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GEAR COMPOSITION OF STABLE SET POLYTOPES
AND $\bar{\gamma}$-PERFECTION

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Abstract

Graphs obtained by applying the gear composition to a given graph $H$ are called geared graphs. We show how a linear description of the stable set polytope $STAB(G)$ of a geared graph $G$ can be obtained by extending the linear inequalities defining $STAB(H)$ and $STAB(H^e)$, where $H^e$ is the graph obtained from $H$ by subdividing the edge $e$.

We also introduce the class of $G$-perfect graphs, i.e., graphs whose stable set polytope is described by: nonnegativity inequalities, rank inequalities, lifted 5-wheel inequalities, and some special inequalities called geared inequalities and $g$-lifted inequalities. We prove that graphs obtained by repeated applications of the gear composition to a given graph $H$ are $G$-perfect, provided that any graph obtained from $H$ by subdividing a subset of its simplicial edges is $G$-perfect. In particular, we show that a large subclass of claw-free graphs is $G$-perfect, thus providing a partial answer to the well-known problem of finding a defining linear system for the stable set polytope of claw-free graphs.

Key words: stable set polytope, graph composition, polyhedral combinatorics.
1. Introduction

Given a graph $G = (V, E)$ and a vector $w \in \mathbb{Q}^V_+$ of node weights, the stable set problem is the problem of finding a set of pairwise nonadjacent nodes (stable set) of maximum weight.

The stable set polytope, denoted by $STAB(G)$, is the convex hull of the incidence vectors of the stable sets of $G$; it is known to be full dimensional. A linear system $Ax \leq b$ is said to be defining for $STAB(G)$ if $STAB(G) = \{x \in \mathbb{R}^V : Ax \leq b\}$. The facet defining inequalities for $STAB(G)$ are those inequalities that constitute the unique nonredundant defining linear system of $STAB(G)$.

So, finding the defining linear system for $STAB(G)$ is equivalent to transform the original optimization problem into the linear program

$$\max \{w^T x : Ax \leq b, x \geq 0\}.$$  

Indeed the existence of a “good” defining linear system for $STAB(G)$ is equivalent to the existence of a polynomial time algorithm for optimizing over $STAB(G)$ (where “good” means that the separation problem for this linear system can be solved in polynomial time). Since the stable set problem is $NP$-hard, it is unlikely to find such a system for general graphs. Nevertheless there are classes of graphs for which such systems are known, as bipartite graphs, line graphs [5], series-parallel graphs [14], odd $K_4$-free [9], and others. It is known that, for these classes of graphs, the weighted stable set problem is polynomial time solvable [12].

In [12], Grötschel, Lovász and Schrijver present a more general point of view. Instead of looking for classes of graphs having a well-defined linear system describing $STAB(G)$, they consider a set $\mathcal{L}$ of valid inequalities for $STAB(G)$ and the following polyhedron

$$\mathcal{L}STAB(G) = \{x \in \mathbb{R}^V_+ \mid x \text{ satisfies } \mathcal{L}\}.$$  

Further they name $\mathcal{L}$-perfect the graphs $G$ having $\mathcal{L}STAB(G) = STAB(G)$. Two basic questions arise in this context: the first one is whether the optimization problem for $\mathcal{L}STAB(G)$ can be solved in polynomial time (equivalently whether the separation problem for $\mathcal{L}STAB(G)$ is polynomial time solvable [11]); the second one is which graphs belong to the class of $\mathcal{L}$-perfect graphs. Different sets $\mathcal{L}$ of inequalities have been considered in literature together with the corresponding classes of $\mathcal{L}$-perfect graphs. We mention some of them in a non exhaustive list: edge plus odd-hole inequalities and $t$-perfect graphs [4]; clique plus odd-hole inequalities and $h$-perfect graphs; rank inequalities and rank-perfect graphs [23].

Here, we consider a family $\mathcal{G}$ consisting of the following (lifted) inequalities: rank inequalities, 5-wheel inequalities, geared inequalities and $g$-lifted inequalities. The definition of rank and 5-wheel inequalities is given later. The geared and the $g$-lifted inequalities are generated by the graph composition named gear composition introduced in [7]. This composition starts from a given graph $H$ and builds a new graph $G$ by replacing a suitable edge of $H$ with the fixed graph $B$ (gear) shown in Fig. 1. This new graph $G$ is called geared graph generated by $H$ and $B$.

![Figure 1: The gear with nodes $d_1, b_1, h_1, b_2, c, a, d_2, b_2$.](image)

The gear composition has an important polyhedral property: it preserves the property of an inequality of being facet defining. This means that a facet defining inequality of $STAB(H)$ can be “properly extended” to a facet defining inequality of $STAB(G)$ when $G$ is a geared graph. The geared inequalities
were introduced in [7]; in this paper we identify another class of inequalities generated by the gear composition, the so-called g-lifted inequalities. Both classes of inequalities are essential in the linear description of \( STAB(G) \) when \( G \) is a geared graph and we provide sufficient conditions for them being facet defining. Then, we investigate the relations between the polyhedron

\[
GSTAB(G) = \{ x \in \mathbb{R}^V_+ | x \text{ satisfies } G \}.
\]

and the stable set polytope of a graph \( G \) obtained as the gear composition of \( H \) and \( B \). Clearly, \( STAB(G) \subseteq GSTAB(G) \); here, we provide sufficient conditions to have equality, i.e., we exhibit classes of graphs which are \( G \)-perfect. In particular, we consider the class of graphs \( G_H \) obtained by iteratively applying the gear composition to a given graph \( H \). We show that if the gear composition is applied to “suitable” simplicial edges of a line graph \( H \), then the graphs in \( G_H \) are claw-free and \( G \)-perfect. This allows us to exhibit the linear description of the polytope \( STAB(G) \) for a large subclass of claw-free graphs with stability number at least 4, thus providing a partial answer to the well-known problem of finding a defining linear system for the stable set polytope of claw-free graphs.

In Section 2, we recall the definition of gear composition and we show some of its polyhedral properties. In particular we show under which conditions the gear composition preserves the property of a graph of being facet producing. In Section 3, we show that, apart from clique and 5-wheel inequalities, geared inequalities and g-lifted inequalities are the only new linear inequalities involving \( B \) that are necessary to describe \( STAB(G) \) when \( G \) is a geared graph generated by \( H \) and \( B \) along \( e \). Finally in Section 4, we introduce the class of inequalities \( G \). Then we prove under which conditions the stable set polytope of a geared graph is described by nonnegativity constraints plus inequalities in \( G \) and we provide interesting examples of \( G \)-perfect graphs.

We denote by \( G = (V_G, E_G) \) any graph with node set \( V_G \) and edge set \( E_G \). An edge \( e \in E_G \) with endnodes \( u \) and \( v \) will be denoted by \( uv \). We denote by \( \delta(v) \) the set of edges of \( G \) having \( v \) as endnode and by \( N(v) \) the set of nodes of \( V_G \) adjacent to \( v \). A clique-cutset of \( G \) is a complete subgraph whose removal disconnects \( G \).

A k-hole \( C_k = (v_1, v_2, \ldots, v_k) \) is a chordless cycle of length \( k \). A 5-wheel \( W = (h : v_1, \ldots, v_5) \) is a graph consisting of a 5-hole \( C = (v_1, \ldots, v_5) \), called rim of \( W \), and a node \( h \) (hub of \( W \)) adjacent to every node of \( C \). A claw is the graph \( K_{1,3} \).

A gear \( B \) is a graph of eight nodes \( \{a, b_1, b_2, c, d_1, d_2, h_1, h_2\} \) such that \( W_1 = (h_1 : a, d_1, b_1, c, h_2) \) and \( W_2 = (h_2 : a, d_2, b_2, c, h_1) \) are 5-wheels (see Fig. 1); moreover, the edges of these wheels are the only edges of \( B \). When no confusion arises we shall denote as \( W_i = (h_i : C_i) \) for \( i = 1, 2 \), the two 5-wheels contained in the gear \( B \).

If \( w : V_G \rightarrow \mathbb{Q}_+ \) is any weighting of the nodes of \( G \), then \( \alpha(G, w) \) denotes the maximum weight of a stable set of \( G \). We refer to \( \alpha(G) = \alpha(G, 1) \) (1 being the vector of all ones) as the stability number of \( G \).

Given a vector \( \beta \in \mathbb{R}^m \) and a subset \( S \subseteq \{1, \ldots, m\} \), define \( \beta_S \in \mathbb{R}^{|S|} \) as the subvector of \( \beta \) restricted on the indices of \( S \) and \( \beta(S) = \sum_{i \in S} \beta_i \). Given a subset \( S \subseteq \{1, \ldots, m\} \), we denote by \( x_S \in \mathbb{R}^m \) the incidence vector of \( S \).

A linear inequality \( \sum_{j \in V_G} \pi_j x_j \leq \pi_0 \) is said to be valid for \( STAB(G) \) if it holds for all \( x \in STAB(G) \). For short, we also denote a linear inequality \( \pi^T x \leq \pi_0 \) as \( (\pi, \pi_0) \). A valid inequality for \( STAB(G) \) defines a facet of \( STAB(G) \) if and only if it is satisfied as an equality by \( |V_G| \) affinely independent incidence vectors of stable sets of \( G \) (called roots or tight solutions). We also say that a stable set \( S \) is tight for \( (\pi, \pi_0) \) if its incidence vector \( x_S \) is a tight solution of \( (\pi, \pi_0) \).

If the support of a facet defining inequality \( (\pi, \pi_0) \) coincides with \( V_G \), we say that the graph \( G \) supports (or produces) the corresponding facet or equivalently that \( (\pi, \pi_0) \) has full support on \( V_G \).

A linear inequality \( \sum_{j \in V_G} \pi_j x_j \leq \pi_0 \) is said to be a rank inequality for \( STAB(G) \) if \( \pi_i = 1 \) for each \( i \in S \subseteq V_G, \pi_i = 0 \) for each \( i \in V_G \setminus S \) and \( \pi_0 = \alpha(G[S]) \) where \( G[S] \) is the subgraph of \( G \) induced
by $S$. Given a 5-wheel $W = (h : v_1, v_2, v_3, v_4, v_5)$, then the inequality $\sum_{i=1}^{5} x_{v_i} + 2x_h \leq 2$ is called 5-wheel inequality.

We recall the definition of the sequential lifting procedure defined in [16] that will be used in the following sections. Let $\mathcal{S}(G)$ denote the family of the stable sets of $G$. If $\sum_{j \in V_G \setminus \{v\}} \pi_j x_j \leq \pi_0$ is a facet defining inequality of $STAB(G \setminus \{v\})$, then the inequality
\[
\sum_{j \in V_G \setminus \{v\}} \pi_j x_j + \pi_v x_v \leq \pi_0 \quad \text{with} \quad \pi_v = \pi_0 - \max_{S \in \mathcal{S}(G \setminus (N(v) \cup \{v\}))} \pi(S)
\]
is facet defining for $STAB(G)$. This inequality will be called sequential lifting of $(\pi_{V_G \setminus \{v\}}, \pi_0)$ and $\pi_v$ will be called the lifting coefficient of $v$. This procedure can be iterated to generate facet defining inequalities, simply called lifted inequalities, in a higher dimensional space.

2. Geared inequalities and g-lifted inequalities

An edge $v_1v_2$ of a graph $H$ is said to be simplicial if $K_1 = N(v_1) \setminus \{v_2\}$ and $K_2 = N(v_2) \setminus \{v_1\}$ are nonempty cliques of $H$. Notice that $K_1$ and $K_2$ may have nonempty intersection. Simplicial edges have a trivial though very useful polyhedral property:

**Proposition 2.1.** Let $H$ be a graph and $H'$ be a subgraph of $H$ that supports a facet defining inequality $(\pi, \pi_0)$ of $STAB(H)$ which is not a clique inequality. If $H'$ contains a simplicial edge $v_1v_2$, then $\pi_{v_1} = \pi_{v_2}$. If $H'$ contains a simplicial edge $v_1v_2$ subdivided with a node $t$, then $\pi_{v_1} = \pi_{v_2} = \pi_t$.

**Proof.** Since $v_1v_2$ is simplicial we have that $K_1 = N(v_1) \setminus \{v_2\}$ and $K_2 = N(v_2) \setminus \{v_1\}$ are nonempty cliques of $H'$. Let us consider a tight stable set $S_1$ missing $K_1 \cup \{v_1\}$ (it exists since $(\pi, \pi_0)$ is not a clique inequality). Clearly, $v_2 \in S_1$ (since otherwise $S_1 \cup \{v_1\}$ would violate $(\pi, \pi_0)$). Hence, $\pi_{v_2} \geq \pi_{v_1}$ (since otherwise $S_1 \setminus \{v_2\} \cup \{v_1\}$ would violate $(\pi, \pi_0)$). A symmetric argument proves that $\pi_{v_1} \geq \pi_{v_2}$ and the first claim follows.

Consider now a simplicial edge $v_1v_2$ subdivided with a node $t$. Obviously, $v_1t$ and $tv_2$ are both simplicial. Hence, we have that $\pi_{v_1} = \pi_t = \pi_{v_2}$ and the proposition follows.

We recall the definition of gear composition given in [7] together with a picture describing how it works:

**Definition 2.2.** Let $H = (V_H, E_H)$ be a graph with a simplicial edge $e = v_1v_2$ and let $B = (V_B, E_B)$ be a gear. The gear composition of $H$ and $B$ along $v_1v_2$ generates a new graph $G$ such that:

\[
V_G = V_H \setminus \{v_1, v_2\} \cup V_B,
E_G = E_H \setminus (\delta(v_1) \cup \delta(v_2)) \cup E_B \cup F_1 \cup F_2, \text{ where } F_i = \{d_iu | u \in K_i\} \cup \{b_iu | u \in K_i\} \text{ for } i = 1, 2.
\]

The graph $G$ will be called the geared graph generated by $H$ and $B$ along $e$ and denoted by $G = (H, B, e)$.

