Submodular Functions, Matroids, and Certain Polyhedra*

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I

The viewpoint of the subject of matroids, and related areas of lattice theory, has always been, in one way or another, abstraction of algebraic dependence or, equivalently, abstraction of the incidence relations in geometric representations of algebra. Often one of the main derived facts is that all bases have the same cardinality. (See Van der Waerden, Section 33.)

From the viewpoint of mathematical programming, the equal cardinality of all bases has special meaning — namely, that every basis is an optimum-cardinality basis. We are thus prompted to study this simple property in the context of linear programming.

It turns out to be useful to regard "pure matroid theory", which is only incidentally related to the aspects of algebra which it abstracts, as the study of certain classes of convex polyhedra.

(1) A *matroid* $M = (E, F)$ can be defined as a finite set E and a nonempty family F of so-called *independent* subsets of E such that

(a) Every subset of an independent set is independent, and
(b) For every $A \subseteq E$, every maximal independent subset of A, i.e., every *basis* of A, has the same cardinality, called the *rank*, $r(A)$, of A (with respect to M).

(This definition is not standard. It is prompted by the present interest).

(2) Let $\mathbb{R}_E$ denote the space of real-valued vectors $x = [x_j]$, $j \in E$.
Let $\mathbb{R}_E^+ = \{x : 0 \leq x \in \mathbb{R}_E\}$.

(3) A *polymatroid* P in the space $\mathbb{R}_E$ is a compact non-empty subset of $\mathbb{R}_E^+$ such that

(a) $0 \leq x^0 \leq x^1 \in P \implies x^0 \in P$.
(b) For every $a \in \mathbb{R}_E^+$, every maximal $x \in P$ such that $x \leq a$, i.e., every *basis* x of a, has the same sum $\sum_{j \in E} x_j$, called the rank, $r(a)$, of a (with respect to P).

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Here *maximal* x means that there is no $x' > x$ having the properties of x.

(4) A polymatroid is called *integral* if (b) holds also when a and x are restricted to being integer-valued, i.e., for every integer-valued vector $a \in \mathbb{R}_E^+$, every maximal integer-valued x, such that $x \in P$ and $x \leq a$, has the same sum $\sum_{j \in E} x_j = r(a)$.

(Sometimes it may be convenient to regard an *integral polymatroid* as consisting only of its integer-valued members).

(5) Clearly, the 0–1 valued vectors in an integral polymatroid are the "incidence vectors" of the sets $J \in F$ of a matroid $M = (E, F)$.

II

(6) Let f be a real-valued function on a lattice L. Call it a β_0-function if

(a) $f(a) \geq 0$ for every $a \in K = L - \{\emptyset\}$;

(b) is non-decreasing: $a \leq b \implies f(a) \leq f(b)$; and

(c) submodular:

$$f(a \vee b) + f(a \wedge b) \leq f(a) + f(b)$$

for every $a \in L$ and $b \in L$.

(d) Call it a β-function if, also, $f(\emptyset) = 0$. In this case, f is also subadditive, i.e., $f(a \vee b) \leq f(a) + f(b)$.

(We take the liberty of using the prefixes *sub* and *super* rather than "upper semi" and "lower semi". *Semi* refers to either. The term *semi-modular* is taken from lattice theory where it refers to a type of lattice on which there exists a semimodular function f such that if a is a maximal element less than element b then $f(a) + 1 = f(b)$. See [1].)

(7) For any $x = [x_j] \in \mathbb{R}_E$, and any $A \subseteq E$, let $x(A)$ denote $\sum_{j \in A} x_j$.

(8) **Theorem.** *Let L be a family of subsets of E, containing E and $\emptyset$, and closed under intersections, $A \cap B = A \wedge B$. Let f be a β_0-function on L. Then the following polyhedron is a polymatroid:*

$$P(E, f) = \{x \in \mathbb{R}_E^+ : x(A) \leq f(A) \textit{ for every } A \in L - \emptyset = K\}.$$

Its rank function r is, for any $a = [a_j] \in \mathbb{R}_E^+$,

$$r(a) = \min \left(\sum_{j \in E} a_j z_j + \sum_{A \in K} f(A) y_A \right)$$

where the z_j's and y_A's are 0's and 1's such that for every $j \in E$,

$$z_j + \sum_{j \in A \in K} y_A \geq 1.$$

Where $f(\emptyset) \geq 0$, only one non-zero y_A is needed.

Where f is integer-valued, $P(E, f)$ is an integral polymatroid.

