Equilibrium traffic assignment models exist to deal with a range of observational uncertainties. Perhaps the most well known is the stochastic user equilibrium model, incorporating random perceptual terms to represent components of drivers’ disutility that are not explained through observed attributes (Sheffi, 1985). The uncertainty represented by such a model is from the viewpoint of the modeller; it provides no hypothesis as to how the drivers in the network might themselves consider uncertainty. This latter point is the focus of the present paper. In the context of traffic assignment, the primary source of uncertainty for the individual driver is surely the fact that for any particular trip, the driver will not know in advance the precise traffic conditions he/she would encounter on the alternative routes available. This uncertainty derives from the ambient trip-to-trip variation in travel times, both within and between days, due to factors such as incidents, breakdowns, weather and the variability in activity patterns. These factors lead to variations in, among other things, flows and capacities, which in turn impact on travel times.

At one level, empirical work is clearly needed to address how drivers respond to such variability in particular case-study situations (see, for example, Abdel-Aty et al, 1997; De Palma & Rochat, 1999). However, in order to have practical value, such empirical work needs appropriate hypothetical models with adjustable parameters, in order that the behavioural information may be exploited in a network assignment model. The purpose of the present paper is to present such a hypothetical model. The development of such a model has particular relevance for a number of reasons, ranging from applications in the emerging topic of road network reliability, to methods for quantifying the negative societal impacts of variability, and understanding the part driver information systems may have in mitigating this effect.

The research presented in this paper builds on the work of Noland (1999), where the effect was considered of stochastic travel times on a binary route choice problem. In particular the implication for general network problems is considered here, and more plausible/flexible travel time density functions are introduced. The approach links the additional disutility of variable travel times to scheduling considerations of the drivers, in meeting some preferred arrival time. A illustrative example is used to illustrate the technique.
2. Model Formulation

The approach is based on the ‘schedule delay’ concept proposed by Vickrey (1969), which has enjoyed some considerable use in the transportation field. Its use has primarily been in the context of travellers’ departure time choice decisions (e.g. Arnott et al., 1990); even though a number of studies have additionally included route choice, the scheduling considerations have been assumed not to affect route choice once a departure time has been chosen. In addition, its use has been mainly in a deterministic travel time setting.

Consider a particular individual on a given inter-zonal movement, with a fixed departure time \(d\) from their trip origin. It is supposed that the individual has in mind a range of times \([a_E(d), a_L(d)]\) within which it would be acceptable to arrive at the destination, given their already-chosen departure time \(d\). The dependence on \(d\) signifies the individual may take a different view of what is acceptable, depending on the time of day at which they are travelling (this is the reason for terming them acceptable as opposed to preferred arrival times, the latter not conditioning on the departure time). These assumptions imply a range of acceptable travel times \([a_E(d)−d, a_L(d)−d]\), which we shall assume are independent of \(d\), and denoted \([\tau_E, \tau_L]\).

The next component is to assume that the individual has a view on the variability in travel times. Indeed, they are assumed to possess such information that they perceive (or can be represented as perceiving) a full joint probability distribution for the route travel times, with the random variable \(T_r\) denoting the travel time on route \(r\). The perceived generalized travel cost of route \(r\) is assumed to be \(\theta_1 + \theta_2 E[T_r]\), for some coefficients \(\theta_1\) and \(\theta_2\), where \(\theta_1\) represents the combined effect of all attributes that do not depend on travel time. In a conventional traffic assignment model, it would be precisely these generalized costs that would be used to determine route choices. However, here we suppose that individuals form a composite disutility \(u_r\) for each route \(r\), comprising both the generalized travel cost and the expected degree to which the travel time may be unacceptable:

\[
 u_r = \theta_1 + \theta_2 E[T_r] + \theta_3 E[\max(0, \tau_E - T_r)] + \theta_4 E[\max(0, T_r - \tau_L)] . 
\]  

(2.1)

In (2.1), \(\theta_4\) reflects the value of being later than acceptable, and clearly will generally be valued negatively (implies \(\theta_4 > 0\) since \(u_r\) a disutility). In contrast, \(\theta_3\) reflects the value of being earlier than acceptable. In the analogous problem of departure time choice, empirical evidence does indeed suggest that late and early arrival is valued differently, but that the magnitude of the early-arrival coefficient is somewhat less than the late-arrival one (Small, 1982). In our present context this also seems reasonable, and indeed one may argue that (since departure times are fixed) individuals might be almost indifferent to early-arrival (\(\theta_3 = 0\)).
Let us suppose, then, that *link* travel times are jointly distributed as a multivariate Normal distribution, then by a convolution the route travel times will also be multivariate Normal, implying univariate Normal marginal route travel times, say \( T_r \sim \text{Normal}(\mu_r, \sigma_r^2) \). Then it may be shown that (2.1) becomes:

\[
    u_r = \theta_1 + \theta_2 \mu_r + \theta_3 \sigma_r L \left( -\frac{\tau_E - \mu_r}{\sigma_r} \right) + \theta_4 \sigma_r L \left( \frac{\tau_L - \mu_r}{\sigma_r} \right) \tag{2.2}
\]

where the function \( L \) is the unit normal linear loss integral:

\[
    L(x) = \int_x^\infty (u-x) \phi(u) \, du = \phi(x) + x \Phi(x) - x \quad (-\infty < x < \infty) \tag{2.3}
\]

where \( \phi() \) and \( \Phi() \) are respectively the density function and cumulative distribution function of a Normal(0,1) variate. It may be verified that (2.3) operates in a plausible way. For example, the expected late arrival time increases with a decrease in \( \tau_L (> \mu_r) \), reflecting less late arrival flexibility. It is also notable that the utility is non-linear in \( \mu_r \) and \( \sigma_r \), since the ‘loss’ terms are themselves functions of \( \mu_r \) and \( \sigma_r \) (though a linear simplification could be achieved with route-specific acceptable travel times, at the loss of a direct link to scheduling).