**Definition 2.3.** Let $H$ be a graph with a simplicial edge $e = v_1v_2$ and let $H^e$ be the graph obtained from $H$ by subdividing $e$ with a new node $t$.

An inequality $(\pi, \pi_0)$ which is valid for $STAB(H)$ is said to be g-extendable (with respect to $e$) if $\pi_{v_1} = \pi_{v_2} = \lambda > 0$ and it is not the inequality $x_{v_1} + x_{v_2} \leq 1$.

An inequality $(\pi, \pi_0)$ which is valid for $STAB(H^e)$ is said to be g-liftable (with respect to $e$) if $\pi_{v_1} = \pi_{v_2} = \pi_t = \lambda > 0$. 
Definition 2.4. Let $H = (V_H, E_H)$ be a graph containing the simplicial edge $e = v_1v_2$, let $B = (V_B, E_B)$ be a gear and let $(\pi, \pi_0)$ be a valid inequality for $STAB(H)$ that is g-extendable with respect to $e$. Then the inequalities

$$\sum_{i \in V_H \{v_1, v_2\}} \pi_i x_i + \lambda \sum_{i \in V_H \{h_1, h_2\}} x_i + 2\lambda(x_{h_1} + x_{h_2}) \leq \pi_0 + 2\lambda (1)$$

$$\sum_{i \in V_H \{v_1, v_2\}} \pi_i x_i + \lambda \sum_{i \in V_H \setminus A} x_i \leq \pi_0 + \lambda (2)$$

where $A \in \{\{b_1, c\}, \{b_2, c\}, \{d_1, a\}, \{d_2, a\}, \{a, c\}\}$ are called geared inequalities associated with $(\pi, \pi_0)$. The unique geared inequality that has full support on $V_B$ is (1) and it will be called proper geared inequality.

Geared inequalities are essential in the linear description of the stable set polytope of geared graphs. Indeed, it was proved that:

Theorem 2.5. [7] Let $G = (H, B, e)$ be a geared graph generated by $H$ and $B$ along $e$ and let $(\pi, \pi_0)$ be an inequality that is g-extendable with respect to $e$. If $(\pi, \pi_0)$ is facet defining for $STAB(H)$, then the proper geared inequality (1) associated with $(\pi, \pi_0)$ is facet defining for $STAB(G)$.

The above theorem can be extended to the geared inequalities (2) as follows:

Theorem 2.6. Let $G = (H, B, e)$ be a geared graph generated by $H$ and $B$ along $e$ and let $(\pi, \pi_0)$ be an inequality that is g-extendable with respect to $e$. If $(\pi, \pi_0)$ is facet defining for $STAB(H)$, then the geared inequalities (2) associated with $(\pi, \pi_0)$ are facet defining for $STAB(G)$ for each $A \in \{\{b_1, c\}, \{b_2, c\}, \{d_1, a\}, \{d_2, a\}, \{a, c\}\}$.

Proof. A sketch of the proof for the case $A = \{a, c\}$ was given in [7]. For the sake of completeness, we recall here the arguments used in that proof. Consider the graph $G'$ obtained from $H$ by subdividing the edge $e = v_1v_2$ with two nodes $h_1$ and $h_2$ and renaming $v_i$ as $d_i$, $i = 1, 2$. Clearly $G'$ is a subgraph of $G$ and, by a result of Wolsey [24] on edge subdivisions, the following inequality

$$\sum_{i \in V_{G'} \{v_1, v_2\}} \pi_i x_i + \lambda \sum_{i \in \{d_1, h_1, h_2, d_2\}} x_i \leq \pi_0 + \lambda$$
is facet defining for $STAB(G')$. This inequality can be lifted to yield a facet defining inequality of $STAB(G)$ by observing that $b_1$ and $b_2$ can be lifted with coefficient $\lambda$, and then $a$ and $c$ can be lifted with coefficient zero. This completes the proof of case $A = \{a, c\}$. The facet defining defining inequality corresponding to $A = \{b_1, c\}$ is obtained by first lifting the nodes $a$ and $b_2$ with coefficient $\lambda$ and then the nodes $b_1$ and $c$ with coefficient zero. The remaining cases can be proved analogously by changing the order of the lifted nodes.

**Example 2.1.** Consider the 5-hole $C_5$ and the geared graph $G$ obtained as the gear composition of $C_5$ and $B$ along the simplicial edge $e = v_1v_2$ (see Fig. 3). Thus, we write $G = (C_5, B, e)$.

![Figure 3: A 5-hole $C_5$ and a geared 5-hole $G$](image)

As the 5-hole inequality $x(V_{C_5}) \leq 2$ is valid for $STAB(C_5)$ and it is g-extendable with respect to $e$, the following inequality

$$x(V_G \setminus \{h_1, h_2\}) + 2x_{h_1} + 2x_{h_2} \leq 4$$

is a proper geared inequality associated with $x(V_{C_5}) \leq 2$. Since $x(V_{C_5}) \leq 2$ is facet defining for $STAB(C_5)$, the proper geared inequality associated with $x(V_{C_5}) \leq 2$ is facet defining for $STAB(G)$, by Theorem 2.5. Furthermore, the following five inequalities

$$x(V_G \setminus A) \leq 3, \text{ where } A \in \{\{d_2, a\}, \{d_1, a\}, \{b_2, c\}, \{b_1, c\}, \{a, c\}\},$$

are geared inequality associated with $x(V_{C_5}) \leq 2$ and are facet defining for $STAB(G)$, by Theorem 2.6.

The inequalities (1) and (2) (see Example 2.1) are not the only inequalities generated by the gear composition. In the remaining of this section we present another class of valid inequalities for $STAB(G)$ called g-lifted inequalities.

**Definition 2.7.** Let $H = (V_H, E_H)$ be a graph containing the simplicial edge $e = v_1v_2$, let $B = (V_B, E_B)$ be a gear and let $(\pi, \pi_0)$ be a valid inequality for $STAB(H^e)$ that is g-liftable with respect to $e$. Then the inequalities

$$\sum_{i \in V_H \setminus \{v_1, v_2\}} \pi_i x_i + \lambda \sum_{i \in V_B} x_i \leq \pi_0 + \lambda,$$  \hspace{1cm} (3)

$$\sum_{i \in V_H \setminus \{v_1, v_2\}} \pi_i x_i + \lambda \sum_{i \in V_B \setminus A} x_i \leq \pi_0,$$  \hspace{1cm} (4)

where $A \in \{\{b_1, c, b_2, h_1, h_2\}, \{d_1, a, d_2, h_1, h_2\}\}$

are called g-lifted inequalities associated with $(\pi, \pi_0)$. The unique g-lifted inequality that has full support on $V_B$ is (3) and it will be called proper g-lifted inequality.
Inequalities 4 are clearly valid, as their supporting graph $G \setminus A$ is isomorphic to $H^e$. We then prove that the proper g-lifted inequality is valid for $STAB(G)$.

**Lemma 2.8.** Let $G = (H, B, e)$ be a geared graph generated by $H$ and $B$ along $e$ and let $(\pi, \pi_0)$ be an inequality that is g-liftable with respect to $e$. Then the proper g-lifted inequality (3) associated with $(\pi, \pi_0)$ is valid for $STAB(G)$.

**Proof.** Let $(\bar{\pi}, \bar{\pi}_0)$ denote the proper g-lifted inequality (3) and let $S$ be a maximal stable set of $G$. To prove the lemma we distinguish three cases depending on the intersection of $S$ with the subset \{b_1, b_2, d_1, d_2\} of $V_B$. If $|S \cap \{b_1, b_2, d_1, d_2\}| = 2$, then $K_1 \cap S = K_2 \cap S = \emptyset$ and the set $S \setminus V_B$ is a stable set of $H^e$. It follows that $\pi(S \setminus V_B) = \pi(S \setminus V_B) \leq \pi_0 - 2\lambda$, since otherwise the stable set $S \setminus V_B \cup \{v_1, v_2\}$ of $H^e$ would violate $(\pi, \pi_0)$. Moreover, $\pi(S \setminus V_B) \leq 3\lambda$ and thus, $\pi(S \setminus V_B) + \pi(S \setminus V_B) \leq \pi_0 - 2\lambda + 3\lambda = \pi_0 + \lambda$.

If $|S \cap \{b_1, b_2, d_1, d_2\}| = 1$, we first suppose that $b_1 \in S$; then, $b_2, h_1, c, d_1, d_2 \notin S$ and $S \setminus V_B$ contains exactly one node in $\{b_2, a\}$. Thus, $\pi(S \setminus V_B) = 2\lambda$. Since $S \cap K_1 = \emptyset$, $(S \setminus V_B) \cup \{v_1\}$ is a stable set of $H^e$, and so $\pi(S \setminus V_B) = \pi(S \setminus V_B) \leq \pi_0 - \lambda$. Hence, $\pi(S \setminus V_B) + \pi(S \setminus V_B) \leq \pi_0 - \lambda + 2\lambda = \pi_0 + \lambda$ and the result follows. The cases with $b_2 \in S, d_1 \in S$, or $d_2 \in S$ are analogous.

In the last case, $|S \cap \{b_1, b_2, d_1, d_2\}| = 0$ and $S \setminus V_B$ is a stable set in $H^e$. We have that $\pi(S \setminus V_B) = \pi(S \setminus V_B) \leq \pi_0 - \lambda$ since otherwise the stable set $(S \setminus V_B) \cup \{t\}$ of $H^e$ would violate $(\pi, \pi_0)$. By the maximality of $S$, exactly one among the sets $\{h_1\}, \{h_2\}$, and $\{a, c\}$, is contained in $S$, thus implying that $\pi(S \setminus V_B) \leq 2\lambda$. Hence, $\pi(S \setminus V_B) + \pi(S \setminus V_B) \leq \pi_0 - \lambda + 2\lambda$ and the thesis follows. $lacksquare$

In the following we provide sufficient conditions for the class of g-lifted inequalities to be facet defining. Next theorem is the analogous of theorems 2.5 and 2.6 for g-lifted inequalities.

**Theorem 2.9.** Let $G = (H, B, e)$ be a geared graph generated by $H$ and $B$ along $e$ and let $(\pi, \pi_0)$ be an inequality that is g-liftable with respect to $e$. If $(\pi, \pi_0)$ is facet defining for $STAB(H^e)$, then the proper g-lifted inequality (3) and the g-lifted inequalities (4) for $A \in \{\{b_1, c, b_2, h_1, h_2\}, \{d_1, a, d_2, h_1, h_2\}\}$ associated with $(\pi, \pi_0)$, are facet defining for $STAB(G)$.

**Proof.** We first prove the theorem for the proper g-lifted inequality. Suppose that $\beta^T x \leq \beta_0$ is a facet defining inequality for $STAB(G)$ that contains all the roots of (3): we prove below that such inequality is equivalent to (3).

We first show that we have that the coefficients $\beta_v$ associated with nodes $v \in V_B$ are equal. Let $x^S_i, i = 1, 2$, be roots of $(\pi, \pi_0)$ such that $S_i \cap (K_i \cup \{v_i\}) = \emptyset$. These roots always exist because $(\pi, \pi_0)$ has $\pi_{v_1} = \pi_{v_2} = \lambda > 0$ and so, it is not the clique inequality defined by $K_i \cup \{v_i\}, i = 1, 2$. Now $t$ must belong to $S_i$ since otherwise $S_i \cup \{v_i\}$ would violate $(\pi, \pi_0)$. Consider the following stable sets whose incidence vectors are roots of (3):

\[
S^1_1 = S_1 \setminus \{t\} \cup \{a, c\}, \\
S^2_1 = S_1 \setminus \{t\} \cup \{a, b_1\}, \\
S^3_1 = S_1 \setminus \{t\} \cup \{h_2, b_1\}, \\
S^4_1 = S_1 \setminus \{t\} \cup \{h_2, d_1\}, \\
S^5_1 = S_1 \setminus \{t\} \cup \{c, d_1\}.
\]

From $\beta(S^1_i) = \beta(S^2_i) = \beta(S^3_i) = \beta(S^4_i) = \beta(S^5_i)$, it follows that $\beta_v = \beta_{v_1} = \beta_{v_2} = \lambda$. Analogously, using $S_2$ it can be proved that $\beta_v = \beta_{v_1} = \beta_{v_2} = \lambda$.

Let $M$ be a matrix whose rows are $|V_H|$ incidence vectors of stable sets of $H^e$ which are linearly independent roots of $(\pi, \pi_0)$, i.e.,

\[ M \pi = \pi_0 \mathbb{1}. \] (5)
Any stable set $\tilde{S}$ of $H^e$ can be transformed into a stable set $S$ of $G$ as follows: set $S = \tilde{S} \setminus \{v_1, v_2, t\} \cup S_B$, where $S_B$ is a stable set of $B$ such that $d_i \in S_B$ if and only if $v_i \in \tilde{S}$ for $i = 1, 2$ and moreover, $a \in S_B$ if and only if $t \in \tilde{S}$. It is not difficult to verify that if $x^\tilde{S}$ defines a root of $(\pi, \pi_0)$ then $S_B$ can be chosen so that $x^S$ defines a root of (3) such that $\beta(S \cap \{h_1, h_2, c\}) = \beta_{h_1}$, since $\{h_1, h_2, c\}$ is a clique and $\beta_{h_1} = \beta_{h_2} = \beta_c$. By replacing $V_{H^e}$ with $V' = V_{H^e} \setminus \{v_1, v_2, t\} \cup \{d_1, d_2, a\}$, we have $M \beta_{V'} = (\beta_0 - \beta_{h_1}) \mathbb{1}$ and by (5),

$$\beta_{V'} = (\beta_0 - \beta_{h_1})M^{-1} \mathbb{1} = \frac{\beta_0 - \beta_{h_1}}{\pi_0} \pi.$$

In particular, since $\beta_{d_1} = \beta_{h_1}$ we have

$$\beta_{d_1} = \frac{\beta_0 - \beta_{d_1}}{\pi_0} \pi v_1 = \frac{\beta_0 - \beta_{d_1}}{\pi_0} \lambda.$$

Then $\beta_{d_1} > 0$ and, without loss of generality, we can fix $\beta_{d_1} = \lambda$; as a consequence, we have that

$$\beta_0 = \pi_0 + \lambda,$$

$$\beta_u = \pi_u \quad \text{for each } u \in V_{H^e} \setminus \{v_1, v_2, t\},$$

and the first part of the theorem follows.