(9) **Theorem.** *A function f of all sets $A \subseteq E$ is itself the rank function of a matroid $M = (E, F)$ iff it is an integral β-function such that $f(\{j\}) = 1$ or 0 for every $j \in E$. Such an f determines M by:*

$$J \in F \iff J \subseteq E \text{ and } |J| = f(J).$$

(10) For any $a = [a_j] \in \mathbb{R}^+_E$ and $b = [b_j] \in \mathbb{R}^+_E$, let $a \vee b = [u_j] \in \mathbb{R}^+_E$ and $a \wedge b = [v_j] \in \mathbb{R}^+_E$, where

$$u_j = \max(a_j, b_j) \quad \text{and} \quad v_j = \min(a_j, b_j).$$

(11) **Theorem.** *The rank function $r(a)$, $a \in \mathbb{R}^+_E$, for any polymatroid $P \subset \mathbb{R}^+_E$, is a β-function on $\mathbb{R}^+_E$ relative to the above $\vee$ and $\wedge$.*

(12) For any $x = [x_j] \in \mathbb{R}^+_E$ and any $A \subseteq E$, let $x/A = [(x/A)_j] \in \mathbb{R}^+_E$ denote the vector such that $(x/A)_j = x_j$ for $j \in A$, and $(x/A)_j = 0$ for $j \notin A$.

(13) Given a polymatroid $P \subset \mathbb{R}^+_E$, let $\alpha \in \mathbb{R}^+_E$ be an integer-valued vector such that $x < \alpha$ for every $x \in P$. Where r is the rank function of P, let $f_P(A) = r(\alpha/A)$ for every $A \subseteq E$.

Let $L_E = \{A : A \subseteq E\}$. Clearly, by (11), f_P is a β-function on L_E. Furthermore, if P is integral, then f is integral.

(14) **Theorem.** *For any polymatroid $P \subset \mathbb{R}^+_E$,*

$$P = P(E, f_P).$$

Thus, all polymatroids $P \in \mathbb{R}^+_E$ are polyhedra, and they correspond to certain β-functions on L_E.

Theorem 8 provides a useful way of constructing matroids which is quite different from the usual algebraic constructions.

(15) For any given integral β_0-function f as in (8), let a set $J \subseteq E$ be a member of F iff for every $A \in K = L - \{\emptyset\}$, $|J \cap A| \leq f(A)$. In particular, where $L = L_E$, let a set $J \subseteq E$ be a member of F when for every $\emptyset \neq A \subseteq J$, $|A| \leq f(A)$. Then (8) implies that $M = (E, F)$ is a matroid, and gives a formula for its rank function in terms of f. (This generalizes a construction given by Dilworth [1]).

III

In this section, K will denote $L_E - \{\emptyset\} = \{A : \emptyset \neq A \subseteq E)$.

(16) Given any $c = \{c_j\} \in \mathbb{R}_E$, and given a β-function f on L_E, we show how to solve the linear program:

$$\text{maximize } c \cdot x = \sum_{j \in E} c_j x_j \text{ over } x \in P(E, f).$$

(17) Let $j(1)$, $j(2)$, $\ldots$ be an ordering of E such that

$$c_{j(1)} \geq c_{j(2)} \geq \cdots c_{j(k)} > 0 \geq c_{j(k+1)} \geq \cdots$$

(18) For each integer i, $1 \leq i \leq k$, let

$$A_i = \{j(1), j(2), \ldots, j(i)\}.$$

(19) **Theorem.** (The Greedy Algorithm). *$c \cdot x$ is maximized over $x \in P(E, f)$ by the following vector x^0:*

$$\begin{aligned} x^0_{j(1)} &= f(A_1); & \\ x^0_{j(i)} &= f(A_i) - f(A_{i-1}) & \textit{for } 2 \leq i \leq k; \\ x^0_{j(i)} &= 0 & \textit{for } k < i \leq |E|. \end{aligned}$$

(There is a well-known non-polyhedral version of this for graphs, given by Kruskal [9]. A related theorem for matroids is given by Rado [15]).

The dual *l.p.* is to minimize

$$f \cdot y = \sum_{A \in K} f(A) y(A) \text{ where}$$

(20) $y(A) \geq 0$; and for every $j \in E$, $\sum_{j \in A} y(A) \geq c_j$.

(21) **Theorem.** *An optimum solution, $y^0 = [y^0(A)]$, $A \in K$, to the dual* l.p. *is*

$$\begin{aligned} &y^0(A_i) = c_{j(i)} - c_{j(i+1)} & &\textit{for } 1 \leq i \leq k-1; \\ &y^0(A_k) = c_{j(k)}; \quad \textit{and} \quad y^0(A) = 0 & &\textit{for all other } A \in K. \end{aligned}$$

(22) **Theorem.** Corollary to (19). *The vertices of the polyhedron $P(E, f)$ are precisely the vectors of the form x^0 in* (19) *for some sequence $j(1), j(2), \ldots, j(k)$.*

(23) Where f is the rank function of a matroid $M = (E, F)$, (9) and (22) imply that the vertices of $P(E, f)$ are precisely the incidence vectors of the members of F, i.e., the independent sets of M. Such a $P(E, f)$ is called a *matroid polyhedron.*

(24) Let f be a β-function on L_E. A set $A \in L_E$ is called f*-closed* or an f*-flat*, when, for any $C \in L_E$ which properly contains A, $f(A) < f(C)$.

