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GEAR COMPOSITION OF STABLE SET POLYTOPES AND $\mathcal{G}$-PERFECTION

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Abstract

Graphs obtained by applying the gear composition to a given graph H are called *geared graphs*. We show how a linear description of the stable set polytope $STAB(G)$ of a geared graph G can be obtained by extending the linear inequalities defining $STAB(H)$ and $STAB(H^e)$, where H^e is the the graph obtained from H by subdividing the edge e.

We also introduce the class of $\mathcal{G}$-perfect graphs, i.e., graphs whose stable set polytope is described by: nonnegativity inequalities, rank inequalities, lifted 5-wheel inequalities, and some special inequalities called *geared inequalities* and *g-lifted inequalities*. We prove that graphs obtained by repeated applications of the gear composition to a given graph H are $\mathcal{G}$-perfect, provided that any graph obtained from H by subdividing a subset of its simplicial edges is $\mathcal{G}$-perfect. In particular, we show that a large subclass of claw-free graphs is $\mathcal{G}$-perfect, thus providing a partial answer to the well-known problem of finding a defining linear system for the stable set polytope of claw-free graphs.

Key words: stable set polytope, graph composition, polyhedral combinatorics.

1. Introduction

Given a graph $G = (V, E)$ and a vector $w \in \mathbb{Q}_+^V$ of node weights, the *stable set problem* is the problem of finding a set of pairwise nonadjacent nodes *(stable set)* of maximum weight.

The *stable set polytope*, denoted by $STAB(G)$, is the convex hull of the incidence vectors of the stable sets of G; it is known to be full dimensional. A linear system $Ax \leq b$ is said to be *defining* for $STAB(G)$ if $STAB(G) = \{x \in \mathbb{R}^V : Ax \leq b\}$. The *facet defining inequalities* for $STAB(G)$ are those inequalities that constitute the unique nonredundant defining linear system of $STAB(G)$.

So, finding the defining linear system for $STAB(G)$ is equivalent to transform the original optimization problem into the linear program $\max\{w^T x : Ax \leq b, x \geq 0\}$. Indeed the existence of a "good" defining linear system for $STAB(G)$ is equivalent to the existence of a polynomial time algorithm for optimizing over $STAB(G)$ (where "good" means that the separation problem for this linear system can be solved in polynomial time). Since the stable set problem is NP-hard, it is unlikely to find such a system for general graphs. Nevertheless there are classes of graphs for which such systems are known, as bipartite graphs, line graphs [5], series-parallel graphs [14], odd K_4-free [9], and others. It is known that, for these classes of graphs, the weighted stable set problem is polynomial time solvable [12].

In [12], Grötschel, Lovász and Schrijver present a more general point of view. Instead of looking for classes of graphs having a well-defined linear system describing $STAB(G)$, they consider a set $\mathcal{L}$ of valid inequalities for $STAB(G)$ and the following polyhedron

$$\mathcal{L}STAB(G) = \{x \in \mathbb{R}_+^V | \ x \text{ satisfies } \mathcal{L}\}.$$

Further they name *$\mathcal{L}$-perfect* the graphs G having $\mathcal{L}STAB(G) = STAB(G)$. Two basic questions arise in this context: the first one is whether the optimization problem for $\mathcal{L}STAB(G)$ can be solved in polynomial time (equivalently whether the separation problem for $\mathcal{L}STAB(G)$ is polynomial time solvable [11]); the second one is which graphs belong to the class of $\mathcal{L}$-perfect graphs. Different sets $\mathcal{L}$ of inequalities have been considered in literature together with the corresponding classes of $\mathcal{L}$-perfect graphs. We mention some of them in a non exhaustive list: edge plus odd-hole inequalities and t-perfect graphs [4]; clique plus odd-hole inequalities and h-perfect graphs; rank inequalities and rank-perfect graphs [23].

Here, we consider a family $\mathcal{G}$ consisting of the following (lifted) inequalities: *rank inequalities, 5-wheel inequalities, geared inequalities* and *g-lifted inequalities*. The definition of rank and 5-wheel inequalities is given later. The geared and the g-lifted inequalities are generated by the graph composition named *gear composition* introduced in [7]. This composition starts from a given graph H and builds a new graph G by replacing a suitable edge of H with the fixed graph B (*gear*) shown in Fig. 1. This new graph G is called *geared graph* generated by H and B.

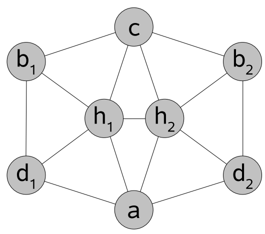


Figure 1: The gear with nodes $d_1, b_1, h_1, h_2, c, a, d_2, b_2$.

The gear composition has an important polyhedral property: it preserves the property of an inequality of being facet defining. This means that a facet defining inequality of $STAB(H)$ can be "properly extended" to a facet defining inequality of $STAB(G)$ when G is a geared graph. The geared inequalities were introduced in [7]; in this paper we identify another class of inequalities generated by the gear composition, the so-called *g-lifted inequalities*. Both classes of inequalities are essential in the linear description of $STAB(G)$ when G is a geared graph and we provide sufficient conditions for them being facet defining. Then, we investigate the relations between the polyhedron

$$\mathcal{G}STAB(G) = \{x \in \mathbb{R}^V_+ | \ x \text{ satisfies } \mathcal{G}\}.$$

and the stable set polytope of a graph G obtained as the gear composition of H and B. Clearly, $STAB(G) \subseteq \mathcal{G}STAB(G)$; here, we provide sufficient conditions to have equality, i.e., we exhibit classes of graphs which are $\mathcal{G}$-perfect. In particular, we consider the class of graphs $\mathcal{G}_H$ obtained by iteratively applying the gear composition to a given graph H. We show that if the gear composition is applied to "suitable" simplicial edges of a line graph H, then the graphs in $\mathcal{G}_H$ are claw-free and $\mathcal{G}$-perfect. This allows us to exhibit the linear description of the polytope $STAB(G)$ for a large subclass of claw-free graphs with stability number at least 4, thus providing a partial answer to the well-known problem of finding a defining linear system for the stable set polytope of claw-free graphs.

In Section 2, we recall the definition of gear composition and we show some of its polyhedral properties. In particular we show under which conditions the gear composition preserves the property of a graph of being facet producing. In Section 3, we show that, apart from clique and 5-wheel inequalities, geared inequalities and g-lifted inequalities are the only new linear inequalities involving B that are necessary to describe $STAB(G)$ when G is a geared graph generated by H and B along e. Finally in Section 4, we introduce the class of inequalities $\mathcal{G}$. Then we prove under which conditions the stable set polytope of a geared graph is described by nonnegativity constraints plus inequalities in $\mathcal{G}$ and we provide interesting examples of $\mathcal{G}$-perfect graphs.

We denote by $G = (V_G, E_G)$ any graph with node set V_G and edge set E_G. An edge $e \in E_G$ with endnodes u and v will be denoted by uv. We denote by $\delta(v)$ the set of edges of G having v as endnode and by $N(v)$ the set of nodes of V_G adjacent to v. A clique-cutset of G is a complete subgraph whose removal disconnects G.

A *k-hole* $C_k = (v_1, v_2, \ldots, v_k)$ is a chordless cycle of length k. A 5-*wheel* $W = (h : v_1, \ldots, v_5)$ is a graph consisting of a 5-hole $C = (v_1, \ldots, v_5)$, called *rim* of W, and a node h (*hub* of W) adjacent to every node of C. A *claw* is the graph $K_{1,3}$.

A *gear* B is a graph of eight nodes $\{a, b_1, b_2, c, d_1, d_2, h_1, h_2\}$ such that $W_1 = (h_1 : a, d_1, b_1, c, h_2)$ and $W_2 = (h_2 : a, d_2, b_2, c, h_1)$ are 5-wheels (see Fig. 1); moreover, the edges of these wheels are the only edges of B. When no confusion arises we shall denote as $W_i = (h_i : C_i)$ for $i = 1, 2$, the two 5-wheels contained in the gear B.

If $w : V_G \to \mathbb{Q}_+$ is any weighting of the nodes of G, then $\alpha(G, w)$ denotes the maximum weight of a stable set of G. We refer to $\alpha(G) = \alpha(G, \mathbb{1})$ ($\mathbb{1}$ being the vector of all ones) as the *stability number* of G.