Consider now the inequalities (4). They are isomorphic to the original g-liftable inequality $(\pi, \pi_0)$ and hence they are trivially valid. If $A = \{b_1, c, b_2, h_1, h_2\}$, it is easy to check that the lifting coefficients of the nodes, e.g., in the order $h_1, h_2, b_1, b_2, c$, are all equal to zero. This argument proves that these inequalities are facet defining for $STAB(G)$. □

**Example 2.2.** Consider the 4-hole $C_4$ and the geared graph $G$ obtained as the gear composition of $C_4$ and $B$ along the simplicial edge $e = v_1v_2$ (see Fig. 3). Thus, we write $G = (C_4, B, e)$.

![Figure 4: A 4-hole $C_4$ and a geared 4-hole $G$](image)

The subdivision of the simplicial edge $e = v_1v_2$ with a new node $t$ generates a 5-hole $C_5^2$. Since $x(V_{C_5^2}) \leq 2$ is valid for $STAB(C_5^2)$ and it is g-liftable with respect to $e$, the inequality $x(V_G) \leq 3$ is a proper g-lifted inequality associated with $x(V_{C_5^2}) \leq 2$. Since $x(V_{C_5^2}) \leq 2$ is facet defining for $STAB(C_5^2)$, this proper g-lifted inequality is also facet defining for $STAB(G)$, by Theorem 2.9. Moreover, the following two inequalities

$$x(V_G \setminus A) \leq 3 \text{ where } A \in \{\{b_1, c, b_2, h_1, h_2\}, \{d_1, a, d_2, h_1, h_2\}\},$$

are non proper g-lifted inequalities associated with $x(V_{C_5^2}) \leq 2$ and they are also facet defining for $STAB(G)$, by Theorem 2.9. □
The above results show that facet defining inequalities for $STAB(H)$ and $STAB(H^c)$ generate geared and g-lifted inequalities, respectively, that are facet defining for $STAB(G)$ when $G = (H, B, e)$ is the geared graph generated by $H$ and $B$ along $e$. This implies that geared and g-lifted inequalities are necessary for the linear description of $STAB(G)$. Next section will be devoted to prove that they are also sufficient.

3. Gear composition of polyhedra

In this section we show that, apart from clique and 5-wheel inequalities, geared inequalities and g-lifted inequalities are the only new linear inequalities involving $B$ that are necessary to describe $STAB(G)$ when $G$ is a geared graph generated by $H$ and $B$ along $e$.

Throughout this section, we indicate by $(\beta, \beta_0)$ a generic facet defining inequality for $STAB(G)$; we split the vector of coefficients $\beta$ into two subvectors $(\beta_{V \setminus B}, \beta_B)$ where $\beta_{V \setminus B}$ is the vector of coefficients associated with nodes $V_G \setminus V_B$ and $\beta_B$ is the vector of coefficients associated with nodes $V_B$. Moreover, the components of $\beta_B$ will be indexed as follows: $\beta_B = (\beta_{d_1}, \beta_{h_1}, \beta_{h_2}, \beta_c, \beta_{a_2}, \beta_{d_2}, \beta_{h_2})$.

We first observe that if $e$ is a simplicial edge and $K_1 = K_2$ then the geared graph $G$ generated by $H$ and $B$ along $e$ has a clique-cutset $K_1 = K_2$. When this happens the results of Chvátal on the composition of polyhedra [4] explain how to find a defining linear system for $STAB(G)$ from the defining linear systems of $STAB(H)$ and $STAB(K_1 \cup \{v_1, v_2\}, B, e)$. So, in the rest of the paper we will focus on the composition of polyhedra resulting from applying the gear composition along a simplicial edge that has $K_1 \neq K_2$.

We state now the main theorem of this paper.

**Theorem 3.1.** Let $G = (H, B, e)$ be a geared graph generated by $H$ and $B$ along the simplicial edge $e$. Then the stable set polytope $STAB(G)$ is described by the following linear inequalities:

- nonnegativity inequalities,
- clique inequalities,
- (lifted) 5-wheel inequalities,
- geared inequalities associated with facet defining inequalities of $STAB(H)$ having nonzero coefficients on the endnodes of $e$,
- g-lifted inequalities associated with facet defining inequalities of $STAB(H^c)$ having nonzero coefficients on the endnodes of $e$,
- facet defining inequalities of $STAB(H)$ having zero coefficients on the endnodes of $e$.

**Proof.** Since the proof of this result is quite technical and up to some extent repetitive, we arrange it into three main steps that are illustrated below (each step is proved in a separate subsection). We consider a facet defining inequality $(\beta, \beta_0)$ for $STAB(G)$ that is neither a clique inequality nor a lifted 5-wheel inequality. We denote as $V' = V_G \setminus V_B$ and by $\lambda$ a positive scalar number. We also assume that the components of $\beta_B$ are not all zero. Then we show that:

1) If $(\beta, \beta_0)$ does not have full support on $V_B$ and we denote by $A \subset V_B$ the set $\{u \in V_B : \beta_u = 0\}$, then $(\beta, \beta_0)$ has the form:

$$\beta_{V'}^T x_{V'} + \lambda x_{E \setminus A} \leq \beta_0$$

where $A \in \{\{b_1, c\}, \{b_2, c\}, \{d_1, a\}, \{d_2, a\}, \{a, c\}, \{b_1, c, b_2, h_1, h_2\}, \{d_1, a, d_2, h_1, h_2\}\}$ (by Theorem 3.4 in Subsection 3.1).
2) If \((\beta, \beta_0)\) has full support on \(V_B\) then it has one of the following forms:

   a) \(\beta^T V \nu + \lambda x_{B\setminus \{h_1, h_2\}} + 2\lambda (x_{h_1} + x_{h_2}) \leq \beta_0,\)

   b) \(\beta^T V \nu + \lambda x_B \leq \beta_0,\)

(by Theorem 3.9 in Subsection 3.2)

3) If \((\beta, \beta_0)\) has the form described in 1) with \(A \in \{\{b_1, c\}, \{b_2, c\}, \{d_1, a\}, \{d_2, a\}, \{a, c\}\}\) or the form described in 2a) then it is a geared inequality associated with a facet defining inequality of \(STAB(H)\) (by Theorem 3.10 and Theorem 3.11 in Subsection 3.3);

If \((\beta, \beta_0)\) has the form described in 1) with \(A \in \{\{b_1, c, b_2, h_1, h_2\}, \{d_1, a, d_2, h_1, h_2\}\}\) or the form described in 2b) then it is a g-lifted inequality associated with a facet defining inequality of \(STAB(H^e)\) (by Theorem 3.12 in Subsection 3.3.)

As a consequence of the above results, we have that each facet defining inequality for \(STAB(G)\) which is different from clique inequalities and 5-wheel inequalities and has \(\beta_B \neq 0\) is:

either an inequality of type (1) or (2) where \((\pi, \pi_0)\) is a g-extendable facet defining inequality of \(STAB(H)\),

or an inequality of type (3) or (4) where \((\pi, \pi_0)\) is a g-liftable facet defining inequality of \(STAB(H^e)\).

Finally, Proposition 2.1 establishes that every facet defining inequality for \(STAB(H)\), that is not a clique inequality, cannot have a zero coefficient on one endnode of \(e\) and a nonzero coefficient on the other endnode. Hence, facet defining inequalities for \(STAB(H)\) with zero coefficient on the endnodes of \(e\) have a supporting graph that is a subgraph of \(G\) and may be lifted with zero coefficients. Thus the thesis follows.

### 3.1. Inequalities not having full support on \(V_B\)

In this section we deal with inequalities that do not have full support on \(V_B\). Throughout this section we shall denote by \(A\) the set \(\{u \in V_B : \beta_u = 0\}\). If an inequality \((\beta, \beta_0)\) does not have full support on \(V_B\) then \(A \neq \emptyset\). We start by recalling the arguments that will often be used in the proofs of this subsection. The first one is a well-known result of Chvátal:

**Theorem 3.2.** [4] The supporting graph of a facet defining inequality for \(STAB(G)\) does not have a clique-cutset.

The next observation concerns the lifting coefficients of nodes in \(A\). More precisely,

**Observation 1.** Let \(G = (H, B, e)\) be a geared graph and let \(G \setminus A\) be the subgraph of \(G\) supporting the facet defining inequality \((\beta, \beta_0)\) of \(STAB(G)\), namely \(A = \{u \in V_B : \beta_u = 0\}\). Then every node of \(u \in A\) has lifting coefficient \(\beta_u = 0\).

As a consequence of Observation 1 we have that if \((\beta, \beta_0)\) is facet defining for \(STAB(G)\) that does not have full support on \(V_B\) then each node \(u \in A\) has lifting coefficient \(\beta_u = 0\) (for short, has 0-lifting coefficient). By the definition of lifting this implies that:

**Observation 2.** For each node \(u \in A\), there exists a tight stable set \(S_u\) in \(G \setminus (A \cup N(u))\).
Notice also that if there exist two adjacent nodes \( u \in A \) and \( v \in V_G \setminus A \) with \( N(v) \setminus \{u\} \subseteq N(u) \), then every stable set \( S' \) in \( G \setminus (A \cup N(u)) \) can be augmented by adding the node \( v \). This implies that a tight stable set in \( G \setminus (A \cup N(u)) \) satisfying Observation 2 does not exist, a contradiction. Hence,

**Observation 3.** No node \( u \in A \) is adjacent to a node \( v \in V_G \setminus A \) with \( N(v) \setminus \{u\} \subseteq N(u) \).

Moreover, we will also use the following arguments:

**Observation 4.** Let \( G \) be a graph and let \((\pi, \pi_0)\) and \((\beta, \beta_0)\) be two facet defining inequalities for \( STAB(G) \). If \((\beta, \beta_0)\) is not a positive scalar multiple of \((\pi, \pi_0)\) then there exists a stable set \( S \) that is tight for \((\beta, \beta_0)\), i.e., \( \beta x^S = \beta_0 \), but not for \((\pi, \pi_0)\), i.e., \( \pi x^S < \pi_0 \).

In the next proofs clique inequalities or 5-wheel inequalities will play the role of \((\pi, \pi_0)\). In these cases, we will say that there exists a tight stable set \( S \) for \((\beta, \beta_0)\) that misses a certain clique in \( V_B \cup K_1 \cup K_2 \) or one of the two 5-wheels contained in \( B \).

**Observation 5.** Let \( G \) be a graph and let \((\beta, \beta_0)\) be a facet defining inequality for \( STAB(G) \). Then for any \( u \in V_G \) there exists at least a root of \((\beta, \beta_0)\) containing \( u \).

We are now ready to prove that, if \( G \) is a geared graph, then for any facet defining inequality \((\beta_{V \setminus B}, \beta_B, \beta_0)\) for \( STAB(G) \) that has not full support on \( V_B \), the vector \( \beta_B \) can assume only 7 different values (listed in Theorem 3.4). This will be proved in two steps: first we show which are the zero components of \( \beta_B \) (Lemma 3.3); then we prove that all the nonzero components of \( \beta_B \) are equal (Theorem 3.4).

**Lemma 3.3.** Let \( G = (H, B, e) \) be a geared graph and let \((\beta_{V \setminus B}, \beta_B, \beta_0)\) be a facet defining inequality for \( STAB(G) \) with both \( \beta_{V \setminus B} \) and \( \beta_B \) different from the zero vector. If \((\beta, \beta_0)\) does not have full support on \( V_B \) and it is neither a clique inequality nor a 5-wheel inequality, then one of the following cases occurs:

1) \( A \in \{\{b_1, c\}, \{b_2, c\}, \{d_1, a\}, \{d_2, a\}, \{a, c\}\} \),

2) \( A \in \{\{b_1, c, b_2, h_1, h_2\}, \{d_1, a, d_2, h_1, h_2\}\} \).

**Proof.** Without loss of generality, let \( G \setminus A \) denote the supporting graph of the inequality \((\beta, \beta_0)\), i.e., the subgraph induced by the nodes of \( G \) associated with nonzero components of \( \beta \). Clearly, \( G \setminus A \) has to be connected and, by Theorem 1, it has no clique-cutsets. If \( G \setminus A \) satisfies these two properties we say that \( G \setminus A \) is admissible. The proof consists in showing that, apart from those listed in the thesis, all admissible configurations of \( A \) yield a contradiction.

Observe that if there does not exist a path between \( K_1 \) and \( K_2 \) contained in \( B \), then \( K_1 \) and \( K_2 \) are clique-cutsets of \( G \setminus A \). It is not difficult to check that if \( G \setminus A \) is admissible then \(|A| \leq 5 \). If \(|A| = 5 \) and \( G \setminus A \) is admissible then case 2) occurs. If \(|A| = 4 \) and \( G \setminus A \) is admissible then \( A \) is isomorphic to one of the following configurations (that are derived by enumeration as described in Appendix A):

i) \( A = \{b_1, a, c, b_2\} \),

ii) \( A = \{b_1, a, c, d_2\} \),

iii) \( A = \{b_1, c, h_1, d_2\} \).

In the first two cases Observation 3 is contradicted by nodes \( a \) and \( h_1 \); in the third case the nodes \( d_1 \) and \( d_2 \) contradict Observation 3. Hence, \(|A| = 4 \) cannot occur.