(25) **Theorem.** *If A and B are f-closed then $A \cap B$ is f-closed.*

(In particular, for the f of (9), the f-flats form a "geometric" or "matroid" lattice.)

Proof: Suppose that C properly contains $A \cap B$. Then either $C \not\subseteq A$ or $C \not\subseteq B$. Since f is non-decreasing we have $f(A \cap B) \leq f(A \cap C)$ and $f(A \cap B) \leq f(B \cap C)$. Thus, since f is submodular, we have either

$$0 < f(A \cup C) - f(A) \leq f(C) - f(A \cap C) \leq f(C) - f(A \cap B), \quad \text{or}$$
$$0 < f(B \cup C) - f(B) \leq f(C) - f(B \cap C) \leq f(C) - f(A \cap B).$$

(26) A set $A \in K$ is called f*-separable* when

$$f(A) = f(A_1) + f(A_2)$$

for some partition of A into non-empty subsets A_1 and A_2. Otherwise A is called f-inseparable.

(27) **Theorem.** *Any $A \in K$ partitions in only one way into a family of f-inseparable sets A_i such that $f(A) = \sum f(A_i)$.* The A_i's are called the f*-blocks* of A.

If a polyhedron $P \subset \mathbb{R}_E$ has dimension equal to $|E|$ then there is a unique minimal system of linear inequalities having P as its set of solutions. These inequalities are called the *faces* of P.

It is obvious that a polymatroid $P \subset \mathbb{R}_E^+$ has dimension $|E|$ if and only if, where f is the β-function which determines it, and set $\emptyset$ is f-closed. It is obvious that inequality $x(A) \leq f(A)$, $A \in K$, is a face of $P(E, f)$ only if A is f-closed and f-inseparable.

(28) **Theorem.** *Where f is a β-function on L_E such that the empty set is f-closed, the faces of polymatroid $P(E, f)$ are: $x_j \geq 0$ for every $j \in E$; and $x(A) \leq f(A)$ for every $A \in K$ which is f-closed and f-inseparable.*

IV

(29) Let each V_p, $p = 1$ and 2, be a family of disjoint subsets of H. Where $[a_{ij}]$, $i \in E$, $j \in E$, is the 0–1 incidence matrix of $V_1 \cup V_2 = H$, the following *l.p.* is known as the Hitchcock problem.

(30) Maximize $c \cdot x = \sum_{j \in E} c_j x_j$, where

(31) $x_j \geq 0$ for every $j \in E$, and $\sum_{j \in E} a_{ij} x_j \leq b_i$ for every $i \in H$.

The dual *l.p.* is

(32) Minimize $b \cdot y = \sum b_i y_i$, where

(33) $y_i \geq 0$ for every $i \in H$, and $\sum_{i \in H} a_{ij} y_i \geq c_j$ for every $j \in E$.

Denote the polyhedron of solutions of a system Q by $P[Q]$.

The following properties of the Hitchcock problem are important in its combinatorial use.

(34) **Theorem.** (*a*) *Where the b_i's are integers, the vertices of* $P[(31)]$ *are integer-valued.* (*b*) *Where the c_j's are integers, the vertices of* $P[(33)]$ *are integer-valued.*

Theorem (34a) generalizes to the following.

(35) **Theorem.** *For any two integral polymatroids P_1 and P_2 in $\mathbb{R}^+_E$, the vertices of $P_1 \cap P_2$ are integer-valued.*

The following technique for proving theorems like (34) is due to Alan Hoffman [7].

(36) **Theorem.** *The matrix $[a_{ij}]$ of the Hitchcock problem is totally unimodular — that is, the determinant of every square submatrix has value 0, 1, or -1.*

(37) **Theorem.** *Theorem* (34) *holds whenever $[a_{ij}]$ is totally unimodular.*

(38) Let each V_p, $p = 1$ and 2, be a family of subsets of E such that any two members of P are either disjoint or else one is a subset of the other.

(39) **Theorem.** *The incidence matrix of the $V_1 \cup V_2$ of* (38) *is totally unimodular.*

Property (29) is a special case of (38). Property (38) is a special case of the following.

(40) Let each V_p, $p = 1$ and 2, be a family of subsets of E such that for any $R \in V_p$ and $S \in V_p$ either $R \cap S = \emptyset$ or $R \cap S \in V_p$.

The incidence matrix of the $V_1 \cup V_2$ of (40) is generally not totally unimodular. However,

(41) **Theorem.** *From the incidence matrix of each V_p of* (40)*, once can obtain, by subtracting certain rows from others, the incidence matrix of a family of mutually disjoint subsets of E. Thus, in the same way, one can obtain from the incidence matrix of the $V_1 \cup V_2$ of* (40)*, a matrix of the Hitchcock type.*

(42) **Theorem.** *For any polymatroid $P(E, f)$ and any $x \in P(E, f)$, if $x(A) = f(A)$ and $x(B) = f(B)$ then either $A \cap B = \emptyset$ or $x(A \cap B) = f(A \cap B)$.*

Theorems (42), (41), and (34a) imply (35).