Given a vector $\beta \in \mathbb{R}^m$ and a subset $S \subseteq \{1, \ldots, m\}$, define $\beta_S \in \mathbb{R}^{|S|}$ as the subvector of β restricted on the indices of S and $\beta(S) = \sum_{i \in S} \beta_i$. Given a subset $S \subseteq \{1, \ldots, m\}$, we denote by $x^S \in \mathbb{R}^m$ the incidence vector of S.

A linear inequality $\sum_{j \in V_G} \pi_j x_j \leq \pi_0$ is said to be *valid* for $STAB(G)$ if it holds for all $x \in STAB(G)$. For short, we also denote a linear inequality $\pi^T x \leq \pi_0$ as (π, π_0). A valid inequality for $STAB(G)$ *defines* a facet of $STAB(G)$ if and only if it is satisfied as an equality by $|V_G|$ affinely independent incidence vectors of stable sets of G (called *roots* or *tight solutions*). We also say that a stable set S is tight for (π, π_0) if its incidence vector x^S is a tight solution of (π, π_0).

If the support of a facet defining inequality (π, π_0) coincides with V_G, we say that the graph G *supports* (or *produces*) the corresponding facet or equivalently that (π, π_0) has full support on V_G.

A linear inequality $\sum_{j \in V_G} \pi_j x_j \leq \pi_0$ is said to be a *rank inequality* for $STAB(G)$ if $\pi_i = 1$ for each $i \in S \subseteq V_G$, $\pi_i = 0$ for each $i \in V_G \setminus S$ and $\pi_0 = \alpha(G[S])$ where $G[S]$ is the subgraph of G induced by S. Given a 5-wheel $W = (h : v_1, v_2, v_3, v_4, v_5)$, then the inequality $\sum_{i=1}^{5} x_{v_i} + 2x_h \leq 2$ is called *5-wheel inequality*.

We recall the definition of the *sequential lifting* procedure defined in [16] that will be used in the following sections. Let $\mathscr{S}(G)$ denote the family of the stable sets of G. If $\sum_{j \in V_G \setminus \{v\}} \pi_j x_j \leq \pi_0$ is a facet defining inequality of $STAB(G \setminus \{v\})$, then the inequality

$$\sum_{j \in V_G \setminus \{v\}} \pi_j x_j + \pi_v x_v \leq \pi_0 \quad \text{with} \quad \pi_v = \pi_0 - \max_{S \in \mathscr{S}(G \setminus (N(v) \cup \{v\}))} \pi(S)$$

is facet defining for $STAB(G)$. This inequality will be called *sequential lifting of* $(\pi_{V_G \setminus \{v\}}, \pi_0)$ and π_v will be called the *lifting coefficient of* v. This procedure can be iterated to generate facet defining inequalities, simply called *lifted inequalities*, in a higher dimensional space.

2. Geared inequalities and g-lifted inequalities

An edge $v_1 v_2$ of a graph H is said to be *simplicial* if $K_1 = N(v_1) \setminus \{v_2\}$ and $K_2 = N(v_2) \setminus \{v_1\}$ are nonempty cliques of H. Notice that K_1 and K_2 may have nonempty intersection. Simplicial edges have a trivial though very useful polyhedral property:

Proposition 2.1. *Let H be a graph and H' be a subgraph of H that supports a facet defining inequality (π, π_0) of $STAB(H)$ which is not a clique inequality. If H' contains a simplicial edge $v_1 v_2$, then $\pi_{v_1} = \pi_{v_2}$. If H' contains a simplicial edge $v_1 v_2$ subdivided with a node t, then $\pi_{v_1} = \pi_{v_2} = \pi_t$.*

Proof. Since $v_1 v_2$ is simplicial we have that $K_1 = N(v_1) \setminus \{v_2\}$ and $K_2 = N(v_2) \setminus \{v_1\}$ are nonempty cliques of H'. Let us consider a tight stable set S_1 missing $K_1 \cup \{v_1\}$ (it exists since (π, π_0) is not a clique inequality). Clearly, $v_2 \in S_1$ (since otherwise $S_1 \cup \{v_1\}$ would violate (π, π_0)). Hence, $\pi_{v_2} \geq \pi_{v_1}$ (since otherwise $S_1 \setminus \{v_2\} \cup \{v_1\}$ would violate (π, π_0)). A symmetric argument proves that $\pi_{v_1} \geq \pi_{v_2}$ and the first claim follows.

Consider now a simplicial edge $v_1 v_2$ subdivided with a node t. Obviously, $v_1 t$ and $t v_2$ are both simplicial. Hence, we have that $\pi_{v_1} = \pi_t = \pi_{v_2}$ and the proposition follows. ∎

We recall the definition of gear composition given in [7] together with a picture describing how it works:

Definition 2.2. *Let $H = (V_H, E_H)$ be a graph with a simplicial edge $e = v_1 v_2$ and let $B = (V_B, E_B)$ be a gear. The* gear composition *of H and B along $v_1 v_2$ generates a new graph G such that:*

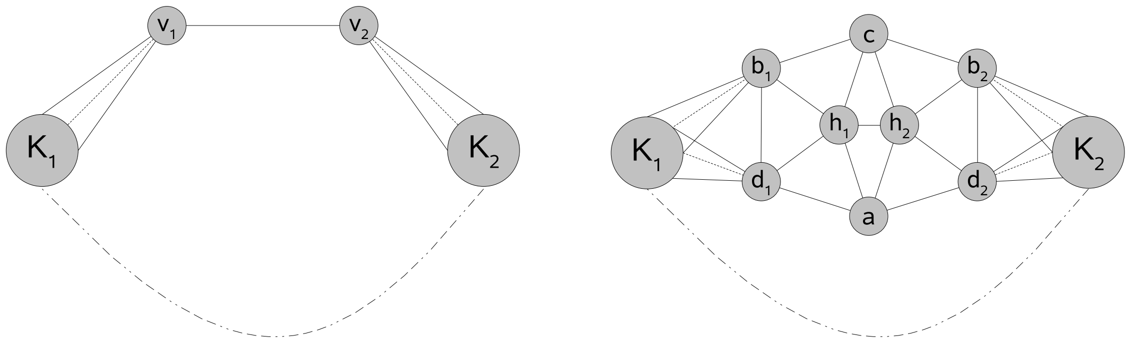
$V_G = V_H \setminus \{v_1, v_2\} \cup V_B$,
$E_G = E_H \setminus (\delta(v_1) \cup \delta(v_2)) \cup E_B \cup F_1 \cup F_2$, *where* $F_i = \{d_i u | u \in K_i\} \cup \{b_i u | u \in K_i\}$ *for* $i = 1, 2$.

The graph G will be called the geared graph *generated by H and B along e and denoted by $G = (H, B, e)$.*

Definition 2.3. *Let H be a graph with a simplicial edge $e = v_1 v_2$ and let H^e be the graph obtained from H by subdividing e with a new node t.*

An inequality (π, π_0) which is valid for $STAB(H)$ is said to be g-extendable *(with respect to e) if $\pi_{v_1} = \pi_{v_2} = \lambda > 0$ and it is not the inequality $x_{v_1} + x_{v_2} \leq 1$.*

An inequality (π, π_0) which is valid for $STAB(H^e)$ is said to be g-liftable *(with respect to e) if $\pi_{v_1} = \pi_{v_2} = \pi_t = \lambda > 0$.*



(a) (b)

Figure 2: (a) A graph H with a simplicial edge v_1v_2; (b) The geared graph $G = (H, B, v_1v_2)$.

Definition 2.4. *Let $H = (V_H, E_H)$ be a graph containing the simplicial edge $e = v_1v_2$, let $B = (V_B, E_B)$ be a gear and let (π, π_0) be a valid inequality for $STAB(H)$ that is g-extendable with respect to e. Then the inequalities*

$$\diamond \quad \sum_{i \in V_H \setminus \{v_1,v_2\}} \pi_i x_i + \lambda \sum_{i \in V_B \setminus \{h_1,h_2\}} x_i + 2\lambda(x_{h_1} + x_{h_2}) \leq \pi_0 + 2\lambda \tag{1}$$

$$\diamond \quad \sum_{i \in V_H \setminus \{v_1,v_2\}} \pi_i x_i + \lambda \sum_{i \in V_B \setminus A} x_i \leq \pi_0 + \lambda \tag{2}$$

where $A \in \{\{b_1, c\}, \{b_2, c\}, \{d_1, a\}, \{d_2, a\}, \{a, c\}\}$

are called geared inequalities *associated with (π, π_0). The unique geared inequality that has full support on V_B is (1) and it will be called* proper geared inequality.