If \(|A| = 3 \) and \( G \setminus A \) contains no clique-cutset then \( A \) is isomorphic to one of the following configurations:

i) \( A = \{b_1, a, c\} \),
In cases (i) and (ii), the $b_1$ and $d_1$ contradict Observation 3, and so, they cannot occur. Consider case (iii): let $S_{h_1}$ be a tight stable set in $G \setminus (A \cup N(h_1))$ (it exists by Observation 2). Clearly, $S_{h_1}$ contains $d_2$ (since otherwise $S_{h_1} \cup \{a\}$ would violate $(\beta, \beta_0)$) and a node in $K_1$ (since otherwise $S \cup \{d_1\}$ would violate $(\beta, \beta_0)$). It follows that $\beta_{d_2} \geq \beta_{h_2} + \beta_a$. Now let $T$ be a tight stable set missing the clique $\{a, h_2, d_2\}$. Then $b_2 \in T$ (otherwise $T \cup \{h_2\}$ is feasible and violates $(\beta, \beta_0)$) and, consequently, $\beta_{b_2} \geq \beta_{d_2}$; as $\beta_a > 0$, this is a contradiction. Finally consider case (iv): let $S_{h_1}$ a tight stable set in $G \setminus (A \cup N(h_1))$ (it exists by Observation 2). Clearly, $S_{h_1}$ contains $a$ (otherwise $S_{h_1} \cup \{d_1\}$ would violate $(\beta, \beta_0)$) and so, $\beta_a \geq \beta_{h_2} + \beta_{d_1}$. Let $T$ be a tight stable set missing the clique $\{d_1, a, h_1\}$. Then $h_2 \in T$ and $\beta_{h_2} \geq \beta_a$, a contradiction.

If $|A| = 2$ and $G \setminus A$ contains no clique-cutset then $A$ is isomorphic to one of the following configurations:

i) $A = \{a, c\}$,
ii) $A = \{b_1, c\}$,
iii) $A = \{b_1, a\}$,
iv) $A = \{b_1, h_2\}$,
v) $A = \{h_1, h_2\}$,
vi) $A = \{b_1, d_2\}$.

The cases (i) and (ii) are listed in 1) of the thesis. Notice that all the remaining cases of the thesis are isomorphic to case (ii) and so they can be proved by symmetry.

In case (iii), the nodes $b_1$ and $d_1$ contradict Observation 3; in case (iv), Observation 3 is contradicted by $h_2$ and $c$. In case (v), as the node $h_1$ has a 0-lifting coefficient, i.e., there exists a tight stable set $S_{h_1}$ in $G \setminus (A \cup N(h_1))$; it is not difficult to see that $S_{h_1}$ contains either $d_2$ or $b_2$. But then either $S_{h_1} \cup \{c\}$ or $S_{h_1} \cup \{a\}$ violates $(\beta, \beta_0)$, a contradiction. So, all cases (iii)-(v) yield a contradiction.

It remains to consider the case (vi). By Observation 2, as the node $d_2 \in A$, there exists a tight stable set $S_{d_2}$ in $G \setminus (A \cup N(d_2))$. It is not difficult to see that $S_{d_2} \supseteq \{c, d_1\}$ and so, $\beta_{h_2} \leq \beta_c$. Now let $S$ be a tight stable set missing $\{h_1, d_1, a\}$. Then $h_2 \in S$ and so, $\beta_{h_2} = \beta_c$. But then $S \setminus \{h_2\} \cup \{a, c\}$ violates $(\beta, \beta_0)$, a contradiction.

If $|A| = 1$ then there are three nonisomorphic cases to be considered: $A = \{b_1\}$, $A = \{c\}$, and $A = \{h_1\}$.

**Case 1.** $A = \{b_1\}$.

Let $T$ be a tight stable set missing the clique $\{b_2, h_2, c\}$. Clearly $h_1 \in T$ (since otherwise $T \cup \{c\}$ would violate $(\beta, \beta_0)$). By Observation 2, the node $b_1$ has a 0-lifting coefficient, i.e., there exists a tight stable set $S_{b_1}$ in $G \setminus (A \cup N(h_1))$. It is not difficult to see that $S_{b_1}$ contains $\{a, b_2\}$. Then $\beta_a \geq \beta_{h_1}$, $\beta_{h_2} \geq \beta_{d_1}$ and $\beta_{h_2} \geq \beta_c$.

Since $S_{b_1} \supseteq \{a, b_2\}$ and $S_{b_1} \setminus \{a, b_2\} \cup \{d_1, c, d_2\}$ is a stable set, it follows that $\beta_a + \beta_{h_2} \geq \beta_{d_1} + \beta_{h_2} + \beta_c$. If $\beta_a = \beta_{d_1}$ then $\beta_{h_2} \geq \beta_c + \beta_{d_2}$. Since all coefficients of $\beta_B$ apart from $\beta_{b_1}$ are positive, we have that $\beta_{h_2} > \beta_{d_2}$. This implies that $d_2 \notin T$ (since otherwise $T \setminus \{d_2\} \cup \{b_2\}$ would violate $(\beta, \beta_0)$) and so, $T \setminus \{h_1\} \cup \{a, c\}$ violates $(\beta, \beta_0)$, a contradiction.

Hence, $\beta_a > \beta_{d_1}$. Thus every tight stable set $S$ containing $b_2$ contains either $a$ or $h_1$ and every tight stable set $S$ containing $c$ contains either $a$ or $d_2$. In fact, in all other cases, $d_1 \in S$ and $S \setminus \{d_1\} \cup \{a\}$ violates $(\beta, \beta_0)$, a contradiction. Moreover every tight stable set $S$ containing $a$ contains either $b_2$ or $c$ and every tight stable set containing $d_2$ contains either $h_1$ or $c$. Finally every tight stable set containing $h_1$ contains either $b_2$ or $d_2$ (since otherwise $S \setminus \{h_1\} \cup \{a, c\}$ would violate $(\beta, \beta_0)$).
As a consequence, every tight solution of \((\beta, \beta_0)\) is also a tight solution of the 5-wheel inequality 
\((\pi, \pi_0)\) supported by \(W_2 = (b_2 : a, d_2, b_2, c, h_1)\), contradicting Observation 4. \(\text{(End of Case 1)}\)

Case 2. \(A = \{c\}\).

By Observation 3, the node \(c\) has a 0-lifting coefficient, i.e., there exists a tight stable set \(S_c\) in \(G \setminus (A \cup N(c))\). It is not difficult to see that \(S_c\) contains \(\{d_1, d_2\} \cup \{a\}\).

Suppose first that \(\{d_1, d_2\} \subseteq S_c\). From this, it follows that \(\beta_{d_1} \geq \beta_{h_1}\) and \(\beta_{d_2} \geq \beta_{h_1}\), \(i = 1, 2\). If \(\beta_{h_1} = \beta_{d_i}, i = 1, 2\), then \(S_c \setminus \{d_1, d_2\} \cup \{b_1, b_2, a\}\) violates \((\beta, \beta_0)\), a contradiction. Without loss of generality, suppose that \(\beta_{h_1} < \beta_{d_1}\).

Let \(S\) be a tight stable set missing the clique \(\{b_1, d_1\} \cup K_1\). Clearly, \(S\) contains \(h_1\). If \(\beta_{d_1} > \beta_{h_1}\), then \(S \setminus \{h_1\} \cup \{d_1\}\) would violate \((\beta, \beta_0)\), a contradiction. Hence, \(\beta_{d_1} = \beta_{h_1}\). We distinguish now three different cases:

- \(\beta_{h_1} < \beta_a\).

Let us consider a tight stable set \(S\) missing the clique \(\{a, h_2, d_2\}\). Then \(d_1\) or \(h_1\) belongs to \(S\) and, the stable set obtained by replacing \(d_1\) or \(h_1\) in \(S\) with \(a\) violates \((\beta, \beta_0)\), a contradiction.

- \(\beta_{h_1} > \beta_a\).

Since \(\beta_{h_1} < \beta_{d_1}\), every tight stable set \(S\) containing \(b_1\) contains \(a\) (since otherwise \(S \setminus \{b_1\} \cup \{d_1\}\) would violate \((\beta, \beta_0)\)) and every tight stable set containing \(a\) (since otherwise \(S \setminus \{a\} \cup \{h_1\}\) would violate \((\beta, \beta_0)\)), thus implying that the tight solutions of \((\beta, \beta_0)\) are not linearly independent, a contradiction.

- \(\beta_{h_1} = \beta_a\).

Since \(S_c \setminus \{d_1, d_2\} \cup \{a, b_1, b_2\}\) is a stable set, we have that \(\beta_a + \beta_{h_1} + \beta_{b_2} \leq \beta_{d_1} + \beta_{d_2}\). Since \(\beta_a = \beta_{h_1} = \beta_{b_2}\), we have that \(\beta_{b_2} < \beta_{d_2}\). Consider a tight stable set \(S'\) missing \(\{h_1, d_1, a\}\). Since \(\beta_{b_2} < \beta_{d_1}\), we have that \(b_1 \not\in S'\) (otherwise \(S' \setminus \{b_1\} \cup \{d_1\}\) violates \((\beta, \beta_0)\)) and \(h_2 \in S'\).

So, \(\beta_{h_2} \geq \beta_{d_2}\). Let \(S''\) be a tight stable set missing \(\{h_2, d_2, b_2\}\). It contains \(h_1\) or \(a\) and so, \(\beta_{h_1} = \beta_a \geq \beta_{h_2}\). Moreover, every tight stable set missing \(\{b_2, d_2\}\) \(K_2\) clearly contains \(h_2\) and yields \(\beta_{h_2} \geq \beta_{d_2}\). Hence, \(\beta_{h_2} = \beta_a \geq \beta_{d_2}\). But then every tight stable set \(S\) containing \(b_1\) contains \(b_2\). In fact, \(S\) contains \(a\) (since otherwise \(S \setminus \{b_1\} \cup \{d_1\}\) would violate \((\beta, \beta_0)\)) and \(b_2\) (since otherwise \(S \setminus \{b_1, a\}\) \(\cup \{d_1, h_2\}\) would violate \((\beta, \beta_0)\)). A symmetric argument shows that every tight stable set containing \(b_2\) also contains \(b_1\), thus implying that the tight solutions of \((\beta, \beta_0)\) are not linearly independent, a contradiction.

Suppose now that \(\{d_1, d_2\} \not\subseteq S_c\) and so, \(a \in S_c\). Let \(S_1\) be a tight stable set containing \(d_i, i = 1, 2\). Since \(S_1 \setminus \{d_i\} \cup \{b_1\}\) is a stable set, we have that \(\beta_{d_i} \geq \beta_{h_i}, i = 1, 2\). Let \(S'\) be a tight stable set missing \(\{a, h_1, h_2\}\). Since, by hypothesis, there does not exist a tight stable set containing both \(d_1\) and \(d_2\), we have that \(S'\) contains neither \(\{d_1, b_2\}\) nor \(\{d_2, b_1\}\). It follows that \(S' \cup \{a\}\) violates \((\beta, \beta_0)\), a contradiction. \(\text{(End of Case 2)}\)

Case 3. \(A = \{h_1\}\).

By Observation 3, the node \(h_1\) has a 0-lifting coefficient, i.e., there exists a tight stable set \(S_{h_1}\) in \(G \setminus (A \cup N(h_1))\). It is not difficult to see that \(S_{h_1}\) contains either \(d_2\) or \(b_2\). But then either \(S_{h_1} \cup \{c\}\) or \(S_{h_1} \cup \{a\}\) violates \((\beta, \beta_0)\), a contradiction. \(\text{(End of Case 3)}\)

Thus the lemma follows.

The next theorem shows that all the nonzero components of \(\beta_B\) are equal.

**Theorem 3.4.** Let \(G = (H, B, e)\) be a geared graph generated by \(H\) and \(B\) along the simplicial edge \(e = v_1v_2\) and let \(V' = V_H \setminus \{v_1, v_2\}\). Then each facet defining inequality \((\beta_{V'}, \beta_B, \beta_0)\) of \(STAB(G)\)
that does not have full support on $V_B$, that is neither a clique inequality nor a 5-wheel inequality, and has $\beta_B \neq 0$, is of the following form:

$$\beta^T_V x_V + \lambda x_{B \setminus A} \leq \beta_0$$

where $A \in \{\{b_1, c\}, \{b_2, c\}, \{d_1, a\}, \{d_2, a\}, \{a, c\}, \{b_1, c, b_2, h_1, h_2\}, \{d_1, a, d_2, h_1, h_2\}\}$ and $\lambda > 0$.

**Proof.** By Lemma 3.3, we know that $\beta_B$ has either five zero components or two zero components like in the thesis. In the first case we have that the supporting graph of $(\beta, \beta_0)$ is a subgraph of $H^c$ containing the subdivision of the simplicial edge $e$ and, by Proposition 2.1, we are done.

It remains to show that if $\beta_B$ has two zero components then all the remaining components are equal. By Lemma 3.3, the vector $\beta_B$ satisfies one of the following conditions: $A = \{a, c\}$, $A = \{b_1, c\}$, $A = \{b_2, c\}$, $A = \{d_1, a\}$, $A = \{d_2, a\}$.

Suppose first that $A = \{a, c\}$. We have that $\beta_{d_i} = \beta_{b_i}$, $i = 1, 2$ otherwise there would not exist stable sets containing each of the nodes $d_1, d_2, b_1, b_2$ which are tight for $(\beta, \beta_0)$ (such stable sets must exist for Observation 5). Now, if $\beta_h > \beta_1$ then the tight stable set missing $\{h_1, h_2\}$ would violate $(\beta, \beta_0)$ after replacing $d_1$ or $b_1$ with $h_1$. Moreover, if $\beta_h < \beta_1$ then the tight stable set missing $K_1 \cup \{d_1, b_1\}$ would violate $(\beta, \beta_0)$ after replacing $h_1$ with $d_1$. Hence, $\beta_h = \beta_1$ and similar arguments prove that $\beta_{h_2} = \beta_2$. As the edge $h_1 h_2$ is simplicial in $G \setminus A$, we have, by Proposition 2.1, that $\beta_{h_1} = \beta_{h_2}$. Thus all nonzero coefficients of $\beta_B$ are equal and we are done.