(43) Assuming that each V_p of (38) contains the set E, $L_p = V_p \cup \{\emptyset\}$ is a particularly simple lattice. For any non-negative non-decreasing function $f(i) = b_i$, $i \in V_p$, let $f(\emptyset) = -f(E)$. Then f is a β_0-function on L_p.

(44) The only integer vectors in a matroid polyhedron P are the vectors of the independent sets of the matroid, and these vectors are all vertices of P. Thus, (35) implies:

(45) **Theorem.** *Where P_1 and P_2 are the polyhedra of any two matroids M_1 and M_2 on E, the vertices of $P_1 \cap P_2$ are precisely the vectors which are vertices of both P_1 and P_2 — namely, the incidence vectors of sets which are independent in both M_1 and M_2.*

Where P_1, P_2, and P_3 are the polyhedra of three matroids on E, polyhedron $P_1 \cap P_2 \cap P_3$ generally has many vertices besides those which are vertices of P_1, P_2, and P_3.

Let $c = [c_j]$, $j \in E$, be any numerical weighting of the elements of E. In view of (45), the problem:

(46) Find a set J, independent in both M_1 and M_2, that has maximum weight-sum, $\sum_{j \in J} c_j$, is equivalent to the *l.p.* problem:

(47) Find a vertex x of $P_1 \cap P_2$ that maximizes $c \cdot x$.

(48) Assuming there is a good algorithm for recognizing whether of not a set $J \subseteq E$ is independent in M_1 or in M_2, there is a good algorithm for problem (46). This seems remarkable in view of the apparent complexity of matroid polyhedra in other respects. For example, a good algorithm is not known for the problem:

(49) Given a matroid $M_1 = (E, F_1)$ and given an element $e \in E$, minimize $|D|$, $D \subseteq E$, where $e \in D \notin F_1$;

Or the problems:

(50) Given three matroids M_1, M_2, and M_3, on E, and given an objective vector $c \in \mathbb{R}_E$, maximize $c \cdot x$ where $x \in P_1 \cap P_2 \cap P_3$.

Or maximize $\sum_{j \in J} c_j$ where $J \in F_1 \cap F_2 \cap F_3$.

V

Where f_1 and f_2 are β-functions on L_E, the dual of the *l.p.*:

(51) Maximize $c \cdot x = \sum_{j \in E} c_j x_j$, where

(52) For every $j \in E$, $x_j \geq 0$; and for every $A \in K$, $x(A) \leq f_1(A)$ and $x(A) \leq f_2(A)$; is the *l.p.*:

(53) Minimize $f \cdot y = \sum_{A \in K} [f_1(A) y_1(A) + f_2(A) y_2(A)]$

where

(54) For every $A \in K$, $y_1(A) \geq 0$ and $y_2(A) \geq 0$; and for every $j \in E$,

$$\sum_{j \in A \in K} [y_1(A) + y_2(A)] \geq c_j.$$

Combining systems (52) and (54) we get,

$$\begin{aligned} &\sum_{j \in E} x_j \left(\sum_{j \in A \in K} [y_1(A) + y_2(A)] - c_j \right) \\ &+ \sum_{A \in K} y_1(A)[f_1(A) - \sum_{j \in A} x_j] \\ &+ \sum_{A \in K} y_2(A)[f_2(A) - \sum_{j \in A} x_j] \ \geq \ 0. \end{aligned} \tag{55}$$

Expanding and cancelling we get

$$c \cdot x \leq f \cdot y \tag{56}$$

for any x satisfying (52) and any $y = (y_1, y_2)$ satisfying (54).

(57) Equality holds in (56) if and only if equality holds in (55).

The *l.p.* duality theorem says that

(58) If there is an x^0, a vertex of $P[(52)]$, which maximizes $c \cdot x$, then there is a $y^0 = (y_1^0, y_2^0)$, a vertex of $P[(54)]$, such that

$$c \cdot x^0 = f \cdot y^0, \tag{59}$$

and hence such that y^0 minimizes $f \cdot y$.

For the present problem obviously there is such an x^0.

The vertices of (54) are not generally all integer-valued when the c_j's are. However,

(60) **Theorem.** *If the c_j's are all integers, then, regardless of whether f_1 and f_2 are integral, there is an integer-valued solution $y^4 = (y_1^4, y_2^4)$ of (54) which minimizes $f \cdot y$.*

Let $y^3 = (y_1^3, y_2^3)$ be any solution of (54) which minimizes $f \cdot y$.