Geared inequalities are essential in the linear description of the stable set polytope of geared graphs. Indeed, it was proved that:

Theorem 2.5. *[7] Let $G = (H, B, e)$ be a geared graph generated by H and B along e and let (π, π_0) be an inequality that is g-extendable with respect to e. If (π, π_0) is facet defining for $STAB(H)$, then the proper geared inequality (1) associated with (π, π_0) is facet defining for $STAB(G)$.*

The above theorem can be extended to the geared inequalities (2) as follows:

Theorem 2.6. *Let $G = (H, B, e)$ be a geared graph generated by H and B along e and let (π, π_0) be an inequality that is g-extendable with respect to e. If (π, π_0) is facet defining for $STAB(H)$, then the geared inequalities (2) associated with (π, π_0) are facet defining for $STAB(G)$ for each $A \in \{\{b_1, c\}$, $\{b_2, c\}$, $\{d_1, a\}$, $\{d_2, a\}$, $\{a, c\}\}$.*

Proof. A sketch of the proof for the case $A = \{a, c\}$ was given in [7]. For the sake of completeness, we recall here the arguments used in that proof. Consider the graph G' obtained from H by subdividing the edge $e = v_1v_2$ with two nodes h_1 and h_2 and renaming v_i as d_i, $i = 1, 2$. Clearly G' is a subgraph of G and, by a result of Wolsey [24] on edge subdivisions, the following inequality

$$\sum_{i \in V_G \setminus \{v_1,v_2\}} \pi_i x_i + \lambda \sum_{i \in \{d_1,h_1,h_2,d_2\}} x_i \leq \pi_0 + \lambda$$

is facet defining for $STAB(G')$. This inequality can be lifted to yield a facet defining inequality of $STAB(G)$ by observing that b_1 and b_2 can be lifted with coefficient λ, and then a and c can be lifted with coefficient zero. This completes the proof of case $A = \{a, c\}$. The facet defining defining inequality corresponding to $A = \{b_1, c\}$ is obtained by first lifting the nodes a and b_2 with coefficient λ and then the nodes b_1 and c with coefficient zero. The remaining cases can be proved analogously by changing the order of the lifted nodes. ∎

Example 2.1. Consider the 5-hole C_5 and the geared graph G obtained as the gear composition of C_5 and B along the simplicial edge $e = v_1v_2$ (see Fig. 3). Thus, we write $G = (C_5, B, e)$.

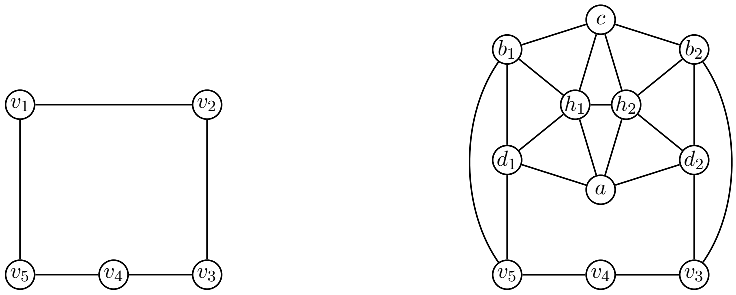


Figure 3: A 5-hole C_5 and a geared 5-hole G

As the 5-hole inequality $x(V_{C_5}) \leq 2$ is valid for $STAB(C_5)$ and it is g-extendable with respect to e, the following inequality

$$x(V_G \setminus \{h_1, h_2\}) + 2x_{h_1} + 2x_{h_2} \leq 4$$

is a proper geared inequality associated with $x(V_{C_5}) \leq 2$. Since $x(V_{C_5}) \leq 2$ is facet defining for $STAB(C_5)$, the proper geared inequality associated with $x(V_{C_5}) \leq 2$ is facet defining for $STAB(G)$, by Theorem 2.5. Furthermore, the following five inequalities

$$x(V_G \setminus A) \leq 3, \text{ where } A \in \{\{d_2, a\}, \{d_1, a\}, \{b_2, c\}, \{b_1, c\}, \{a, c\}\},$$

are geared inequality associated with $x(V_{C_5}) \leq 2$ and are facet defining for $STAB(G)$, by Theorem 2.6. □

The inequalities (1) and (2) (see Example 2.1) are not the only inequalities generated by the gear composition. In the remaining of this section we present another class of valid inequalities for $STAB(G)$ called *g-lifted inequalities*.

Definition 2.7. *Let $H = (V_H, E_H)$ be a graph containing the simplicial edge $e = v_1v_2$, let $B = (V_B, E_B)$ be a gear and let (π, π_0) be a valid inequality for $STAB(H^e)$ that is g-liftable with respect to e. Then the inequalities*

$$\diamond \quad \sum_{i \in V_H \setminus \{v_1, v_2\}} \pi_i x_i + \lambda \sum_{i \in V_B} x_i \leq \pi_0 + \lambda, \tag{3}$$

$$\diamond \quad \sum_{i \in V_H \setminus \{v_1, v_2\}} \pi_i x_i + \lambda \sum_{i \in V_B \setminus A} x_i \leq \pi_0 \tag{4}$$

where $A \in \{\{b_1, c, b_2, h_1, h_2\}, \{d_1, a, d_2, h_1, h_2\}\}$

are called g-lifted inequalities *associated with* (π, π_0). *The unique g-lifted inequality that has full support on V_B is (3) and it will be called* proper g-lifted inequality.

Inequalities 4 are clearly valid, as their supporting graph $G \setminus A$ is isomorphic to H^e. We then prove that the proper g-lifted inequality is valid for $STAB(G)$.

Lemma 2.8. *Let $G = (H, B, e)$ be a geared graph generated by H and B along e and let (π, π_0) be an inequality that is g-liftable with respect to e. Then the proper g-lifted inequality (3) associated with (π, π_0) is valid for $STAB(G)$.*

Proof. Let $(\bar{\pi}, \bar{\pi}_0)$ denote the proper g-lifted inequality (3) and let S be a maximal stable set of G. To prove the lemma we distinguish three cases depending on the intersection of S with the subset $\{b_1, b_2, d_1, d_2\}$ of V_B. If $|S \cap \{b_1, b_2, d_1, d_2\}| = 2$, then $K_1 \cap S = K_2 \cap S = \emptyset$ and the set $S \setminus V_B$ is a stable set of H^e. It follows that $\pi(S \setminus V_B) = \bar{\pi}(S \setminus V_B) \leq \pi_0 - 2\lambda$, since otherwise the stable set $S \setminus V_B \cup \{v_1, v_2\}$ of H^e would violate (π, π_0). Moreover, $\bar{\pi}(S \cap V_B) \leq 3\lambda$ and thus, $\bar{\pi}(S \setminus V_B) + \bar{\pi}(S \cap V_B) \leq \pi_0 - 2\lambda + 3\lambda = \pi_0 + \lambda$.

If $|S \cap \{b_1, b_2, d_1, d_2\}| = 1$, we first suppose that $b_1 \in S$; then, $b_2, h_1, c, d_1, d_2 \notin S$ and $S \cap V_B$ contains exactly one node in $\{h_2, a\}$. Thus, $\bar{\pi}(S \cap V_B) = 2\lambda$. Since $S \cap K_1 = \emptyset$, $(S \setminus V_B) \cup \{v_1\}$ is a stable set of H^e, and so $\pi(S \setminus V_B) = \bar{\pi}(S \setminus V_B) \leq \pi_0 - \lambda$. Hence, $\bar{\pi}(S \setminus V_B) + \bar{\pi}(S \cap V_B) \leq \pi_0 - \lambda + 2\lambda = \pi_0 + \lambda$ and the result follows. The cases with $b_2 \in S$, $d_1 \in S$, or $d_2 \in S$ are analogous.