This modified route disutility may then be embedded in a user equilibrium assignment model, by assuming that the variance terms are fixed (independent of flow), and equilibrating link flows. The link flows impact on mean link travel times through the link performance functions, whereby revised mean route travel times may be input to (2.2). An equilibrium state prevails when all used routes have equal disutility, this being less than or equal to the disutility on any unused route. Such an equilibrium will be termed a Schedule-Based User Equilibrium (SBUE). In the special case that \( \theta_3 = \theta_4 = 0 \), there is no value placed on early/late arrival, and SBUE collapses to a standard User Equilibrium (UE). Similarly, as \( \tau_L \to \infty \) and \( \tau_L \to \infty \), then even for non-zero \( \theta_3 \) and \( \theta_4 \), SBUE approaches UE as drivers increase in flexibility. In the example below, it can be seen how an increasing travel time standard deviation on path 1 leads, in the SBUE model, to an increasingly deterrent effect on the flow using path 1, this flow substantially different from the UE solution. However, this result also depends on the late arrival flexibility \( \tau_L \); as \( \tau_L \) is increased the SBUE graph will approach the UE line, as the perceived lateness penalty diminishes.

### 3. Equilibrium With Mixture Distributions

Assuming multivariate Normal link travel times allows quite a general specification of travel time variability, yet it also comes with some disadvantages. In particular, there is the implicit
assumption of symmetry, whereas many sources of variability (in traffic flows and link capacities, for example) will likely have an asymmetric effect on travel times, yielding a positively skewed distribution. This would occur even if, say, flows followed a symmetric distribution, due to the non-linear relationship with travel times. One approach to this problem is to assume that flows are themselves random variables, and allow the variability in both flows and travel times to be endogenous to the model. While generalised equilibrium models of this kind have been proposed (Watling, 2002), they have yet to be linked to scheduling considerations, and we shall not pursue this avenue in the present paper.

Instead, we shall continue to assume exogenously-specified travel time densities. In particular, we shall suppose that the travel time on a link follows a Normal mixture distribution. That is to say, we define a number of Normal distributions which the travel times follow conditional on some given probability levels. Then the joint route travel times density will be a mixture of multivariate Normals, with the marginal route travel time densities univariate Normal mixtures. If the resulting travel time density for route $r$ is then a mixture of $m$ Normal($\mu_{ir}, \sigma_{ir}^2$) densities ($i = 1,2,\ldots,m$) with mixing parameters $p_{ir}$ ($i = 1,2,\ldots,m$), then the generalisation of (2.2) is:

$$u_r = \theta_1 + \theta_2 \sum_{i=1}^{m} p_{ir} \mu_{ir} + \theta_3 \sum_{i=1}^{m} p_{ir} \sigma_{ir} L \left( \frac{\tau - \mu_{ir}}{\sigma_{ir}} \right) + \theta_4 \sigma_{ir} \sum_{i=1}^{m} p_{ir} \sigma_{ir} L \left( \frac{\tau - \mu_{ir}}{\sigma_{ir}} \right).$$

(3.1)

4. **EXAMPLE: EQUILIBRIUM WITH LINK INCIDENTS**

An application of the generalised model in section 3 is considered here. With probability $p_a$ an incident occurs on link $a$. When an incident does not occur, the travel time density is Nor($\mu_a, \sigma_a^2$) (by an abuse of notation, we here allow $\mu$ and $\sigma$ to denote properties of a link, rather than a path). When an incident does occur, the density is Nor($k_a \mu_a, \sigma_a^2$), for some given $k_a \geq 1$. The overall mean travel time for each link $a$ is then $(1 - p_a) \mu_a + p_a k_a \mu_a$, and it is these values that are equilibrated with the link travel time functions in the generalised SBUE model.

![Network 1](image)

$T_1 \sim \text{Nor}(8+\left(f_1/200\right)^2, \sigma_1^2)$

$T_2 \sim \text{Nor}(10+f_2/350, 4)$

$\theta_1 = 0, \theta_2 = 1, \theta_3 = 0, \theta_4 = 2, \tau_L = 15.$
For the network illustrated, there are three paths:

The UE path flows are (300, 100, 600) at path travel times of 22.0. For pure Normal link travel times, SBUE path flows are (338, 0, 662) at disutilities (22.0, 22.1, 22.0). Therefore the active paths are different in the UE and SBUE cases. Note also that in the SBUE state, the path mean travel times are unequal (20.9, 20.7, 21.9), with the unused path actually having the lowest mean time (but unappealing as it uses two ‘risky’ links). For the same example, but assuming incidents now occur on link 1 with probability $p_1 = 0.1$, the graph below illustrates mean travel times for the two SBUE used paths at varying values of $k_1$, the inflation factor for mean travel times. While an increase in $k_1$ would *increase* the path 1 mean travel time at *fixed* flows, the SBUE model predicts that drivers would mitigate this effect by diverting away from the risky route, to the extent that the *equilibrium* path 1 travel time *decreases* with $k_1$. 
REFERENCES