(61) For every $j \in E$, and $p = 1, 2$ let $c_j^p = \sum_{j \in A \in K} y_p^3$

For each $p = 1, 2$ consider the problem,

(62) Minimize $f_p \cdot y_p = \sum_{A \in K} f_p(A) y_p(A)$ where

(63) for every $A \in K$, $y_p(A) \geq 0$; and for every $j \in E$,

$$\sum_{j \in A \in K} y_p(A) \geq c_j^p.$$

(64) By (21) for each p, there is an optimum solution, say y_p^4, to (62) having the following form:

(65) The sets $A \in K$, such that $y_p^4(A) > 0$, form a nested sequence,

$$A_1 \subset A_2 \subset A_3 \subset \ldots.$$

Since y_p^3 is a solution of (63), we have $f_p y_p^4 \leq f_p y_p^3$, for each p, and thus $f \cdot y^4 \leq f \cdot y^3$. Since $c_j^1 + c_j^2 \geq c_j$ for every $j \in E$, y^4 is a solution of (54), and hence y^4 is an optimum solution of (54). Thus, we have that

(66) **Theorem.** *There exists a solution y^4 of* (54) *which minimizes $f \cdot y$ and which has property* (65) *for each $p = 1, 2$.*

The problem, minimize $f \cdot y$ subject to (54) and also subject to $y_p(A) = 0$ for every $y_p^4(A) = 0$, has the form [(32), (33)] where $[a_{ij}]$ is the incidence matrix of a $V_1 \cup V_2$ as in (38). Thus, by (39) and (37), we have:

(67) **Theorem.** *If the c_j's are all integers then the y^4 of* (66) *can be taken to be integer-valued.*

In particular this proves (60).

An immediate consequence of (35), (60), and the *l.p.* duality theorem is

(68) **Theorem.** $\max c \cdot x = \min f \cdot y$ *where $x \in P[(52)]$ and $y \in P[(54)]$.*
If f is integral, x can be integral.
If c is integral, y can be integral.

In particular, where f_1 and f_2 are the rank functions, r_1 and r_2, of any two matroids, $M_1 = (E, F_1)$ and $M_2 = (E, F_2)$, and where every $c_j = 1$, (68) implies:

(69) **Theorem.** $\max |J| = \min[r_1(S) + r_2(E - S)]$, *where $J \in F_1 \cap F_2$, and where $S \subseteq E$.*

(A related result is given by Tutte [16]).

VI

(70) **Theorem.** *For each $i \in E'$, let Q_i be a subset of E. For each $A' \subseteq E'$, let $u(A') = \bigcup_{i \in A'} Q_i$. Let f be any integral β-function on L_E.*

Then $f'(A') = f(u(A'))$ *is an integral* β*-function on* $L_{E'} = \{A' : A' \subseteq E'\}$.

(71) This follows from the relations

$$u(A' \cup B') = u(A') \cup u(B') \quad \text{and}$$
$$u(A' \cap B') \subseteq u(A') \cap u(B').$$

(72) Applying (15) to f' yields a matroid on E'.

(73) In particular, taking f to mean cardinality, if we let $J' \subseteq E'$ be a member of F' iff $|A'| \leq |u(A')|$ for every $A' \subseteq J'$, then $M' = (E', F')$ is a matroid.

(74) Hall's SDR theorem says that: $|A'| \leq |u(A')|$ for every $A' \subseteq J'$ iff the family $\{Q_j\}$, $i \in J'$, has a system of distinct representatives, i.e., a transversal. A transversal of a family $\{Q_i\}$, $i \in J'$ is a set $\{j_i\}$, $i \in J'$, of distinct elements such that $j_i \in Q_i$. Thus,

(75) **Theorem.** *For any finite family* $\{Q_i\}$, $i \in E'$, *of subsets of* E, *the sets* $J' \subseteq E'$ *such that* $\{Q_i\}$, $i \in J'$, *has a transversal are the independent sets of a matroid on* E' (called a *transversal matroid*).

There are a number of interesting ways to derive (75). Some others are in [2], [3], [5], and [12]. The present derivation is the way (75) was first obtained and communicated.

The following is the same result with the roles of elements and sets interchanged.

(76) *Let* $J \in F_0$ *iff, for some* $J' \subseteq E'$, J *is a transversal of* $\{Q_i\}$, $i \in J'$. That is, let $J \in F_0$ iff J is a *partial transversal* of $\{Q_i\}$, $i \in E'$. *Then* $M_0 = (E, F_0)$ *is a matroid.*

(77) Thus, where P_0 is the polyhedron of M_0 and where P is the polyhedron of any other matroid, $M = (E, F)$, on E, the vertices of $P_0 \cap P$ are the incidence vectors of the M-independent partial transversals of $\{Q_i\}$, $i \in E'$.

By (8), the rank function r_0 of M_0 is, for each $A \subseteq E$,

(78) $r_0(A) = \min\,[|A_0| + |\{i : (A - A_0) \cap Q_i \neq \emptyset\}|]$
where $A_0 \subseteq A$.