In the last case, $|S \cap \{b_1, b_2, d_1, d_2\}| = 0$ and $S \setminus V_B$ is a stable set in H^e. We have that $\pi(S \setminus V_B) = \bar{\pi}(S \setminus V_B) \leq \pi_0 - \lambda$ since otherwise the stable set $(S \setminus V_B) \cup \{t\}$ of H^e would violate (π, π_0). By the maximality of S, exactly one among the sets $\{h_1\}$, $\{h_2\}$, and $\{a, c\}$, is contained in S, thus implying that $\bar{\pi}(S \cap V_B) \leq 2\lambda$. Hence, $\bar{\pi}(S \setminus V_B) + \bar{\pi}(S \cap V_B) \leq \pi_0 - \lambda + 2\lambda$ and the thesis follows. ∎

In the following we provide sufficient conditions for the class of g-lifted inequalities to be facet defining. Next theorem is the analogous of theorems 2.5 and 2.6 for g-lifted inequalities.

Theorem 2.9. *Let $G = (H, B, e)$ be a geared graph generated by H and B along e and let (π, π_0) be an inequality that is g-liftable with respect to e. If (π, π_0) is facet defining for $STAB(H^e)$, then the proper g-lifted inequality (3) and the g-lifted inequalities (4) for $A \in \{\{b_1, c, b_2, h_1, h_2\}, \{d_1, a, d_2, h_1, h_2\}\}$ associated with (π, π_0), are facet defining for $STAB(G)$.*

Proof. We first prove the theorem for the proper g-lifted inequality. Suppose that $\beta^T x \leq \beta_0$ is a facet defining inequality for $STAB(G)$ that contains all the roots of (3): we prove below that such inequality is equivalent to (3).

We first show that the coefficients β_v associated with nodes $v \in V_B$ are equal. Let x^{S_i}, $i = 1, 2$, be roots of (π, π_0) such that $S_i \cap (K_i \cup \{v_i\}) = \emptyset$. These roots always exist because (π, π_0) has $\pi_{v_1} = \pi_t = \pi_{v_2} = \lambda > 0$ and so, it is not the clique inequality defined by $K_i \cup \{v_i\}$, $i = 1, 2$. Now t must belong to S_i since otherwise $S_i \cup \{v_i\}$ would violate (π, π_0). Consider the following stable sets whose incidence vectors are roots of (3):

$$\begin{aligned} S_1^1 &= S_1 \setminus \{t\} \cup \{a, c\} \\ S_1^2 &= S_1 \setminus \{t\} \cup \{a, b_1\} \\ S_1^3 &= S_1 \setminus \{t\} \cup \{h_2, b_1\} \\ S_1^4 &= S_1 \setminus \{t\} \cup \{h_2, d_1\} \\ S_1^5 &= S_1 \setminus \{t\} \cup \{c, d_1\}. \end{aligned}$$

From $\beta(S_1^1) = \beta(S_1^2) = \beta(S_1^3) = \beta(S_1^4) = \beta(S_1^5)$, it follows that $\beta_c = \beta_{b_1} = \beta_a = \beta_{d_1} = \beta_{h_2}$. Analogously, using S_2 it can be proved that $\beta_c = \beta_{b_2} = \beta_a = \beta_{d_2} = \beta_{h_1}$.

Let M be a matrix whose rows are $|V_{H^e}|$ incidence vectors of stable sets of H^e which are linearly independent roots of (π, π_0), i.e.,

$$M\pi = \pi_0 \mathbb{1}. \tag{5}$$

Any stable set $\tilde{S}$ of H^e can be transformed into a stable set S of G as follows: set $S = \tilde{S} \setminus \{v_1, v_2, t\} \cup S_B$, where S_B is a stable set of B such that $d_i \in S_B$ if and only if $v_i \in \tilde{S}$ for $i = 1, 2$ and moreover, $a \in S_B$ if and only if $t \in \tilde{S}$. It is not difficult to verify that if $x^{\tilde{S}}$ defines a root of (π, π_0) then S_B can be chosen so that x^S defines a root of (3) such that $\beta(S \cap \{h_1, h_2, c\}) = \beta_{h_1}$, since $\{h_1, h_2, c\}$ is a clique and $\beta_{h_1} = \beta_{h_2} = \beta_c$. By replacing V_{H^e} with $V' = V_{H^e} \setminus \{v_1, v_2, t\} \cup \{d_1, d_2, a\}$, we have $M\beta_{V'} = (\beta_0 - \beta_{h_1})\mathbb{1}$ and by (5),

$$\beta_{V'} = (\beta_0 - \beta_{h_1})M^{-1}\mathbb{1} = \frac{\beta_0 - \beta_{h_1}}{\pi_0}\pi.$$

In particular, since $\beta_{d_1} = \beta_{h_1}$ we have

$$\beta_{d_1} = \frac{\beta_0 - \beta_{d_1}}{\pi_0}\pi_{v_1} = \frac{\beta_0 - \beta_{d_1}}{\pi_0}\lambda.$$

Then $\beta_{d_1} > 0$ and, without loss of generality, we can fix $\beta_{d_1} = \lambda$; as a consequence, we have that

$$\begin{array}{ll} \beta_0 = \pi_0 + \lambda, & \\ \beta_u = \pi_u & \text{for each } u \in V_{H^e} \setminus \{v_1, v_2, t\}, \\ \beta_u = \lambda & \text{for each } u \in V_B, \end{array}$$

and the first part of the theorem follows.

Consider now the inequalities (4). They are isomorphic to the original g-liftable inequality (π, π_0) and hence they are trivially valid. If $A = \{b_1, c, b_2, h_1, h_2\}$, it is easy to check that the lifting coefficients of the nodes, e.g., in the order h_1, h_2, b_1, b_2, c, are all equal to zero. This argument proves that these inequalities are facet defining for $STAB(G)$. ∎

Example 2.2. Consider the 4-hole C_4 and the geared graph G obtained as the gear composition of C_4 and B along the simplicial edge $e = v_1v_2$ (see Fig. 3). Thus, we write $G = (C_4, B, e)$.

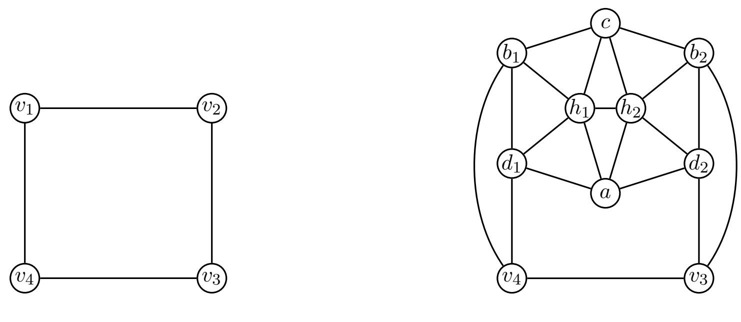


Figure 4: A 4-hole C_4 and a geared 4-hole G

The subdivision of the simplicial edge $e = v_1v_2$ with a new node t generates a 5-hole C_4^e. Since $x(V_{C_4^e}) \leq 2$ is valid for $STAB(C_4^e)$ and it is g-liftable with respect to e, the inequality $x(V_G) \leq 3$ is a proper g-lifted inequality associated with $x(V_{C_4^e}) \leq 2$. Since $x(V_{C_4^e}) \leq 2$ is facet defining for $STAB(C_4^e)$, this proper g-lifted inequality is also facet defining for $STAB(G)$, by Theorem 2.9. Moreover, the following two inequalities

$$x(V_G \setminus A) \leq 3 \text{ where } A \in \{\{b_1, c, b_2, h_1, h_2\}, \{d_1, a, d_2, h_1, h_2\}\}.$$

are non proper g-lifted inequalities associated with $x(V_{C_4^e}) \leq 2$ and they are also facet defining for $STAB(G)$, by Theorem 2.9. □

The above results show that facet defining inequalities for $STAB(H)$ and $STAB(H^e)$ generate geared and g-lifted inequalities, respectively, that are facet defining for $STAB(G)$ when $G = (H, B, e)$ is the geared graph generated by H and B along e. This implies that geared and g-lifted inequalities are necessary for the linear description of $STAB(G)$. Next section will be devoted to prove that they are also sufficient.