The last four cases are symmetric, so we prove in detail the first one and symmetric arguments will prove the remaining cases. Suppose that $A = \{b_1, c\}$ and all components of $\beta_B$ different from $\beta_b$ and $\beta_{c}$ are nonzero. Since $b_1$ has a 0-lifting coefficient with respect to $(\beta, \beta_0)$, we have that there exists a stable set $S_b$ in $G \setminus (A \cup N(b_1))$ which is tight for $(\beta, \beta_0)$. Clearly, $a \in S_b$; it follows that $\beta_b \leq \beta_a$ and $\beta_{b_1} \leq \beta_a$. If $b_2 \notin S_b$, then $\beta_a \geq \beta_{h_2} + \beta_{d_1}$ (since otherwise $S_a \setminus \{a\}$ contains $\{d_2, h_1\}$ we would violate $(\beta, \beta_0)$). Thus, $\beta_a > \beta_{h_2}$. Now, consider a stable set $S'$ which is tight for $(\beta, \beta_0)$ and misses the clique $\{a, d_1, h_1\}$. It has to contain $h_2$, but then $S' \setminus \{h_2\} \cup \{a\}$ violates $(\beta, \beta_0)$, a contradiction. Hence, $b_2 \in S_b$.

Since the node $c$ has a 0-lifting coefficient with respect to $(\beta, \beta_0)$, there exists a stable set $S_c$ in $G \setminus (A \cup N(c))$ which is tight for $(\beta, \beta_0)$. Two possibilities may occur: either $\{d_1, d_2\} \subseteq S_c$ or $a \in S_c$.

Suppose first that $\{d_1, d_2\} \subseteq S_c$. Clearly, $\beta_{h_2} \leq \beta_{d_2}$ for $i = 1, 2$. Moreover, since $S_{d_2} \setminus \{a, b_2\} \cup \{d_1, d_2\}$ and $S_a \setminus \{d_1, d_2\} \cup \{a, b_2\}$ are both feasible for $(\beta, \beta_0)$, we have that $\beta_a + \beta_{b_2} = \beta_{d_1} + \beta_{d_2}$.

Now, if $\beta_{d_1} < \beta_a$ then $\beta_{b_2} < \beta_{d_2}$. Consider a stable set $S'$ which is tight for $(\beta, \beta_0)$ and misses the clique $\{a, d_2, h_2\}$. Then, $b_2 \notin S'$ (since otherwise $S' \setminus \{b_2\} \cup \{d_2\}$ would violate $(\beta, \beta_0)$) and $h_1 \in S'$ (since otherwise $S' \setminus \{h_2\}$ would violate $(\beta, \beta_0)$). It follows that $S' \setminus \{h_1\} \cup \{a\}$ violates $(\beta, \beta_0)$ because $\beta_h \leq \beta_{d_1} < \beta_a$, a contradiction.

Hence, $\beta_{d_1} = \beta_a$ and $\beta_{b_2} = \beta_{d_2}$. If $\beta_{h_1} < \beta_a$ then consider a stable set $S'$ which is tight for $(\beta, \beta_0)$ and contains $h_1$. We have that $S'$ contains $d_2$ (since otherwise $S' \setminus \{h_1\} \cup \{a\}$ would violate $(\beta, \beta_0)$) and so, $S' \setminus \{h_1, d_2\} \cup \{a, b_2\}$ violates $(\beta, \beta_0)$, a contradiction. Thus, $\beta_{h_1} = \beta_a$. If $\beta_{h_2} < \beta_{d_2}$ then consider a stable set $S'$ which is tight for $(\beta, \beta_0)$ and misses the clique $K_2 \cup \{d_2, h_2\}$. Clearly $S'$ has to contain $h_2$ but then $S' \setminus \{h_2\} \cup \{d_2\}$ would violate $(\beta, \beta_0)$). Thus, $\beta_{h_2} = \beta_{d_2}$. Finally, if $\beta_{h_2} > \beta_{d_2}$ then consider a stable set $S'$ which is tight for $(\beta, \beta_0)$ and misses the clique $\{a, d_1, h_1\}$. $S'$ contains $h_2$ and $S' \setminus \{h_2\} \cup \{h_1\}$ violates $(\beta, \beta_0)$, a contradiction. If $\beta_{h_1} < \beta_{h_2}$ then consider a stable set $S'$ which is tight for $(\beta, \beta_0)$ and misses the clique $\{h_2, b_2, d_2\}$. $S'$ contains either $a$ or $h_1$. So, either $S' \setminus \{a\} \cup \{h_2\}$ or $S' \setminus \{h_1\} \cup \{h_2\}$ violates $(\beta, \beta_0)$, a contradiction. Hence $\beta_{h_1} = \beta_{h_2}$. This implies that all non zero components of $\beta_B$ are equal.

Suppose now that there does not exist $S_c$ such that $\{d_1, d_2\} \subseteq S_c$. Then $S_c$ contains $a$. Since $S_{d_1} \setminus \{a, b_2\} \cup \{d_1, d_2\}$ is a stable set which is not tight, then $\beta_{d_1} + \beta_{d_2} < \beta_a + \beta_{h_2}$. Let $S'$ be a stable set which is tight for $(\beta, \beta_0)$ and contains $d_2$. Then $\beta_{d_2} \geq \beta_{b_2}$ (since otherwise $S' \setminus \{d_2\} \cup \{b_2\}$ would
violate \((\beta, \beta_0)\), and so \(\beta_1 < \beta_a\). Now, consider a stable set \(S''\) which is tight for \((\beta, \beta_0)\) and misses \(\{a, h_1, h_2\}\). Clearly, \(d_1 \in S''\) and \(d_2 \notin S''\), so \(S'' \setminus \{d_1\} \cup \{a\}\) violates \((\beta, \beta_0)\), a contradiction. 

3.2. Inequalities having full support on \(V_B\)

Now, we turn our attention to facet defining inequalities of \(STAB(G)\) having full support on \(V_B\). Let \((\beta, \beta_0)\) be any facet defining inequality for \(STAB(G)\) when \(G = (H, B, e)\) is a geared graph, such that \(\beta_B\) has no zero component, i.e., \(\beta_v > 0\) for each \(v \in V_B\). In particular, \((\beta, \beta_0)\) is not a clique inequality or a 5-wheel inequality.

Let \(\mathcal{F}(G)\) denote the set of stable sets of \(G\). Since \((\beta, \beta_0)\) has full support on \(V_B\) it follows that \(S \cap V_B\) is maximal in \(B\) for any stable set \(S \in \mathcal{F}(G)\) that is tight for \((\beta, \beta_0)\).

Let \(\mathcal{R}\) denote the set of the incidence vectors of stable sets in \(\mathcal{F}(G)\) that are roots of \((\beta, \beta_0)\) and let \(M(\beta, \beta_0)\) be the matrix whose rows are indexed by the nodes of \(V_G\) and whose columns are the vectors in \(\mathcal{R}\). Since \((\beta, \beta_0)\) is facet defining, the matrix \(M(\beta, \beta_0)\) has full rank. Consider now the matrix \(M'(\beta, \beta_0)\) obtained by summing up all rows indexed by the nodes \(u \in K_1\) into a single row indexed by \(k_i, i = 1, 2\). This matrix may be interpreted in terms of graphs as follows: let \(B^*\) be the graph obtained from \(B\) by adding two new nodes to \(V_B\), say \(k_1\) and \(k_2\), such that \(N(k_i) = \{b_1, b_2\}, i = 1, 2\). Then \(S \in \mathcal{F}(B^*)\) if and only if there exists a stable set \(S \in \mathcal{F}(G)\) such that: \(S \setminus \{k_1, k_2\} = S \cap V_B\) and \(K_1 \cap S \neq \emptyset\) if and only if \(k_i \in S\). It is not difficult to verify that if \(\text{rank}(M'(\beta, \beta_0)) < |V_B| − \sum_{i=1,2}(|K_i| − 1)\) then \(\text{rank}(M(\beta, \beta_0)) < |V_G|\).

We say that a stable set \(\tilde{S} \in \mathcal{F}(B^*)\) is a tight configuration of \((\beta, \beta_0)\) if and only if there exists a vector \(x^\tilde{S} \in \mathcal{R}\) such that \(S \cap V_B = \tilde{S} \setminus \{k_1, k_2\}\) and \(K_1 \cap S \neq \emptyset\) if and only if \(k_i \in \tilde{S}\). Accordingly, we denote by \(\mathcal{R}'\) the set of the incidence vectors of the tight configurations of \((\beta, \beta_0)\).

So, let \(M''(\beta, \beta_0)\) be the submatrix of \(M'(\beta, \beta_0)\) whose rows are indexed by the nodes of \(B^*\) and whose columns are vectors in \(\mathcal{R}'\). These columns have many repetitions in \(M''(\beta, \beta_0)\) since all stable sets \(S \in \mathcal{F}(G)\) that differ only on nodes out of \(V_B\) produce the same \((0, 1)\)-column of \(M'(\beta, \beta_0)\). We denote by \(\tilde{M}(\beta, \beta_0)\) the matrix of dimension \(|V_B| \times |\mathcal{R}'|\) obtained by deleting multiple columns from \(M''(\beta, \beta_0)\). Clearly, we have that if \(M(\beta, \beta_0)\) has full rank then \(\tilde{M}(\beta, \beta_0)\) has full rank. In particular, we can state the following:

**Proposition 3.5.** Let \(G = (H, B, e)\) be a geared graph. If \((\beta, \beta_0)\) is facet defining for \(STAB(G)\), then the matrix \(\tilde{M}(\beta, \beta_0)\) has rank 10.

We now study in deeper detail the structure of the elements of \(\mathcal{R}'\) in order to deduce some relations among the components of \(\beta_B = (\beta_{d_1}, \beta_{b_1}, \beta_{b_2}, \beta_c, \beta_a, \beta_{d_2}, \beta_{b_3})\).

First of all we observe that there exist exactly 24 maximal stable sets in \(\mathcal{F}(B^*)\); they are depicted in Fig. 7 of Appendix B and denoted by \(R_i, i = 1, \ldots, 24\) (coloured nodes represent nodes of \(R_i, i = 1, \ldots, 24\)). The tight configurations of \((\beta, \beta_0)\) are those \(R_i \in \mathcal{F}(B^*)\) whose incidence vectors belong to \(\mathcal{R}'\).

Each tight configuration \(R_i\) of \((\beta, \beta_0)\) gives rise to a linear system of inequalities \(\mathcal{L}_i\) on \(\beta_B\) by simply considering maximality of \(R_i\) in \(V_B\). For example, let us suppose that \(R_1\) is a tight configuration for \((\beta, \beta_0)\), i.e., there exists a tight stable \(S\) for the inequality \((\beta, \beta_0)\) such that: \(h_1 \in S, S \cap K_1 = \emptyset\), and \(S \cap K_2 \neq \emptyset\).
We derive the following system $L_1$ for the components of $\beta_B$:

\[
\begin{align*}
\beta_c + \beta_a & \leq \beta_{h_1} \quad (6) \\
\beta_{h_1} + \beta_{h_2} & \leq \beta_{h_1} \quad (7) \\
\beta_{d_1} + \beta_{h_2} & \leq \beta_{h_1} \quad (8) \\
\beta_{h_1} + \beta_{a} & \leq \beta_{h_1} \quad (9) \\
\beta_{d_1} + \beta_{c} & \leq \beta_{h_1}. \quad (10)
\end{align*}
\]

Inequality (6) follows by observing that if $\beta_c + \beta_a > \beta_{h_1}$, then the stable set $S' = S \setminus \{h_1\} \cup \{a, c\}$ has the property that $\beta(S') > \beta(S)$. Therefore $x^S$ is not a tight solution for $(\beta, \beta_0)$, a contradiction. Using similar arguments it is possible to derive inequalities (7) to (10).

The systems of inequalities $L_i$ ($i = 2, \ldots, 24$) associated with the other 23 configurations are shown in Appendix C. Each system $L_i$ describes a cone in $\mathbb{R}^{|V|}$ and its solutions represent the coefficients $\beta_B$ of an inequality $(\beta, \beta_0)$ that admits $R_i$ as a tight configuration. Without loss of generality, we add to each system the normalization conditions $\beta_u \leq 1$ for each $u \in V_B$. Then we define a vector $y \in \{0, 1\}^{24}$ such that

\[
y_i = 1 \text{ if and only if } R_i \text{ is a tight configuration of } (\beta, \beta_0).
\]

Thus, for each $i = 1, \ldots, 24$, if $y_i = 1$ then the vector $\beta_B$ must satisfy the linear system $L_i$. If $A_i \beta_B \leq 0$ represents the system $L_i$, we introduce a big-M representation of the above condition: $A_i \beta_B \leq M_i (1 - y_i)$, where $M_i$ is a vector and $(M_i)_j$ is equal to the number of variables in the $j$-th inequality of system $L_i$ having positive coefficients in $(A_i)_j$.

Moreover, the vectors in $R'$ must satisfy the following set $C$ of conditions:

i) for each $u \in V_B$ there exists a stable set $R_i$ for some $i \in \{1, \ldots, 24\}$, such that $u \in R_i$ and $x_R^i \in R'$ (Observation 5);

ii) for each maximal clique $K$ of $B^*$, there exists a stable set $R_i$, for some $i \in \{1, \ldots, 24\}$, such that $R_i \cap K = \emptyset$ and $x_R^i \in R'$ (Observation 4);

iii) for each $W_j = (h_j : C_j)$ of $B$, $j = 1, 2$, there exists a stable set $R_i$, for some $i \in \{1, \ldots, 24\}$, such that $|R_i \cap C_j| < 2$, $h_j \notin R_i$, and $x_R^i \in R'$ (Observation 4);

iv) the rank of the set $\{x_R^i \in R' : R_i \text{ satisfies (i)} \}$ is 10 (Proposition 3.5).

Conditions (i) to (iii) follow from the hypotheses that $(\beta, \beta_0)$ has full support on $V_B$, it is not a clique inequality and it is not a 5-wheel inequality, respectively. Condition (iv) follows from Proposition 3.5. These properties can be translated into a set of constraints on the vector $y$ as follows:

\[
\begin{align*}
\sum_{i : R_i \ni u} y_i & \geq 1, \quad \forall u \in V_B, \quad (11) \\
\sum_{i : R_i \cap K = \emptyset} y_i & \geq 1, \quad \forall K \text{ clique of } B^*, \quad (12) \\
\sum_{i : R_i \ni h_j \text{ and } |R_i \cap C_j| < 2} y_i & \geq 1, \quad \text{for } W_j = (h_j : C_j) \text{ of } B, \ j = 1, 2, \quad (13) \\
\sum_{i = 1}^{24} y_i & \geq 10. \quad (14)
\end{align*}
\]
Notice that the last inequality is a relaxation of property (iv).