Combining (69) and (78), we get

(79) $\max|J| = \min\,[r(A_1) + |A_0| + |E'| - |\{i : Q_i \subseteq A_1 \cup A_0\}|]$

$= \min\,[r(u(A')) + |E'| - |A'|]$, where $J \in F_0 \cap F$, $A_0 \cup A_1 \subseteq E$, $A_0 \cap A_1 = \emptyset$, and $A' \subseteq E'$.

In particular, (79)implies the following theorem of Rado [14], given in 1942.

(80) *For any matroid* M *on* E, *a family* $\{Q_i\}$, $i \in E'$, *of subsets of* E, *has a transversal which is independent in* M *iff* $|A'| \leq r(u(A'))$ *for every* $A' \subseteq E'$.

Taking the f of (70) to be r, (70), (15), and (80) imply:

(81) **Theorem.** *For any matroid M on E, and any family $\{Q_i\}$, $i \in E'$, of subsets of E, the sets $J' \subseteq E'$ such that $\{Q_i\}$, $i \in J'$, has an M-independent transversal are the independent sets of a matroid on E'.*

(82) A *bipartite graph* G consists of two disjoint finite sets, V_1 and V_2, of nodes and a finite set $E(G)$ of edges such that each member of $E(G)$ meets one node in V_1 and one node in V_2.

The following theorem of König is a prototype of (69).

(83) **Theorem.** *For any bipartite graph G,* $\max|J|$, *$J \subseteq E(G)$, such that*

(*a*) *no two members of J meet the same node in V_1, and*

(*b*) *no two members of J meet the same node in V_2,*

equals $\min(|T_1| + |T_2|)$, *$T_1 \subseteq V_1$, and $T_2 \subseteq V_2$, such that every member of $E(G)$ meets a node in T_1 or a node in T_2.*

(84) To get the Hall theorem, (74), from (83), let V_1 be the E' of (70), let V_2 be the E of (70), and let there be an edge in $E(G)$ which meets $i \in V_1$ and $j \in V_2$ iff $j \in Q_i$.

Clearly, if the family $\{Q_i\}$, $i \in E'$, has no transversal then, in (83), $\max|J| < |V_1|$. If the latter holds, then by (83), the T_1 of $\min(|T_1| + |T_2|)$, in (83), is such that

$$|V_1 - T_1| > |u(V_1 - T_1)|.$$

(85) For the König-theorem instance, (83) of (69), the matroids $M_1 = (E, F_1)$ and $M_2 = (E, F_2)$ are particularly simple: Let $E = E(G)$. For $p = 1$ and $p = 2$, let $J \subseteq E(G)$ be a member of F_p iff no two members of J meet the same node in V_p.

(86) Where P_1 and P_2 are the polyhedra of these two matroids, finding a vertex x of $P_1 \cap P_2$ which maximizes $c \cdot x$ is essentially the *optimal assignment problem.* That is, the Hitchcock problem where every $b_i = 1$.

(87) Clearly, the inequality $x(A) \leq r_p(A)$ is a face of P_p, that is, A is r_p-closed and r_p-inseparable, iff, for some node $v \in V_p$, A is the set of edges which meet v.

VII

(88) Let $\{M_i\}$, $i \in I$, be a family of matroids, $M_i = (E, F_i)$, having rank functions r_i. Let $J \subseteq E$ be a member of F iff:

(89) $|A| \leq \sum_i r_i(A)$ for every $A \subseteq J$.

Since $f(A) = \sum_i r_i(A)$ is a β-function on L_E,

(90) **Theorem.** *The $M = (E, F)$ of* (88) *is a matroid, called the* sum *of the matroids M_i.*

In [5], and in [2], it is shown that

(91) **Theorem.** *$J \subseteq E$ satisfies* (89) *iff J can be partitioned into sets J_i such that $J_i \in F_i$.*

(92) An algorithm, *MPAR*, is given there for either finding such a partition of J or else finding an $A \subseteq J$ which violates (89). That is, for recognizing whether or not $J \in F$.

(93) The algorithm is a good one, assuming:

(94) that a good algorithm is available for recognizing, for any $K \subseteq E$ and for each $i \in I$, whether or not $K \in F_i$.

(95) The definition of a matroid $M = (E, F)$ is essentially that, modulo the ease of recognizing, for any $J \subseteq E$, whether or not $J \in F$, one has what is perhaps the easiest imaginable algorithm for finding, in any $A \subseteq E$, a maximum cardinality subset J of A such that $J \in F$.

(96) In particular, by virtue of (90), assuming (94), *MPAR* provides a good algorithm for finding a maximum cardinality set $J \subseteq E$ which is partitionable into sets $J_i \in F_i$.

(97) Assuming (94), *MPAR* combined with (19) is a good algorithm for, given numbers c_j, $j \in E$, finding a set J which is partitionable into sets $J_i \in F_i$ and such that $\sum_{j \in J} c_j$ is maximum.

Where r is the rank function of matroid $M = (E, F)$, let

(98) $r^*(A) = |A| + r(E - A) - r(E)$ for every $A \subseteq E$.