3. Gear composition of polyhedra

In this section we show that, apart from clique and 5-wheel inequalities, geared inequalities and g-lifted inequalities are the only new linear inequalities involving B that are necessary to describe $STAB(G)$ when G is a geared graph generated by H and B along e.

Throughout this section, we indicate by (β, β_0) a generic facet defining inequality for $STAB(G)$; we split the vector of coefficients β into two subvectors $(\beta_{V\setminus B}, \beta_B)$ where $\beta_{V\setminus B}$ is the vector of coefficients associated with nodes $V_G \setminus V_B$ and β_B is the vector of coefficients associated with nodes V_B. Moreover, the components of β_B will be indexed as follows: $\beta_B = (\beta_{d_1}, \beta_{b_1}, \beta_{h_1}, \beta_{h_2}, \beta_c, \beta_a, \beta_{d_2}, \beta_{b_2})$.

We first observe that if e is a simplicial edge and $K_1 = K_2$ then the geared graph G generated by H and B along e has a clique-cutset $K_1 = K_2$. When this happens the results of Chvátal on the composition of polyhedra [4] explain how to find a defining linear system for $STAB(G)$ from the defining linear systems of $STAB(H)$ and $STAB(K_1 \cup \{v_1, v_2\}, B, e)$. So, in the rest of the paper we will focus on the composition of polyhedra resulting from applying the gear composition along a simplicial edge that has $K_1 \neq K_2$.

We state now the main theorem of this paper.

Theorem 3.1. *Let $G = (H, B, e)$ be a geared graph generated by H and B along the simplicial edge e. Then the stable set polytope $STAB(G)$ is described by the following linear inequalities:*

- *nonnegativity inequalities,*
- *clique inequalities,*
- *(lifted) 5-wheel inequalities,*
- *geared inequalities associated with facet defining inequalities of $STAB(H)$ having nonzero coefficients on the endnodes of e,*
- *g-lifted inequalities associated with facet defining inequalities of $STAB(H^e)$ having nonzero coefficients on the endnodes of e,*
- *facet defining inequalities of $STAB(H)$ having zero coefficients on the endnodes of e.*

Proof. Since the proof of this result is quite technical and up to some extent repetitive, we arrange it into three main steps that are illustrated below (each step is proved in a separate subsection). We consider a facet defining inequality (β, β_0) for $STAB(G)$ that is neither a clique inequality nor a lifted 5-wheel inequality. We denote as $V' = V_G \setminus V_B$ and by λ a positive scalar number. We also assume that the components of β_B are not all zero. Then we show that:

1) If (β, β_0) does not have full support on V_B and we denote by $A \subset V_B$ the set $\{u \in V_B : \beta_u = 0\}$, then (β, β_0) has the form:
$$\beta_{V'}^T x_{V'} + \lambda x_{B\setminus A} \leq \beta_0$$
where $A \in \{\{b_1, c\}, \{b_2, c\}, \{d_1, a\}, \{d_2, a\}, \{a, c\}, \{b_1, c, b_2, h_1, h_2\}, \{d_1, a, d_2, h_1, h_2\}\}$ (by Theorem 3.4 in Subsection 3.1).

2) If (β, β_0) has full support on V_B then it has one of the following forms:

   a) $\beta^T_{V'} x_{V'} + \lambda x_{B \setminus \{h_1, h_2\}} + 2\lambda(x_{h_1} + x_{h_2}) \leq \beta_0$,
   b) $\beta^T_{V'} x_{V'} + \lambda x_B \leq \beta_0$.

   (by Theorem 3.9 in Subsection 3.2)

3) If (β, β_0) has the form described in 1) with $A \in \{\{b_1, c\}, \{b_2, c\}, \{d_1, a\}, \{d_2, a\}, \{a, c\}\}$ or the form described in 2a) then it is a geared inequality associated with a facet defining inequality of $STAB(H)$ (by Theorem 3.10 and Theorem 3.11 in Subsection 3.3);

   If (β, β_0) has the form described in 1) with $A \in \{\{b_1, c, b_2, h_1, h_2\}, \{d_1, a, d_2, h_1, h_2\}\}$ or the form described in 2b) then it is a g-lifted inequality associated with a facet defining inequality of $STAB(H^e)$ (by Theorem 3.12 in Subsection 3.3,).

As a consequence of the above results, we have that each facet defining inequality for $STAB(G)$ which is different from clique inequalities and 5-wheel inequalities and has $\beta_B \neq 0$ is:

either an inequality of type (1) or (2) where (π, π_0) is a g-extendable facet defining inequality of $STAB(H)$,

or an inequality of type (3) or (4) where (π, π_0) is a g-liftable facet defining inequality of $STAB(H^e)$.

Finally, Proposition 2.1 establishes that every facet defining inequality for $STAB(H)$, that is not a clique inequality, cannot have a zero coefficient on one endnode of e and a nonzero coefficient on the other endnode. Hence, facet defining inequalities for $STAB(H)$ with zero coefficient on the endnodes of e have a supporting graph that is a subgraph of G and may be lifted with zero coefficients. Thus the thesis follows. ∎

3.1. Inequalities not having full support on V_B

In this section we deal with inequalities that do not have full support on V_B. Throughout this section we shall denote by A the set $\{u \in V_B : \beta_u = 0\}$. If an inequality (β, β_0) does not have full support on V_B then $A \neq \emptyset$. We start by recalling the arguments that will often be used in the proofs of this subsection. The first one is a well-known result of Chvátal:

Theorem 3.2. *[4] The supporting graph of a facet defining inequality for $STAB(G)$ does not have a clique-cutset.*

The next observation concerns the lifting coefficients of nodes in A. More precisely,

Observation 1. *Let $G = (H, B, e)$ be a geared graph and let $G \setminus A$ be the subgraph of G supporting the facet defining inequality (β, β_0) of $STAB(G)$, namely $A = \{u \in V_B : \beta_u = 0\}$. Then every node of $u \in A$ has lifting coefficient $\beta_u = 0$.*

As a consequence of Observation 1 we have that if (β, β_0) is facet defining for $STAB(G)$ that does not have full support on V_B then each node $u \in A$ has lifting coefficient $\beta_u = 0$ (for short, has 0-lifting coefficient). By the definition of lifting this implies that:

Observation 2. *For each node $u \in A$, there exists a tight stable set S_u in $G \setminus (A \cup N(u))$.*

Notice also that if there exist two adjacent nodes $u \in A$ and $v \in V_G \setminus A$ with $N(v) \setminus \{u\} \subseteq N(u)$, then every stable set S in $G \setminus (A \cup N(u))$ can be augmented by adding the node v. This implies that a tight stable set in $G \setminus (A \cup N(u))$ satisfying Observation 2 does not exist, a contradiction. Hence,

Observation 3. *No node $u \in A$ is adjacent to a node $v \in V_G \setminus A$ with $N(v) \setminus \{u\} \subseteq N(u)$.*

Moreover, we will also use the following arguments:

Observation 4. *Let G be a graph and let (π, π_0) and (β, β_0) be two facet defining inequalities for $STAB(G)$. If (β, β_0) is not a positive scalar multiple of (π, π_0) then there exists a stable set S that is tight for (β, β_0), i.e., $\beta x^S = \beta_0$, but not for (π, π_0), i.e., $\pi x^S < \pi_0$.*

In the next proofs clique inequalities or 5-wheel inequalities will play the role of (π, π_0). In these cases, we will say that there exists a tight stable set S for (β, β_0) that *misses* a certain clique in $V_B \cup K_1 \cup K_2$ or one of the two 5-wheels contained in B.

Observation 5. *Let G be a graph and let (β, β_0) be a facet defining inequality for $STAB(G)$. Then for any $u \in V_G$ there exists at least a root of (β, β_0) containing u.*

We are now ready to prove that, if G is a geared graph, then for any facet defining inequality $(\beta_{V \setminus B}, \beta_B, \beta_0)$ for $STAB(G)$ that has not full support on V_B, the vector β_B can assume only 7 different values (listed in Theorem 3.4). This will be proved in two steps: first we show which are the zero components of β_B (Lemma 3.3); then we prove that all the nonzero components of β_B are equal (Theorem 3.4).