We define the polyhedron $\mathcal{P}(B)$ as the convex hull of all the vectors $(\beta_B, y)$ satisfying the following system:

$$
A_i \beta_B \leq M_i (1 - y_i) \quad i = 1, \ldots, 24
$$

$$
\beta_B \leq 1
$$

$y$ satisfies (11), (12), (13), (14)

$$
y \in \{0, 1\}^{24}
$$

The set of the extreme points of $\mathcal{P}(B)$ was obtained by running the software package PORTA (for details on the procedure see Appendix C). From the list of the extreme points output by PORTA we selected only those satisfying condition (iv), namely for each extreme point $(\beta_B, y)$ of $\mathcal{P}(B)$ we checked whether the set of vectors $\{x^{R_i} : y_i = 1\}$ has rank 10. We called these extreme points $C$-feasible. The results of these computations are summarized in the following two theorems.

**Theorem 3.6.** Let $(\beta_B, y)$ be a $C$-feasible extreme point of $\mathcal{P}(B)$. If all the components of $\beta_B$ are nonzero then $\beta_B$ equals

either $(1, 1, 1, 1, 1, 1, 1, 1)$ or $(\frac{1}{2}, \frac{1}{2}, 1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

**Theorem 3.7.** Let $(\beta_B', y')$ and $(\beta_B'', y'')$ be two $C$-feasible extreme points of $\mathcal{P}(B)$. If $y' = y''$ then one of the following possibilities occurs:

a) $\beta_B' = (1, 1, 1, 1, 1, 1, 1, 1), \beta_B'' \in \{(1, 0, 0, 0, 1, 1, 0, 1), (0, 1, 0, 0, 1, 1, 0, 1)\}$

b) $\beta_B' = (\frac{1}{2}, \frac{1}{2}, 1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), \beta_B'' \in \{(1, 1, 1, 1, 0, 0, 1, 1), (1, 0, 1, 1, 0, 1, 1, 1), (1, 1, 1, 1, 0, 1, 1, 0), (1, 1, 1, 1, 0, 0, 1, 1)\}$.
With each point \((\beta_B, y)\) of \(\mathcal{P}(B)\) such that \(y \in \{0,1\}^{24}\) we associate inequalities of the form

\[
\beta_{V'}x_{V'} + \beta_B x_B \leq \beta_0, \tag{16}
\]

that we denote as \((\beta_{V'}, \beta_B, \beta_0)\), where \(V' = V_G \setminus V_B\).

In the following we show that facet defining inequalities of \(STAB(G)\) having full support on \(B\) are associated only with extreme points of \(\mathcal{P}(B)\). To prove this, we first show that any inequality associated with a point of \(\mathcal{P}(B)\) that is convex combination of two extreme points \((\beta_B', y')\) and \((\beta_B'', y'')\) of \(\mathcal{P}(B)\) is dominated by inequalities of type (16) associated with \((\beta_B', y')\) and \((\beta_B'', y'')\).

**Lemma 3.8.** Let \((\beta_B', y')\) and \((\beta_B'', y'')\) be two extreme points of \(\mathcal{P}(B)\) with \(y', y'' \in \{0,1\}^{24}\). Then no inequality \((\gamma_{V'}, \gamma_B, \gamma_0)\) such that

\[
\gamma_B = \mu \beta_B' + (1 - \mu) \beta_B'', \quad 0 < \mu < 1,
\]

with \((\gamma_B, \mu y' + (1 - \mu)y'') \in \mathcal{P}(B)\) is facet defining for \(STAB(G)\).

**Proof.** First observe that, since \(y', y'' \in \{0,1\}^{24}\), then, in order to have \(\mu y' + (1 - \mu)y'' \in \{0,1\}^{24}\), \(y'\) must be equal to \(y''\). Hence, Theorem 3.7 lists all possible pairs of \(\beta_B'\) and \(\beta_B''\). Suppose now that \((\gamma_{V'}, \gamma_B, \gamma_0)\) is a valid inequality for \(STAB(G)\) and consider the following two inequalities:

\[
\begin{align*}
\gamma_{V'}x_{V'} + \beta_B' x_B &\leq \gamma_0 + (1 - \mu), \\
\gamma_{V'}x_{V'} + \beta_B'' x_B &\leq \gamma_0 - \mu. \tag{17}
\end{align*}
\]

We now prove the lemma only for the case a) of Theorem 3.7 in which we choose \(\beta_B''\) being equal to \((1, 0, 0, 0, 1, 1, 0, 1)\): for the other choice of \(\beta_B''\) as for the case b), the proof will follow the same arguments.

Using Fig. 5 and Fig. 7 it is not difficult to check that any tight stable set \(S\) for \((\gamma_{V'}, \gamma_B, \gamma_0)\) must satisfy \(S \cap V_B \in \{\{a, c\}, \{d_2, h_1\}, \{d_1, h_2\}, \{a, b_1\}, \{d_1, c\}, \{d_2, c\}, \{a, b_2\}, \{d_1, d_2, c\}\}\). Indeed, all other cases lead to stable sets that can be augmented with respect to \(\gamma_B\) (e.g., if \(S \cap V_B = \{h_1, b_2\}\), then \(\gamma(S \setminus \{h_1\} \cup \{a\}) > \gamma(S)\)).

We now show that every tight solution for \((\gamma_{V'}, \gamma_B, \gamma_0)\) is also tight for both inequalities (17) with the help of Fig. 5.
If \( S \cap V_B = \{d_1, d_2, c\} \), then \( \gamma(S \cap V_B) = 2 + \mu \), while \( \beta''_B(S \cap V_B) = 3 = \gamma(S \cap V_B) + (1 - \mu) \) and \( \beta''_B(S \cap V_B) = 2 = \gamma(S \cap V_B) - \mu \) (see Fig. 5(b) and Fig. 5(c)), and thus \( S \) is tight for both inequalities (17). If \( S \cap V_B \in \{d_1, h_2\}, \{d_1, c\}, \{d_2, h_1\}, \{d_2, c\}, \{a, b_1\}, \{a, b_2\}, \{a, c\} \), then \( \gamma(S \cap V_B) = 1 + \mu \), while \( \beta''_B(S \cap V_B) = 2 = \gamma(S \cap V_B) + (1 - \mu) \) and \( \beta''_B(S \cap V_B) = 1 = \gamma(S \cap V_B) - \mu \), and thus \( S \) is tight for both inequalities (17). In a similar way it is possible to assess that if \( (\gamma_{V'}, \gamma_B, \gamma_0) \) is valid for \( STAB(G) \), then both \( (\gamma_{V'}, \beta''_B, \gamma_0 + (1 - \mu)) \) and \( (\gamma_{V'}, \beta''_B, \gamma_0 - \mu) \) are valid for \( STAB(G) \).

Since the inequalities (17) contain all the roots of \( (\gamma_{V'}, \gamma_B, \gamma_0) \) and their convex combination yields \( (\gamma_{V'}, \gamma_B, \gamma_0) \), it follows that \( (\gamma_{V'}, \gamma_B, \gamma_0) \) is not facet defining for \( STAB(G) \).

Finally, we prove that facet defining inequalities for \( STAB(G) \) having full rank on \( V_B \) are associated only with the extreme points of \( \mathcal{P}(B) \) identified in Theorem 3.6.

**Theorem 3.9.** Let \( G = (H, B, v_1v_2) \) be a geared graph and let \( V' = V_B \setminus \{v_1, v_2\} \) and \( B' = B \setminus \{h_1, h_2\} \). Then each facet defining inequality \( (\beta, \beta_0) \) of \( STAB(G) \) having full support on \( V_B \) has one of the following forms:

a) \( \beta''_{V'}x_{V'} + \lambda x_{B'} + 2\lambda(x_{h_1} + x_{h_2}) \leq \beta_0 \),

b) \( \beta''_{V'}x_{V'} + \lambda x_B \leq \beta_0 \).

**Proof.** With every point \( (\beta_B, y) \) of \( \mathcal{P}(B) \) such that \( y \in \{0, 1\} \) it is associated an inequality of the form (16). If \( (\beta, \beta_0) = (\beta_{V'}, \beta_B, \beta_0) \) is associated with the extreme points \( (\frac{1}{2}, \frac{1}{2}, 1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \) or \( (1, 1, 1, 1, 1, 1, 1, 1, 1) \) of \( \mathcal{P}(B) \) then it is an inequality of type a) or b).

Lemma 3.8 shows that no facet defining inequality of \( STAB(G) \) is associated with a point of \( \mathcal{P}(B) \) which is not an extreme point of \( \mathcal{P}(B) \).

Finally, by Theorem 3.6 the only \( C \)-feasible extreme points of \( \mathcal{P}(B) \) having all components of \( \beta_B \) different from zero have either \( \beta_B = (1, 1, 1, 1, 1, 1, 1) \) or \( \beta_B = (\frac{1}{2}, \frac{1}{2}, 1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \); thus the theorem follows.

**3.3. The stable set polytope of a geared graph**

In this section we are given a geared graph \( G \) that is generated by \( H \) and \( B \) along \( e \) and we consider a facet defining inequality \( (\beta, \beta_0) \) of \( STAB(G) \) that has nonzero coefficients on \( V_B \) and that is neither a clique inequality nor a lifted 5-wheel inequality. We prove that \( (\beta, \beta_0) \) is either a geared inequality associated with a facet defining inequality for \( STAB(H) \) or a \( g \)-lifted inequality associated with a facet defining inequality for \( STAB(H^e) \).
From the results in Subsection 3.1, we know that each facet defining inequality of \( STAB(G) \) that does not have full support on \( V_B \) is of the form described in Theorem 3.4. From the results in Subsection 3.2, we know that each facet defining inequality of \( STAB(G) \) with full support on \( V_B \) is either of type a) or of type b) described in Theorem 3.9.

The next theorem shows that the inequalities of type a) are proper geared inequalities associated with facet defining inequalities of \( STAB(H) \).

**Theorem 3.10.** Let \( G = (H, B, v_1v_2) \) be a geared graph and let \( V' = V_H \setminus \{v_1, v_2\} \) and \( B' = B \setminus \{h_1, h_2\} \). If \( (\beta, \beta_0) \) is a facet defining inequality for \( STAB(G) \) of type

\[
\beta_0^H x_{V'} + \lambda x_{B'} + 2\lambda(x_{h_1} + x_{h_2}) \leq \beta_0,
\]

with \( \lambda > 0 \). Then \( (\beta_H, \beta_0 - 2\lambda) \) with \( \beta_{v_1} = \beta_{v_2} = \lambda \) is a facet defining inequality for \( STAB(H) \).

**Proof.** Suppose conversely that \( (\beta_H, \beta_0 - 2\lambda) \) is not facet defining for \( STAB(H) \). Then there exists an inequality \((\pi, \pi_0)\) that is facet defining for \( STAB(H) \) and such that all the roots of \((\beta_H, \beta_0 - 2\lambda)\) are roots of \((\pi, \pi_0)\). By Proposition 2.1, \( \pi_{v_1} = \pi_{v_2} \). If \( \pi_{v_1} = 0 \) then \((\pi, \pi_0)\) can be lifted to a facet defining inequality for \( STAB(G) \) that contains all the roots of \((\beta, \beta_0)\) and has \( \pi_v = 0 \) for each \( v \in V_B \), a contradiction. If \( \pi_{v_1} > 0 \) then we assume without loss of generality that \( \pi_{v_1} = \lambda \) and consider the following proper geared inequality:

\[
\sum_{i \in V_B \setminus \{v_1, v_2\}} \pi_i x_i + \lambda \sum_{i \in V_B \setminus \{h_1, h_2\}} x_i + 2\lambda(x_{h_1} + x_{h_2}) \leq \pi_0 + 2\lambda. \tag{18}
\]

Since \((\pi, \pi_0)\) is g-extendable and facet defining for \( STAB(H) \), it follows, by Theorem 2.5, that (18) is facet defining for \( STAB(G) \).

Let \( x^S \) be a root of \((\beta, \beta_0)\). Notice that \( \beta(S \cap V_B) \) equals either \( 2\lambda \) or \( 3\lambda \); hence, every tight solution \( x^S \)

of \((\beta, \beta_0)\) can be reduced to a tight solution \( x^{S_H} \) of \((\beta_H, \beta_0 - 2\lambda)\) by removing from \( S \) an appropriate stable set \( T \) of weight \( 2\lambda \) contained in \( B \); if \( \beta(S \cap \{a, c, h_1, h_2\}) = 2\lambda \) we remove this subset, i.e., we define \( S' = S \setminus \{a, c, h_1, h_2\} \); otherwise \( S \cap V_B \in \{d_1, d_2, c\}, \{b_1, b_2, a\} \) and we define \( S' = S \setminus \{a, c, b_2, d_2\} \); finally \( S_H \) is built from \( S' \) and \( v_i \in S_H \) if and only if \( S' \cap \{b_i, d_i\} \neq \emptyset \), for \( i = 1, 2 \), while \( S_H \setminus \{v_1, v_2\} = S \setminus V_B \).