Substituting $r(E) = |E| - r^*(E)$, and A for $E - A$, in (98), yields

(99) $r(A) = |A| + r^*(E - A) - r^*(E)$.

(100) It is easy to verify that r^* is the rank function of a matroid $M^* = (E, F^*)$, e.g., that r^* satisfies (9). M^* is called *the dual* of M. By (99), $M^{**} = M$.

(101) By (98), $|J| = r^*(J)$ iff $r(E - J) = r(E)$. Therefore, $J \in F^*$ iff $E - J$ contains an M-basis of E, i.e., a *basis* of M. Thus, it can be determined whether or not $J \in F^*$ by obtaining an M-basis of $E - J$ and observing whether or not its cardinality equals $r(E)$.

Where r is the rank function of a matroid $M = (E, F)$, and where n is a non-negative integer, let

(102) $r^{(n)}(A) = \min[n, r(A)]$ for every $A \subseteq E$.

(103) Clearly, $r^{(n)}$ is the rank function of a matroid $M^{(n)} = (E, F^{(n)})$, called the *n-truncation* of M, such that $J \in F^{(n)}$ iff $J \in F$ and $|J| \leq n$.

(104) For matroids $M_1 = (E, F_1)$ and $M_2 = (E, F_2)$, and any integer $n \leq r_2(E)$, by (103) and (101), there is a set $J \in F_1 \cap F_2$ such that $|J| = n$ iff E can be partitioned into a set $J_1 \in F_1$ and a set $J_2 \in F_2^{(n)^*}$. Theorem (91) says this is possible iff $|A| \leq r_1(A) + r_2^{(n)^*}(n)(A)$ for every $A \subseteq E$. Using (102) and (98), this implies (69).

(105) Using *MPAR*, a maximum cardinality $J \in F_1 \cap F_2$ can be found as follows: Find a maximum cardinality set $H = J_1 \cup J_2$ such that $J_1 \in F_1$ and $J_2 \in F_2^*$. Extend J_2 to B, an M_2^*-basis of H. Clearly, B is an M_2^*-basis of E, and so $H - B \in F_1 \cap F_2$. It is easy to verify that $|H - B| = \max |J|$, $J \in F_1 \cap F_2$.

(106) It is more practical to go in the other direction, obtaining for a given family of matroids $M_i = (E, F_i)$, $i \in I$, an "optimum" family of mutually disjoint sets $J_i \in F_i$, by using the "matroid intersection algorithm" of (48) on the following two matroids $M_1 = (E_I, F_1)$ and $M_2 = (E_I, F_2)$. Let E_I consist of all pairs (j, i), $j \in E$ and $i \in I$. There is a 1–1 correspondence between sets $J \in E_I$ and families $\{J_i\}$, $i \in I$, of sets $J_i \subseteq E$, where J corresponds to the family $\{J_i\}$ such that $j \in J_i \iff (j, i) \in J$. Let $M_1 = (E_I, F_1)$ be the matroid such that $J \subseteq E_I$ is a member of F_1 iff the corresponding sets J_i are mutually disjoint — that is, if and only if the j's of the members of J are distinct. Let $M_2 = (E_I, F_2)$ be the matroid such that $J \subseteq E_I$ is a member of F_2 iff the corresponding sets J_i are such that $J_i \in F_i$.

(Nash-Williams has developed the present subject in another interesting way [13].)

VIII

(107) If $f(a)$ is a β-function on L and k is a not-too-large constant, then $f(a) - k$ is a β_0-function on L. It is useful to apply (15) to, non-β, β_0-functions.

(108) For example, let G be a graph having edge-set $E = E(G)$ and node-set $V = V(G)$. For each $j \in E$, let Q_j be the set of nodes which j meets. For every $A \subseteq E$, let $f(A) = |u(A)| - 1$. Then, by (70), $f(A)$ is a β_0-function on L_E.

(109) Applying (15) to this f yields a matroid, $M(G) = (E, F(G))$.

(110) The minimal dependent sets of a matroid $M = (E, F)$, i.e., the minimal subsets of E which are not members of F, are called the *circuits* of M.

(111) The circuits of $M(G)$ are the minimal non-empty sets $A \subseteq E$ such that $|A| = |u(A)|$.

(112) A set $J \subseteq E$ is a member of $F(G)$ iff J together with the set $u(J)$ of nodes is a forest in G.

IX

(113) Let G be a directed graph. For any $R \subseteq V(G)$, a *branching* B of G *rooted* at R, is a forest of G such that, for every $v \in V(G)$, there is a unique directed path in B (possibly having zero edges) from some node in R to v.

(114) The following problem is solved using matroid intersection

(115) Given any directed graph G, given a numerical weight c_j for each $j \in E = E(G)$, and given sets $R_i \subseteq V(G)$, $i \in I$, find edge-disjoint branchings B_i, $i \in I$, rooted respectively at R_i, which minimize $s = \sum_j c_j$, $j \in \bigcup_{i \in I} B_i$.