Lemma 3.3. *Let $G = (H, B, e)$ be a geared graph and let $(\beta_{V \setminus B}, \beta_B, \beta_0)$ be a facet defining inequality for $STAB(G)$ with both $\beta_{V \setminus B}$ and β_B different from the zero vector. If (β, β_0) does not have full support on V_B and it is neither a clique inequality nor a 5-wheel inequality, then one of the following cases occurs:*

1) $A \in \{\{b_1, c\}, \{b_2, c\}, \{d_1, a\}, \{d_2, a\}, \{a, c\}\}$,
2) $A \in \{\{b_1, c, b_2, h_1, h_2\}, \{d_1, a, d_2, h_1, h_2\}\}$.

Proof. Without loss of generality, let $G \setminus A$ denote the supporting graph of the inequality (β, β_0), i.e., the subgraph induced by the nodes of G associated with nonzero components of β. Clearly, $G \setminus A$ has to be connected and, by Theorem 1, it has no clique-cutsets. If $G \setminus A$ satisfies these two properties we say that $G \setminus A$ is admissible. The proof consists in showing that, apart from those listed in the thesis, all admissible configurations of A yield a contradiction.

Observe that if there does not exist a path between K_1 and K_2 contained in B, then K_1 and K_2 are clique-cutsets of $G \setminus A$. It is not difficult to check that if $G \setminus A$ is admissible then $|A| \leq 5$. If $|A| = 5$ and $G \setminus A$ is admissible then case 2) occurs. If $|A| = 4$ and $G \setminus A$ is admissible then A is isomorphic to one of the following configurations (that are derived by enumeration as described in Appendix A):

i) $A = \{b_1, a, c, b_2\}$,
ii) $A = \{b_1, a, c, d_2\}$,
iii) $A = \{b_1, c, h_1, d_2\}$.

In the first two cases Observation 3 is contradicted by nodes a and h_1; in the third case the nodes d_1 and d_2 contradict Observation 3. Hence, $|A| = 4$ cannot occur.

If $|A| = 3$ and $G \setminus A$ contains no clique-cutset then A is isomorphic to one of the following configurations:

i) $A = \{b_1, a, c\}$,

ii) $A = \{b_1, a, d_2\}$,
iii) $A = \{b_1, c, h_1\}$,
iv) $A = \{b_1, c, b_2\}$.

In cases (i) and (ii), the b_1 and d_1 contradict Observation 3, and so, they cannot occur. Consider case (iii): let S_{h_1} be a tight stable set in $G \setminus (A \cup N(h_1))$ (it exists by Observation 2). Clearly, S_{h_1} contains d_2 (since otherwise $S_{h_1} \cup \{a\}$ would violate (β, β_0)) and a node in K_1 (since otherwise $S \cup \{d_1\}$ would violate (β, β_0)). It follows that $\beta_{d_2} \geq \beta_{b_2} + \beta_a$. Now let T be a tight stable set missing the clique $\{a, h_2, d_2\}$. Then $b_2 \in T$ (otherwise $T \cup \{h_2\}$ is feasible and violates (β, β_0)) and, consequently, $\beta_{b_2} \geq \beta_{d_2}$; as $\beta_a > 0$, this is a contradiction. Finally consider case (iv): let S_{b_1} a tight stable set in $G \setminus (A \cup N(b_1))$ (it exists by Observation 2). Clearly, S_{b_1} contains a (otherwise $S_{b_1} \cup \{d_1\}$ would violate (β, β_0)) and so, $\beta_a \geq \beta_{h_2} + \beta_{d_1}$. Let T be a tight stable set missing the clique $\{d_1, a, h_1\}$. Then $h_2 \in T$ and $\beta_{h_2} \geq \beta_a$, a contradiction.

If $|A| = 2$ and $G \setminus A$ contains no clique-cutset then A is isomorphic to one of the following configurations:

i) $A = \{a, c\}$,
ii) $A = \{b_1, c\}$,
iii) $A = \{b_1, a\}$,
iv) $A = \{b_1, h_2\}$,
v) $A = \{h_1, h_2\}$,
vi) $A = \{b_1, d_2\}$.

The cases (i) and (ii) are listed in 1) of the thesis. Notice that all the remaining cases of the thesis are isomorphic to case (ii) and so they can be proved by symmetry.

In case (iii), the nodes b_1 and d_1 contradict Observation 3; in case (iv), Observation 3 is contradicted by h_2 and c. In case (v), as the node h_1 has a 0-lifting coefficient, i.e., there exists a tight stable set S_{h_1} in $G \setminus (A \cup N(h_1))$; it is not difficult to see that S_{h_1} contains either d_2 or b_2. But then either $S_{h_1} \cup \{c\}$ or $S_{h_1} \cup \{a\}$ violates (β, β_0), a contradiction. So, all cases (iii)÷(v) yield a contradiction.

It remains to consider the case (vi). By Observation 2, as the node $d_2 \in A$, there exists a tight stable set S_{d_2} in $G \setminus (A \cup N(d_2))$. It is not difficult to see that $S_{d_2} \supset \{c, d_1\}$ and so, $\beta_{h_2} \leq \beta_c$. Now let S be a tight stable set missing $\{h_1, d_1, a\}$. Then $h_2 \in S$ and so, $\beta_{h_2} = \beta_c$. But then $S \setminus \{h_2\} \cup \{a, c\}$ violates (β, β_0), a contradiction.

If $|A| = 1$ then there are three nonisomorphic cases to be considered: $A = \{b_1\}$, $A = \{c\}$, and $A = \{h_1\}$.

Case 1. $A = \{b_1\}$.

Let T be a tight stable set missing the clique $\{b_2, h_2, c\}$. Clearly $h_1 \in T$ (since otherwise $T \cup \{c\}$ would violate (β, β_0). By Observation 2, the node b_1 has a 0-lifting coefficient, i.e., there exists a tight stable set S_{b_1} in $G \setminus (A \cup N(b_1))$. It is not difficult to see that S_{b_1} contains $\{a, b_2\}$. Then $\beta_a \geq \beta_{h_1}$, $\beta_a \geq \beta_{d_1}$ and $\beta_{b_2} \geq \beta_c$.

Since $S_{b_1} \supseteq \{a, b_2\}$ and $S_{b_1} \setminus \{a, b_2\} \cup \{d_1, c, d_2\}$ is a stable set, it follows that $\beta_a + \beta_{b_2} \geq \beta_{d_1} + \beta_c + \beta_{d_2}$. If $\beta_a = \beta_{d_1}$ then $\beta_{b_2} \geq \beta_c + \beta_{d_2}$. Since all coefficients of β_B apart from β_{b_1} are positive, we have that $\beta_{b_2} > \beta_{d_2}$. This implies that $d_2 \notin T$ (since otherwise $T \setminus \{d_2\} \cup \{b_2\}$ would violate (β, β_0)) and so, $T \setminus \{h_1\} \cup \{a, c\}$ violates (β, β_0), a contradiction.

Hence, $\beta_a > \beta_{d_1}$. Thus every tight stable set S containing b_2 contains either a or h_1 and every tight stable set S containing c contains either a or d_2. In fact, in all other cases, $d_1 \in S$ and $S \setminus \{d_1\} \cup \{a\}$ violates (β, β_0), a contradiction. Moreover every tight stable set S containing a contains either b_2 or c and every tight stable set containing d_2 contains either h_1 or c. Finally every tight stable set containing h_1 contains either b_2 or d_2 (since otherwise $S \setminus \{h_1\} \cup \{a, c\}$ would violate (β, β_0)).

As a consequence, every tight solution of (β, β_0) is also a tight solution of the 5-wheel inequality (π, π_0) supported by $W_2 = (h_2 : a, d_2, b_2, c, h_1)$, contradicting Observation 4. *(End of Case 1)*

Case 2. $A = \{c\}$.

By Observation 3, the node c has a 0-lifting coefficient, i.e., there exists a tight stable set S_c in $G \setminus (A \cup N(c))$. It is not difficult to see that S_c contains $\{d_1, d_2\}$ or $\{a\}$.

Suppose first that $\{d_1, d_2\} \subseteq S_c$. From this, it follows that $\beta_{d_i} \geq \beta_{h_i}$ and $\beta_{d_i} \geq \beta_{b_i}$, $i = 1, 2$. If $\beta_{b_i} = \beta_{d_i}$, $i = 1, 2$, then $S_c \setminus \{d_1, d_2\} \cup \{b_1, b_2, a\}$ violates (β, β_0), a contradiction. Without loss of generality, suppose that $\beta_{b_1} < \beta_{d_1}$.