By assumption \( x^{S_H} \) is also a tight solution for \((\pi, \pi_0)\). Thus \( x^S \) is also a root of (18) once we reintroduce the previously removed stable set \( T \). Therefore, \((\beta, \beta_0)\) and (18) are equivalent. As \( STAB(G) \) is full dimensional, the two inequalities only differ by a positive scalar factor. Hence, \((\pi, \pi_0)\) is equivalent to \((\beta_H, \beta_0 - 2\lambda)\), contradicting the assumption. \( \blacksquare \)

A similar result holds for facet defining inequalities \((\beta, \beta_0)\) not having full support on \( V_B \). In fact, by Theorem 3.4, all the nonzero components of \( \beta_B \) have the same value, say \( \lambda \), and so, the above proof can be repeated almost literally (using Theorem 2.6 and replacing \( 3\lambda \) and \( 2\lambda \) with \( 2\lambda \) and \( \lambda \), respectively) to show that

**Theorem 3.11.** Let \( G = (H, B, v_1v_2) \) be a geared graph and let \( V' = V_H \setminus \{v_1, v_2\} \) and \( A \in \{b_1, c\}, \{b_2, c\}, \{d_1, a\}, \{d_2, a\}, \{a, c\} \). If \( (\beta, \beta_0) \) is a facet defining inequality for \( STAB(G) \) of type

\[
\beta_0^H x_{V'} + \lambda x_{B \setminus A} \leq \beta_0,
\]

with \( \lambda > 0 \). Then \( (\beta_H, \beta_0 - \lambda) \) with \( \beta_{v_1} = \beta_{v_2} = \lambda \) is a facet defining inequality for \( STAB(H) \).

The next theorem shows that inequalities of type b) in Theorem 3.9 are proper g-lifted inequalities associated with facet defining inequalities of \( STAB(H^c) \).
Theorem 3.12. Let $G = (H, B, e)$ be a geared graph and let $H^e$ be the graph obtained from $H$ by subdividing the edge $e = v_1v_2$ with the new node $t$. Let $V' = V_H \setminus \{v_1, v_2\}$ and $B' = B \setminus \{h_1, h_2\}$. If $(\beta, \beta_0)$ is a facet defining inequality of $STAB(G)$ of type

$$\beta^T_{V'}x_{V'} + \lambda x_B \leq \beta_0,$$

with $\lambda > 0$. Then $(\beta_{H^e}, \beta_0 - \lambda)$ with $\beta_{v_1} = \beta_{v_2} = \beta_t = \lambda$ is a facet defining inequality for $STAB(H^e)$.

Proof. Suppose conversely that $(\beta_{H^e}, \beta_0 - \lambda)$ is not facet defining for $STAB(H^e)$. Then there exists an inequality $(\pi, \pi_0)$ that is facet defining for $STAB(H^e)$ and such that all the roots of $(\beta_{H^e}, \beta_0 - \lambda)$ are roots of $(\pi, \pi_0)$. By Proposition 2.1, $\pi_{v_1} = \pi_{v_2} = \pi_t$. If $\pi_{v_1} = 0$ then $(\pi, \pi_0)$ can be lifted to a facet defining inequality for $STAB(G)$ that contains all the roots of $(\beta, \beta_0)$ and has $\pi_v = 0$ for each $v \in V_B$, a contradiction. If $\pi_{v_1} > 0$ then we assume without loss of generality that $\pi_{v_1} = \lambda$ and consider the following g-lifted inequality:

$$\sum_{i \in V_H \setminus \{v_1, v_2\}} \pi_i x_i + \lambda \sum_{i \in V_B} x_i \leq \pi_0 + \lambda. \quad (19)$$

By Theorem 2.9, the inequality (19) is facet defining for $STAB(G)$.

Let $x^S$ be a root of $(\beta, \beta_0)$. Notice that $\beta(S \cap B) = 3\lambda$ if $S \cap B$ equals $\{d_1, c, d_2\}$ or $\{b_1, a, b_2\}$. In the remaining cases $\beta(S \cap B) = 2\lambda$. It follows that every root $x^S$ of $(\beta, \beta_0)$ can be reduced to a root $x^{S'}$ of $(\beta_{H^e}, \beta_0 - \lambda)$ by removing from $S$ an appropriate stable set $T$ of weight $\lambda$ contained in $B$. By assumption $x^{S'}$ is also a tight solution of $(\pi, \pi_0)$. Hence $x^{S'}$ is also a root of (19) once we reintroduce the stable set $T$ previously removed. Therefore, $(\beta, \beta_0)$ and (19) are equivalent. As $STAB(G)$ is full dimensional, the two inequalities only differ by a positive scalar factor. Hence, $(\pi, \pi_0)$ is equivalent to $(\beta_{H^e}, \beta_0 - \lambda)$, contradicting the assumption. \ \[\square\]

Finally, observe that the g-lifted inequalities (4) are isomorphic to the original facet defining inequality $(\pi, \pi_0)$ of $STAB(H^e)$.

Summing up, theorems 3.4, 3.9, 3.10,3.11, and 3.12 prove Theorem 3.1 as explained in the outline of the proof given at the beginning of Section 3.

4. $G$-perfect graphs

Up to this point we have considered only graphs that are obtained by performing a single gear composition on a given graph $H$. In this section we focus on graphs obtained by repeated applications of the gear composition and we generalize to these graphs the results obtained so far.

We start by extending the definition of geared graphs.

Definition 4.1. Given a graph $H$ which is not a clique, let $\Gamma_H$ be the set of the simplicial edges of $H$ and let a $g$-operation on $e \in \Gamma_H$ be either a gear composition or an edge subdivision applied along $e$. A graph $G \in G_H$ if and only if

either $G = H$,

or $G = (L, B, e)$, where $L \in G_H$, $B$ is an extended gear, and $e \in \Gamma_H \cap E_L$ (i.e., $e$ is a simplicial edge of both $L$ and $H$),

or $G = L^e$, where $L \in G_H$ and $e \in \Gamma_H \cap E_L$.

We call $G_H$ the class of multiple geared graphs generated by $H$. 

Notice that in Definition 4.1 the g-operations, namely gear compositions and edge subdivisions, are performed only along simplicial edges of \( L \) that are also simplicial in the given graph \( H \). This implies that in order to generate graphs in \( \mathcal{G}_H \) we are not allowed to use any of the edges created by an earlier g-operation: in particular, the edges \( v_1t \) and \( tv_2 \), created by an edge subdivision of \( e = v_1v_2 \in \Gamma_H \), cannot be used to perform any g-operation. In fact, these two edges do not belong to \( \Gamma_H \) even though they have the property of being super simplicial. It follows that any graph in \( \mathcal{G}_H \) is obtained by performing at most \( |\Gamma_H| \) g-operations, thus implying that, for any fixed graph \( H \), the family \( \mathcal{G}_H \) contains a finite number of graphs.

Accordingly with Definition 4.1 we need to define a larger family of inequalities that contains the geared and the g-lifted inequalities obtained by repeated applications of the gear composition.

**Definition 4.2.** A facet defining inequality \((\gamma, \gamma_0) \in \mathcal{G}\) if and only if it is (the sequential lifting of)

- either a rank inequality,
- or a 5-wheel inequality,
- or a geared or a g-lifted inequality associated with an inequality in \( \mathcal{G} \).

Consider now the polyhedron

\[
\mathcal{G}_{STAB}(G) = \{ x \in \mathbb{R}_+^V | x \text{ satisfies } \mathcal{G} \}. \tag{20}
\]

Since geared and g-lifted inequalities are valid for \( STAB(G) \), it follows that \( STAB(G) \subseteq \mathcal{G}_{STAB}(G) \) if \( G \) is a geared graph. A graph \( G \) is said to be \( \mathcal{G} \)-perfect if and only if \( STAB(G) = \mathcal{G}_{STAB}(G) \). The results of the previous section state that a defining linear system for \( STAB(G) \) can be easily provided once defining linear systems for \( STAB(H) \) and \( STAB(H^e) \) are known. So, an immediate consequence of Theorem 3.1 is the following:

**Corollary 4.3.** Let \( G = (H, B, e) \) be a geared graph generated by \( H \) and \( B \) along \( e \). If \( H \) and \( H^e \) are \( \mathcal{G} \)-perfect then \( G \) is \( \mathcal{G} \)-perfect.

In the following we denote by \( H^F \) the graph obtained from \( H \) by subdividing all the edges in \( F \subseteq \Gamma_H \).

**Theorem 4.4.** Let \( H \) be a graph, \( F^* = \{e_1, e_2, \ldots, e_k\} \subseteq \Gamma_H \). If \( H \) and \( H^F \) are \( \mathcal{G} \)-perfect for any \( F \subseteq F^* \), and \( G \in \mathcal{G}_H \) is obtained from \( H \) by a sequence of \( k \) g-operations along the edges in \( F^* \), then \( G \) is \( \mathcal{G} \)-perfect.

**Proof.** Let \( G_i \) denote the graph obtained from \( H \) by performing the first \( i \) g-operations on the edges \( e_j \) for \( j = 1, \ldots, i \). Then \( G = G_k \) by hypothesis. We prove the theorem by induction on the number \( k \) of g-operations. If \( k = 1 \) the theorem is true by Corollary 4.3. If \( k > 1 \), then, by induction, the theorem holds for every graph \( L \in \mathcal{G}_H \) obtained by performing at most \( k-1 \) g-operations. Suppose by contradiction that \( G_k \) is not \( \mathcal{G} \)-perfect. If \( G_k \) is obtained as the gear composition of a graph \( G_{k-1} \) and a gear \( B \) along a simplicial edge \( e_k \), namely \( G_k = (G_{k-1}, B, e_k) \), then, by Corollary 4.3, at least one between \( G_{k-1} \) and \( G_{k-1}^{e_k} \) is not \( \mathcal{G} \)-perfect. Since, by induction, \( G_{k-1} \) is \( \mathcal{G} \)-perfect, it follows that \( G_{k-1}^{e_k} \) is not \( \mathcal{G} \)-perfect. Thus, iteratively, if \( G_k \) is not \( \mathcal{G} \)-perfect, then \( G_0 = \{e_k \} = H \{e_1, e_2, \ldots, e_k\} \) is not \( \mathcal{G} \)-perfect, a contradiction. An immediate consequence of Theorem 4.4 is the following:
Corollary 4.5. Let $H$ be a graph and $\Gamma_H$ be the set of its simplicial edges. If $H$ and $H^F$ are $G$-perfect for any $F \subseteq \Gamma_H$, then every graph $G \in \mathcal{G}_H$ is $G$-perfect.

In the following we exhibit a significant class of graphs that is $G$-perfect. This class properly contains the class of line graphs and it is contained in the class of claw-free graphs. To prove this result we need to restrict the application of the $g$-operations to simplicial edges having the further property that: $N(K_1 \cap K_2) \subseteq N(v_1) \cup N(v_2)$. We call these edges super simplicial edges.

Theorem 4.6. Let $H$ be a line graph that is not a clique. Then the graphs belonging to the subfamily of $G_H$ obtained by performing $g$-operations only along super simplicial edges are $G$-perfect.

Proof. By the results of Chvátal on composition of polyhedra [4], we may assume without loss of generality that $H$ does not contain a clique-cutset. This implies that $K_1 \setminus K_2$ and $K_2 \setminus K_1$ are both nonempty.

It is well known that $STAB(H)$ is described only by nonnegativity and rank inequalities [5]; thus, $H$ is $G$-perfect. In order to apply Corollary 4.5 to the line graph $H$ it suffices to guarantee that $H^F$ is a line graph for any subset $F \subseteq \Gamma_H$ of super simplicial edges of $H$. Let $e = v_1v_2$ be a super simplicial edge of $H$. The root graph $R(H)$ of $H$ contains two edges $f_{v_1} = \{w_e, s_1\}$ and $f_{v_2} = \{w_e, s_2\}$ sharing the common node $w_e$. Each node in $K_1 \setminus K_2$ corresponds to an edge of $R(H)$ adjacent to $f_{v_1}$ and not to $f_{v_2}$. Symmetrically each node in $K_2 \setminus K_1$ corresponds to an edge of $R(H)$ adjacent to $f_{v_2}$ and not to $f_{v_1}$.

Finally, since $e$ is super simplicial, it follows that every node in $N(K_1 \cap K_2)$ is completely adjacent to $(K_2 \setminus K_1) \cup (K_1 \setminus K_2) \cup \{v_1, v_2\}$; therefore, each node in $K_1 \cap K_2$ (if any) is associated with an edge of $R(H)$ joining the nodes $s_1$ and $s_2$. Consider now the graph obtained from $R(H)$ by splitting the node $w_e$ into two nodes $w_{1i}$, $w_{2j}$ joined by the edge $w_{1i}w_{2j}$ and such that $w_{1i}$ corresponds to the endnode of the edge $f_{v_i}$ for $i = 1, 2$. This graph is the root graph of $H^e$ and so $H^e$ is a line graph. By iteratively applying the above argument, we prove that $H^F$ is a line graph for any subset $F \subseteq \Gamma_H$ of super simplicial edges of $H$. Thus, $H^F$ is $G$-perfect [5] and Corollary 4.5 holds for the subfamily of $G_H$ obtained from a line graph $H$ by performing $g$-operations only along super simplicial edges. Therefore the graphs belonging to this subfamily are $G$-perfect.$\blacksquare$

In the remaining of this section we explain in a less formal way how $GSTAB(G)$ looks like when $G \in \mathcal{G}_H$ (obtained by performing the $g$-operations only along super simplicial edges) and $H$ is a line graph. Since $H$ is a line graph, a single application of the gear composition to $H$ produces geared inequalities and g-lifted inequalities associated only with rank inequalities. By definitions 2.4 and 2.7, the proper geared inequalities (when associated with rank inequalities) contain at least a pair of coefficients equal to 2 corresponding to the hubs of a gear while the g-lifted and the non-proper geared inequalities (when associated with rank inequalities) have all coefficients equal to 1. By applying the gear composition several times, it is possible to produce g-lifted inequalities associated with geared inequalities; so, it is not true that every g-lifted inequality in $G$ is a rank inequality. Nevertheless, we can say that the inequalities in $G$, which are not 5-wheel inequalities, are only of two types: either they contain pairs of hubs of a gear with coefficients 2 and have all the remaining coefficients equal to 1, or they have all the coefficients equal to 1. We call the former inequalities multiple geared rank inequalities and we refer to the others simply as rank inequalities.