(116) The problem easily reduces to the case where each R_i consists of the same single node, $v_0 \in V(G)$. That is, find $n = |I|$ edge-disjoint branchings B_i, each rooted at node v_0, which minimize s.

(117) Where $F(G)$ is as defined in (109), let $J \subseteq E$ be a member of F_1 iff it is the union of n members of $F(G)$. By (91), $M_1 = (E, F_1)$ is a matroid.

(118) Let $J \subseteq E$ be a member of F_2 iff no more than n edges of J are directed toward the same node in $V(G)$ and no edge of J is directed toward v_0. Clearly, $M_2 = (E, F_2)$ is a matroid:

(119) **Theorem.** *A set $J \subseteq E$ is the edge-set of n edge-disjoint branchings of G, rooted at node $v_0 \in V(G)$, iff $|J| = n(|V(G)| - 1)$ and $J \in F_1 \cap F_2$.*

This is a consequence of the following.

(120) **Theorem.** *The maximum number of edge-disjoint branchings of G, rooted at v_0, equals the minimum over all C, $v_0 \in C \subset V(G)$, of the number of edges having their tails in C and their heads not in C.*

(121) There is an algorithm for finding such a family of branchings in G, and in particular for partitioning a set J as described in (119) into branchings as described in (119).

(122) Let P_1 and P_2 he the polyhedra of matroids M_1 and M_2 respectively. Let $H = \{x : x(E) = n(|V(G)| - 1)\}$.

It follows from (45) that

(123) A vector $x \in \mathbb{R}_E$ is a vertex of $P_1 \cap P_2 \cap H$ iff it is the incidence vector of a set J as described in (119).

(124) A variant of the matroid-intersection algorithm will find such an x which minimizes $c \cdot x$. The case $n = 1$ is treated in [4].

X

(125) Let each L_i be a commutative semigroup. We say $a \leq b$, for $\{a, b\} \subseteq L_i$, iff $a + d = b$ for some $d \in L_i$.

(126) A function f from L_0 into L_1 is called a ψ-function iff

(127) for every $\{a, d\} \subseteq L_0$, $f(a) \leq f(a + d)$; and

(128) for every $\{a, b, c\} \subseteq L_0$, $f(a + b + c) + f(c) \leq f(a + c) + f(b + c)$.

(129) L_i is called a *ψ-semigroup* iff, for $\{a, b, c\} \subseteq L_i$,

$$a + c + c = b + c + c \implies a + c = b + c.$$

For example, L_i is a ψ-semigroup if it is cancellative or if it is idempotent.

(130) **Theorem.** *If $f(\cdot)$ is a ψ-function from L_0 into L_1, $g(\cdot)$ is a ψ-function from L_1 into L_2, and L_1 is a ψ-semigroup, then $g(f(\cdot))$ is a ψ-function from L_0 into L_2.*

(131) **Theorem.** *A function f from a lattice, L_0, into the non-negative reals, L_1, satisfies* (128), *where "+" in L_0 means "$\vee$" and "+" in L_1 means ordinary addition, iff f is non-decreasing, i.e., satisfies* (127), *and f is submodular.*

(132) Thus, β-functions can be obtained by composing ψ-functions.

(133) **Theorem.** *A function f from the non-negative reals into the non-negative reals is a ψ-function, relative to addition in the image and preimage, iff it is non-decreasing and concave.*

(134) **Theorem.** *A function f from a lattice, L_0, into a lattice, L_1, is a ψ-function, relative to joins "$\vee$" in each, iff it is a join-homomorphism, i.e., for every $\{a, b\} \subseteq L_0$, $f(a \vee b) = f(a) \vee f(b)$.*

Let $h(S)$ be any real (integer)-valued function of the elements $S \in L$ of a finite lattice L. In principle, an (integral) non-decreasing submodular function f on L can be obtained recursively from h as follows:

(135) **Theorem.** *For each $S \in L$, let $g(S) = \min[h(S), g(A)+g(B)-g(A\wedge B)]$ where $A < S$, $B < S$, and $A \vee B = S$. Then g is submodular. For each $S \in L$, let $f(S) = \min g(A)$ where $S \leq A \in L$. Then f is submodular and non-decreasing. If h is submodular then $g = h$. If h is submodular and non-decreasing then $f = h$.*

(A similar construction was communicated to me by D. A. Higgs.)

(136) The β-functions on a finite lattice L correspond to the members of a polyhedral cone $\beta(L)$ in the space of vectors $y = [y_A]$, $A \in L - \{\emptyset\}$. Where $y_\emptyset = 0$, $\beta(L)$ is the set of solutions to the system:

(137) $y_A + y_B - y_{A\vee B} - y_{A\wedge B} \geq 0$ and $y_{A\vee B} - y_A \geq 0$ for every $A \in L$ and $B \in L$.

(138) Characterizing the extreme rays of $\beta(L)$, in particular for $L = \{A : A \subseteq E\}$, appears to be difficult.

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