Let S be a tight stable set missing the clique $\{b_1, d_1\} \cup K_1$. Clearly, S contains h_1. If $\beta_{d_1} > \beta_{h_1}$, then $S \setminus \{h_1\} \cup \{d_1\}$ would violate (β, β_0), a contradiction. Hence, $\beta_{d_1} = \beta_{h_1}$. We distinguish now three different cases:

- $\beta_{h_1} < \beta_a$.

  Let us consider a tight stable set S missing the clique $\{a, h_2, d_2\}$. Then d_1 or h_1 belongs to S and, the stable set obtained by replacing d_1 or h_1 in S with a violates (β, β_0), a contradiction.

- $\beta_{h_1} > \beta_a$.

  Since $\beta_{b_1} < \beta_{d_1}$, every tight stable set S containing b_1 contains a (since otherwise $S \setminus \{b_1\} \cup \{d_1\}$ would violate (β, β_0)) and every tight stable set containing a contains b_1 (since otherwise $S \setminus \{a\} \cup \{h_1\}$ would violate (β, β_0)), thus implying that the tight solutions of (β, β_0) are not linearly independent, a contradiction.

- $\beta_{h_1} = \beta_a$.

  Since $S_c \setminus \{d_1, d_2\} \cup \{a, b_1, b_2\}$ is a stable set, we have that $\beta_a + \beta_{b_1} + \beta_{b_2} \leq \beta_{d_1} + \beta_{d_2}$. Since $\beta_a = \beta_{h_1} = \beta_{d_1}$, we have that $\beta_{b_2} < \beta_{d_2}$. Consider a tight stable set S' missing $\{h_1, d_1, a\}$. Since $\beta_{b_1} < \beta_{d_1}$, we have that $b_1 \notin S'$ (otherwise $S' \setminus \{b_1\} \cup \{d_1\}$ violates (β, β_0)) and $h_2 \in S'$. So, $\beta_{h_2} \geq \beta_a$. Let S'' be a tight stable set missing $\{h_2, d_2, b_2\}$. It contains h_1 or a and so, $\beta_{h_1} = \beta_a \geq \beta_{h_2}$. Moreover, every tight stable set missing $\{b_2, d_2\} \cup K_2$ clearly contains h_2 and yields $\beta_{h_2} \geq \beta_{d_2}$. Hence, $\beta_{h_2} = \beta_a \geq \beta_{d_2}$. But then every tight stable set S containing b_1 contains b_2. In fact, S contains a (since otherwise $S \setminus \{b_1\} \cup \{d_1\}$ would violate (β, β_0)) and b_2 (since otherwise $S \setminus \{b_1, a\} \cup \{d_1, h_2\}$ would violate (β, β_0)). A symmetric argument shows that every tight stable set containing b_2 also contains b_1, thus implying that the tight solutions of (β, β_0) are not linearly independent, a contradiction.

Suppose now that $\{d_1, d_2\} \not\subseteq S_c$ and so, $a \in S_c$. Let S_i be a tight stable set containing d_i, $i = 1, 2$. Since $S_i \setminus \{d_i\} \cup \{b_i\}$ is a stable set, we have that $\beta_{d_i} \geq \beta_{b_i}$, $i = 1, 2$. Let S' be a tight stable set missing $\{a, h_1, h_2\}$. Since, by hypothesis, there does not exist a tight stable set containing both d_1 and d_2, we have that S' contains neither $\{d_1, b_2\}$ nor $\{d_2, b_1\}$. It follows that $S' \supset \{b_1, b_2\}$. But then $S' \cup \{a\}$ violates (β, β_0), a contradiction. *(End of Case 2)*

Case 3. $A = \{h_1\}$.

By Observation 3, the node h_1 has a 0-lifting coefficient, i.e., there exists a tight stable set S_{h_1} in $G \setminus (A \cup N(h_1))$. It is not difficult to see that S_{h_1} contains either d_2 or b_2. But then either $S_{h_1} \cup \{c\}$ or $S_{h_1} \cup \{a\}$ violates (β, β_0), a contradiction. *(End of Case 3)*

Thus the lemma follows. ∎

The next theorem shows that all the nonzero components of β_B are equal.

Theorem 3.4. *Let $G = (H, B, e)$ be a geared graph generated by H and B along the simplicial edge $e = v_1 v_2$ and let $V' = V_H \setminus \{v_1, v_2\}$. Then each facet defining inequality $(\beta_{V'}, \beta_B, \beta_0)$ of $STAB(G)$*

that does not have full support on V_B, that is neither a clique inequality nor a 5-wheel inequality, and has $\beta_B \neq 0$, is of the following form:

$$\beta_{V'}^T x_{V'} + \lambda x_{B \setminus A} \leq \beta_0$$

where $A \in \{\{b_1, c\}, \{b_2, c\}, \{d_1, a\}, \{d_2, a\}, \{a, c\}, \{b_1, c, b_2, h_1, h_2\}, \{d_1, a, d_2, h_1, h_2\}\}$ and $\lambda > 0$.

Proof. By Lemma 3.3, we know that β_B has either five zero components or two zero components like in the thesis. In the first case we have that the supporting graph of (β, β_0) is a subgraph of H^e containing the subdivision of the simplicial edge e and, by Proposition 2.1, we are done.

It remains to show that if β_B has two zero components then all the remaining components are equal. By Lemma 3.3, the vector β_B satisfies one of the following conditions: $A = \{a, c\}$, $A = \{b_1, c\}$, $A = \{b_2, c\}$, $A = \{d_1, a\}$, $A = \{d_2, a\}$.

Suppose first that $A = \{a, c\}$. We have that $\beta_{d_i} = \beta_{b_i}$, $i = 1, 2$ otherwise there would not exist stable sets containing each of the nodes d_1, d_2, b_1, b_2 which are tight for (β, β_0) (such stable sets must exist for Observation 5). Now, if $\beta_{h_1} > \beta_{d_1}$ then the tight stable set missing $\{h_1, h_2\}$ would violate (β, β_0) after replacing d_1 or b_1 with h_1. Moreover, if $\beta_{h_1} < \beta_{d_1}$ then the tight stable set missing $K_1 \cup \{d_1, b_1\}$ would violate (β, β_0) after replacing h_1 with d_1. Hence, $\beta_{h_1} = \beta_{d_1}$ and similar arguments prove that $\beta_{h_2} = \beta_{d_2}$. As the edge $h_1 h_2$ is simplicial in $G \setminus A$, we have, by Proposition 2.1, that $\beta_{h_1} = \beta_{h_2}$. Thus all nonzero coefficients of β_B are equal and we are done.

The last four cases are symmetric, so we prove in detail the first one and symmetric arguments will prove the remaining cases. Suppose that $A = \{b_1, c\}$ and all components of β_B different from β_{b_1} and β_c are nonzero. Since b_1 has a 0-lifting coefficient with respect to (β, β_0), we have that there exists a stable set S_{b_1} in $G \setminus (A \cup N(b_1))$ which is tight for (β, β_0). Clearly, $a \in S_{b_1}$; it follows that $\beta_{h_1} \leq \beta_a$ and $\beta_{d_1} \leq \beta_a$. If $b_2 \notin S_{b_1}$, then $\beta_a \geq \beta_{h_2} + \beta_{d_1}$ (since otherwise $S_{b_1} \setminus \{a\} \cup \{h_2, d_1\}$ would violate (β, β_0)). Thus, $\beta_a > \beta_{h_2}$. Now, consider a stable set S' which is tight for (β, β_0) and misses the clique $\{a, d_1, h_1\}$. It has to contain h_2, but then $S' \setminus \{h_2\} \cup \{a\}$ violates (β, β_0), a contradiction. Hence, $b_2 \in S_{b_1}$.

Since the node c has a 0-lifting coefficient with respect to (β, β_0), there exists a stable set S_c in $G \setminus (A \cup N(c))$ which is tight for (β, β_0). Two possibilities may occur: either $\{d_1, d_2\} \subseteq S_c$ or $a \in S_c$.