The iterative application of the gear composition yields some complications; in fact, the same inequality can be seen both as a geared inequality and as a g-lifted inequality depending on the order the gear compositions have been performed. To see an example consider the graph $G$ depicted in Fig. 6 (a) obtained by applying twice the gear composition to the 4-hole $C_4 = (v_1, v_2, w_2, w_1)$. Indeed, there are two ways to generate $G$:

1. Apply the gear composition to $C_4$ and a gear $B_1$ along $w_1w_2$ to generate the graph $H_1$ in Fig. 6 (b). Then apply to $H_1$ another gear composition with a gear $B_2$ along the edge $v_1v_2$ to obtain the
graph $G$. Since the inequality (b) is $g$-extendable with respect to $v_1v_2$, the inequality (a) is a proper geared inequality associated with the inequality (b) (see Definition 2.4). The inequality (a) is also facet defining for $STAB(G)$ by Theorem 2.5.

2. Apply the gear composition to $C_4$ and a gear $B_2$ along $v_1v_2$ to obtain the graph $H_2 = (C_4, B_2, v_1v_2)$. Then apply to $H_2$ another gear composition with a gear $B_1$ along the edge $w_1w_2$ to obtain the graph $G$. Since the inequality (c) is $g$-liftable with respect to $w_1w_2$, the inequality (a) is a proper $g$-liftable inequality associated with the inequality (c) (see Definition 2.7). The inequality (a) is also facet defining for $STAB(G)$ by Theorem 2.9.

As a consequence, the inequalities in $G$ are (the sequential liftings of) either multiple geared rank inequalities or rank inequalities or $5$-wheel inequalities.

If $H$ is a line graph then the graphs in $G_H$ (obtained by performing the $g$-operations only along super simplicial edges) are not quasi-line since they contain $5$-wheels, but they are claw-free. To see this, suppose by contradiction that a graph $L \in G_H$ contains a claw $C$. Since the gear $B$ is claw-free and the only edges that were removed from the original line graph $H$ were super simplicial edges, we have that $C$ must contain at least two nodes, say $v_1$ and $v_2$, corresponding to the endnodes of a super simplicial edge $e$ of $H$. So, $C = (y : v_1, v_2, w)$ where $y$ is the center of the claw. Clearly either $y \in V_B$ or $y \in K_1 \cap K_2$. In both cases we have that $N(y) \subseteq N(v_1) \cup N(v_2)$, and so $w$ is adjacent to $v_1$ or $v_2$, contradicting the hypothesis that $C$ was a claw.

The problem of finding a linear description for $STAB(G)$ when $G$ is claw-free is an open problem which has been studied for decades [8, 15, 19, 13, 22] and for which many conjectures have been stated and disproved [10, 7]. The case when $G$ has stability number 2 has been solved by Cook (see [21]) while for the case $\alpha(G) = 3$ there exists a characterization of the roots of the facet defining inequalities of $STAB(G)$ [17]. The recent decomposition theorem for claw-free graphs of Chudnovsky and Seymour [3] offers new perspectives to face the problem of finding a linear description of the stable set polytope of a claw-free graph. Indeed they identify subclasses of claw-free graphs which might be easier to treat from the polyhedral point of view. For instance, their decomposition theorem restricted to quasi-line [3] graphs led to the settlement of the Ben Rebea’s conjecture [6].

Chudnovsky and Seymour also pointed out in [2] that, when dealing with claw-free graphs with stability number at least 4, it is convenient to assume that they do not admit a 1-join (a graph $G$ admits a 1-join if $V_G$ can be partitioned into four sets $A_1, B_1, A_2, B_2$ such that $A_1 \cup A_2$ is a clique, $B_1$ and $B_2$ are nonempty, and the only edges between $A_1 \cup B_1$ and $A_2 \cup B_2$ are those between $A_1$ and $A_2$). Indeed, this assumption is very convenient also from the polyhedral point of view. In fact, if $G$ admits a 1-join then $G$ has a clique-cutset and so, by Theorem 3.2, it does not support a facet defining inequality of $STAB(G)$.
So, when looking for facet defining inequalities for the stable set polytope, it is quite natural to assume that the graph that supports the inequality does not admit 1-joins.

Hence, a subclass of claw-free graphs that is likely to investigate is that of: claw-free graphs which are not quasi-line, have \( \alpha(G) \geq 4 \) and admit no 1-join. Following [2], these graphs are built from certain quasi-line graphs using only two composition operations which we believe have a polyhedral counterpart. This led us to conjecture that:

**Conjecture 4.1.** The stable set polytope of a claw-free graph \( G \) which is not quasi-line, admits no 1-join and has \( \alpha(G) \geq 4 \) is described by (sequential liftings of):

- nonnegativity inequalities
- rank inequalities
- 5-wheel inequalities
- multiple geared rank inequalities.

An earlier version of this conjecture already appeared in [7] but it was not precisely stated since it did not contain explicitly the hypothesis that \( G \) does not admit 1-joins. This was pointed out to us by Pietropaoli and Wagler [18] who observed that it is possible to compose with a 1-join two claw-free graphs with stability number less than or equal to 3 to obtain a claw-free, not quasi-line graph \( G \) with stability number 4 such that the inequalities listed in the conjecture are not sufficient to describe \( STAB(G) \).

As a final remark notice that the results in this paper support the validity of Conjecture 4.1 since the graphs considered in Theorem 4.6 form a large subclass of claw-free, not quasi-line graphs with stability number at least 4.

We end the paper by observing that Theorem 4.4 also applies to graphs that are not claw-free. As an example, consider a 5-wheel \( W \). Since \( STAB(W) \) and \( STAB(W^F) \) (for any subdivision of a subset \( F \) of simplicial edges of the rim) are described by nonnegativity constraints and inequalities in \( G \), we have that any graph \( G \in G_W \) is \( G \)-perfect, but it is easy to see that a single application of the gear composition to \( W \) creates a claw.
A. Details in the proof of Lemma 3.3

In Lemma 3.3 we prove that there are only 7 possible supporting subgraphs of $B$ for a facet defining inequality $(\beta_1 \cap B, \beta_B, \beta_0)$ of $STAB(G)$.

The proof is by enumeration of all the possible $2^8$ supports and shows that all the supports that are different from the ones listed in the thesis cannot be associated with a facet defining inequality.

Here we examine in detail the case $|A| = 4$, where $A$ is the subset of nodes in $V_B$ that are not included in the support.

First observe that any supporting graph of a facet defining inequality that is neither a clique inequality nor a 5-wheel inequality must contain a path between $K_1$ and $K_2$ whose internal nodes are contained in $B$, otherwise these cliques are clique-cutset and by Theorem 1 $G \setminus A$ is not the supporting graph of a facet defining inequality. This means that $A$ cannot separate $K_1$ from $K_2$.

In particular $A$ contains neither $\{b_1, d_1\}$ nor $\{b_2, d_2\}$, therefore $A \cap \{b_1, b_2, d_1, d_2\}$ is one of the following sets:

- a) $\{b_1, b_2\}$,
- b) $\{b_1, d_2\}$,
- c) $\{d_1, d_2\}$,
- d) $\{d_1, b_2\}$,
- e) $\{b_1\}$,
- f) $\{d_1\}$,
- g) $\{b_2\}$,
- h) $\{d_2\}$,
- j) $\emptyset$.

It is easy to see that the gear $B$ is a highly symmetric graph: if we reverse $B$ upside-down we again obtain a gear with a different order of the nodes, and the same if we reverse $B$ from left to right. This means that if the supporting graph of a facet defining inequality has a nonempty intersection with $B$, there exists a symmetric facet defining inequality with a symmetric supporting graph. Therefore we list the cases up to symmetry.

Clearly, case c) is symmetric to case a) and case d) is symmetric to case b); cases f), g), and h) are symmetric to case e) (with a upside-down and/or left-to-right reversal); finally, case j) implies $A = \{h_1, h_2, a, c\}$ which separates $K_1$ and $K_2$. Thus we are left with only three cases a), b), and e).

Case a) $(A \cap \{b_1, b_2, d_1, d_2\} = \{b_1, b_2\})$ produces the following subcases by considering all the possible subsets of 2 nodes in $\{h_1, h_2, a, c\}$:

- a1) $A = \{b_1, b_2, a, c\}$,
- a2) $A = \{b_1, b_2, h_1, h_2\}$,
- a3) $A = \{b_1, b_2, a, h_1\}$,
- a4) $A = \{b_1, b_2, a, h_2\}$,
- a5) $A = \{b_1, b_2, c, h_1\}$,
- a6) $A = \{b_1, b_2, c, h_2\}$.

Case a1) matches case i) in the proof of Lemma 3.3 (case $|A| = 4$). In case a2) node $c$ is isolated, i.e., $G \setminus A$ is not admissible. In all other cases $G \setminus A$ contains a clique-cutset, i.e., it is not admissible: $K_1$ in a3) and a4), $\{d_1, a\}$ in a5), $\{d_1, a\}$ in a6).

Case b) $(A \cap \{b_1, d_1, b_2, d_2\} = \{b_1, d_2\})$ produces the following subcases by considering all the possible subsets of 2 nodes in $\{h_1, h_2, a, c\}$:

- b1) $A = \{b_1, d_2, a, c\}$,
b2) \( A = \{b_1, d_2, c, h_1\} \),
b3) \( A = \{b_1, d_2, a, h_2\} \),
b4) \( A = \{b_1, d_2, h_1, h_2\} \),
b5) \( A = \{b_1, d_2, a, h_1\} \),
b6) \( A = \{b_1, d_2, c, h_2\} \).

Cases b1) and b2) match cases ii) and iii) of the proof of Lemma 3.3 (case \(|A| = 4\)), respectively. Case b3) is symmetric to case b2) (take the subgraph associated with case b2, first reverse it upside-down and then reverse the resulting graph from left to right and you will obtain a graph isomorphic to case b3). In all other cases \( K_2 \) always defines a clique-cutset of \( G \setminus A \).

Case e) \( (A \cap \{b_1, a_1, b_2, d_2\} = \{b_1\}) \) produces the following subcases by considering all the possible subsets of 3 nodes in \( \{h_1, h_2, a, c\} \):

e1) \( A = \{b_1, a, c, h_1\} \),
e2) \( A = \{b_1, a, c, h_2\} \),
e3) \( A = \{b_1, a, h_1, h_2\} \),
e4) \( A = \{b_1, c, h_1, h_2\} \).

Is it easy to check that \( K_1 \) defines a clique-cutsets set for the cases e1), e2), and e3), and \( K_2 \cup \{d_2\} \) is a clique-cutset for case e4).
B. List of possible tight solutions

Figure 7: The maximal stable sets of $\mathcal{S}(B^*)$
In Subsection 3.2 we stated that each configuration $R_i$ produces a linear system of inequalities $L_i$ on $\beta_L$ by simply considering maximality conditions. In particular, we presented the system $L_i$ together with the rules used to generate it. Similar arguments allow us to derive the systems of inequalities $L_i$ ($i = 2, \ldots, 24$) associated with the other 23 tight configurations, which are listed in Fig 7. The complete list of systems is presented below:

<table>
<thead>
<tr>
<th>$L_1$</th>
<th>$L_2$</th>
<th>$L_3$</th>
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As explained in Subsection 3.2, we considered the polyhedron $P(B)$ that is the convex hull of the feasible solutions of system (15). Theorems 3.6 and 3.7 follow by exhibiting the set of all the extreme points of $P(B)$. This was done with the help of the software package PORTA [1]. This software receives as input a system of linear inequalities and returns the list of the extreme points of the polyhedron described by the given system.

In our case the system is:

$$A_i \beta_B \leq M_i (1 - y_i) \quad i = 1, \ldots, 24$$
$$\beta_B \leq 1$$
$$y \text{ satisfies } (11), (12), (13), (14)$$
$$0 \leq y_i \leq 1 \quad i = 1, \ldots, 24. \quad (21)$$

Unfortunately, PORTA could not run on the whole system (21) in a reasonable amount of time. So, we subdivided the problem in $2^{16}$ subproblems by fixing $y_i$ to zero or to one for $i = 9, \ldots, 24$ as follows. Let $\mathcal{Y} = \{\tilde{y}^0, \tilde{y}^1, \ldots, \tilde{y}^k\}$ with $k = 2^{16} - 1$ denote the set consisting of the vectors $\tilde{y}^j \in \{0, 1\}^{16}$ that are binary encoding of $j$ for $j = 0, \ldots, k$.

We split the vector $y$ into two parts $y = (y_1, \ldots, y_8|\tilde{y})$ where $y_i, i = 1, \ldots, 8$, are variables and $\tilde{y}$ is some vector in $\mathcal{Y}$. Then we ran PORTA on the $2^{16}$ polyhedra $P_j(B)$ obtained from system (21) by fixing $\tilde{y}$ to each vector $\tilde{y}^j$ for $j = 0, \ldots, k$. Namely we applied PORTA to the following $2^{16}$ linear systems (each system is associated with a different vector $\tilde{y}^j$):

$$A_i \beta_B \leq M_i (1 - y_i) \quad i = 1, \ldots, 8$$
$$A_i \beta_B \leq 0 \quad \forall i \in \{9, \ldots, 24\} \text{ such that } \tilde{y}^i_{9-8} = 1$$
$$\beta_B \leq 1$$
$$y \text{ satisfies } (11), (12), (13), (14)$$
$$0 \leq y_i \leq 1 \quad i = 1, \ldots, 8. \quad (22)$$

Let $\mathcal{E}$ be the union of the extreme points of $P_j(B)$ for $j = 0, \ldots, k$ output by PORTA. As a final step, we defined $\mathcal{E}'$ as the set of points of $\mathcal{E}$ such that: i) $y_i \in \{0, 1\}$ for $i = 1, \ldots, 8$; ii) $\text{rank}(x^{Bi} : y_i = 1) = 10$ (this check was carried out using the free software Octave [20]). Finally, $\mathcal{E}'$ corresponds to the set of $C$-feasible of extreme points of $P(B)$ as defined in Subsection 3.2. The codes to replicate the whole computation can be found at the web page: http://www.iasi.cnr.it/~gentile/ClaudioGentileFiles/papers/G-perfect.html.
References


