \ \forall k \in I_0(Y^*) \} \] where \( I_0(X) = \{ b : x_b = 0 \} \).

Engelson (2002) has proved that (asymptotic) stability of equation (11) at steady point \( Y^* \) is equivalent to the (asymptotic) stability of equation (12) at steady point 0. This result considerably simplifies the stability analysis. In particular, the stability can be investigated by the first order criterion using the condition that the corresponding characteristic quasi-polynomial does not have roots with non-negative real parts. This makes it possible to find stability areas in the term of such parameters of the transportation systems as lengths of road links, green times, capacities, travel demand, drivers’ sensitivity to queuing time, share of drivers obtaining the queuing delay information etc.
4 CONCLUSION
Apart from existence, uniqueness and calculation methods of traffic equilibria, the stability properties are important both from the practical and from the theoretical point of view. The dynamic stability is difficult to investigate because of high complexity of interactions in the transportation network and complicated time propagation. The results presented in this paper provide a theoretical justification for using the first order methods in stability analysis of traffic network with side constraints. Additional research is needed to relax the assumptions imposed in this paper on the network topology.

REFERENCES
Engelson L. (2002). On Dynamics of Traffic Queues in a Road Network with Route Choice Based on Real Time Traffic Information. Accepted for publication in Transportation Research, Part C.