Suppose first that $\{d_1, d_2\} \subseteq S_c$. Clearly, $\beta_{h_i} \leq \beta_{d_i}$ for $i = 1, 2$. Moreover, since $S_{b_1} \setminus \{a, b_2\} \cup \{d_1, d_2\}$ and $S_c \setminus \{d_1, d_2\} \cup \{a, b_2\}$ are both feasible for (β, β_0), we have that $\beta_a + \beta_{b_2} = \beta_{d_1} + \beta_{d_2}$. Now, if $\beta_{d_1} < \beta_a$ then $\beta_{b_2} < \beta_{d_2}$. Consider a stable set S' which is tight for (β, β_0) and misses the clique $\{a, d_2, h_2\}$. Then, $b_2 \notin S'$ (since otherwise $S' \setminus \{b_2\} \cup \{d_2\}$ would violate (β, β_0)) and $h_1 \in S'$ (since otherwise $S' \cup \{h_2\}$ would violate (β, β_0)). It follows that $S' \setminus \{h_1\} \cup \{a\}$ violates (β, β_0) because $\beta_{h_1} \leq \beta_{d_1} < \beta_a$, a contradiction.

Hence, $\beta_{d_1} = \beta_a$ and $\beta_{b_2} = \beta_{d_2}$. If $\beta_{h_1} < \beta_a$ then consider a stable set S' which is tight for (β, β_0) and contains h_1. We have that S' contains d_2 (since otherwise $S' \setminus \{h_1\} \cup \{a\}$ would violate (β, β_0)) and so, $S' \setminus \{h_1, d_2\} \cup \{a, b_2\}$ violates (β, β_0), a contradiction. Thus, $\beta_{h_1} = \beta_a$. If $\beta_{h_2} < \beta_{d_2}$ then consider a stable set S' which is tight for (β, β_0) and misses the clique $K_2 \cup \{d_2, b_2\}$. Clearly S' has to contain h_2 but then $S' \setminus \{h_2\} \cup \{d_2\}$ would violate (β, β_0). Thus, $\beta_{h_2} = \beta_{d_2}$. Finally, if $\beta_{h_1} > \beta_{h_2}$ then consider a stable set S' which is tight for (β, β_0) and misses the clique $\{a, d_1, h_1\}$. S' contains h_2 and $S' \setminus \{h_2\} \cup \{h_1\}$ violates (β, β_0), a contradiction. If $\beta_{h_1} < \beta_{h_2}$ then consider a stable set S' which is tight for (β, β_0) and misses the clique $\{h_2, b_2, d_2\}$. S' contains either a or h_1. So, either $S' \setminus \{a\} \cup \{h_2\}$ or $S' \setminus \{h_1\} \cup \{h_2\}$ violates (β, β_0), a contradiction. Hence $\beta_{h_1} = \beta_{h_2}$. This implies that all non zero components of β_B are equal.

Suppose now that there does not exist S_c such that $\{d_1, d_2\} \subseteq S_c$. Then S_c contains a. Since $S_{b_1} \setminus \{a, b_2\} \cup \{d_1, d_2\}$ is a stable set which is not tight, then $\beta_{d_1} + \beta_{d_2} < \beta_a + \beta_{b_2}$. Let S' be a stable set which is tight for (β, β_0) and contains d_2. Then $\beta_{d_2} \geq \beta_{b_2}$ (since otherwise $S' \setminus \{d_2\} \cup \{b_2\}$ would violate (β, β_0)), and so $\beta_{d_1} < \beta_a$. Now, consider a stable set S'' which is tight for (β, β_0) and misses $\{a, h_1, h_2\}$. Clearly, $d_1 \in S''$ and $d_2 \notin S''$, so $S'' \setminus \{d_1\} \cup \{a\}$ violates (β, β_0), a contradiction. ∎

3.2. Inequalities having full support on V_B

Now, we turn our attention to facet defining inequalities of $STAB(G)$ having full support on V_B. Let (β, β_0) be any facet defining inequality for $STAB(G)$ when $G = (H, B, e)$ is a geared graph, such that β_B has no zero component, i.e., $\beta_v > 0$ for each $v \in V_B$. In particular, (β, β_0) is not a clique inequality or a 5-wheel inequality.

Let $\mathscr{S}(G)$ denote the set of stable sets of G. Since (β, β_0) has full support on V_B it follows that $S \cap V_B$ is maximal in B for any stable set $S \in \mathscr{S}(G)$ that is tight for (β, β_0).

Let $\mathcal{R}$ denote the set of the incidence vectors of stable sets in $\mathscr{S}(G)$ that are roots of (β, β_0) and let $M_{(\beta,\beta_0)}$ be the matrix whose rows are indexed by the nodes of V_G and whose columns are the vectors in $\mathcal{R}$. Since (β, β_0) is facet defining, the matrix $M_{(\beta,\beta_0)}$ has full rank. Consider now the matrix $M'_{(\beta,\beta_0)}$ obtained by summing up all rows indexed by the nodes $u \in K_i$ into a single row indexed by k_i, $i = 1, 2$. This matrix may be interpreted in terms of graphs as follows: let B^* be the graph obtained from B by adding two new nodes to V_B, say k_1 and k_2, such that $N(k_i) = \{b_i, d_i\}$, $i = 1, 2$. Then $\tilde{S} \in \mathscr{S}(B^*)$ if and only if there exists a stable set $S \in \mathscr{S}(G)$ such that: $\tilde{S} \setminus \{k_1, k_2\} = S \cap V_B$ and $K_i \cap S \neq \emptyset$ if and only if $k_i \in \tilde{S}$. It is not difficult to verify that if $rank(M'_{(\beta,\beta_0)}) < |V_G| - \sum_{i=1,2}(|K_i| - 1)$ then $rank(M_{(\beta,\beta_0)}) < |V_G|$.

We say that a stable set $\tilde{S} \in \mathscr{S}(B^*)$ is a *tight configuration* of (β, β_0) if and only if there exists a vector $x^S \in \mathcal{R}$ such that $S \cap V_B = \tilde{S} \setminus \{k_1, k_2\}$ and $K_i \cap S \neq \emptyset$ if and only if $k_i \in \tilde{S}$. Accordingly, we denote by $\mathcal{R}'$ the set of the incidence vectors of the tight configurations of (β, β_0).

So, let $M''_{(\beta,\beta_0)}$ be the submatrix of $M'_{(\beta,\beta_0)}$ whose rows are indexed by the nodes of B^* and whose columns are vectors in $\mathcal{R}'$. These columns have many repetitions in $M''_{(\beta,\beta_0)}$ since all stable sets $S \in \mathscr{S}(G)$ that differ only on nodes out of V_{B^*} produce the same $(0, 1)$-column of $M''_{(\beta,\beta_0)}$. We denote by $\tilde{M}_{(\beta,\beta_0)}$ the matrix of dimension $|V_{B^*}| \times |\mathcal{R}'|$ obtained by deleting multiple columns from $M''_{(\beta,\beta_0)}$. Clearly, we have that if $M_{(\beta,\beta_0)}$ has full rank then $\tilde{M}_{(\beta,\beta_0)}$ has full rank. In particular, we can state the following:

Proposition 3.5. *Let $G = (H, B, e)$ be a geared graph. If (β, β_0) is facet defining for $STAB(G)$, then the matrix $\tilde{M}_{(\beta,\beta_0)}$ has rank 10.*

We now study in deeper detail the structure of the elements of $\mathcal{R}'$ in order to deduce some relations among the components of $\beta_B = (\beta_{d_1}, \beta_{b_1}, \beta_{h_1}, \beta_{h_2}, \beta_c, \beta_a, \beta_{d_2}, \beta_{b_2})$.

First of all we observe that there exist exactly 24 maximal stable sets in $\mathscr{S}(B^*)$; they are depicted in Fig. 7 of Appendix B and denoted by R_i, $i = 1, \ldots, 24$ (coloured nodes represent nodes of R_i, $i = 1, \ldots, 24$). The tight configurations of (β, β_0) are those $R_i \in \mathscr{S}(B^*)$ whose incidence vectors belong to $\mathcal{R}'$.

Each tight configuration R_i of (β, β_0) gives raise to a linear system of inequalities $\mathcal{L}_i$ on β_B by simply considering maximality of R_i in V_B. For example, let us suppose that R_1 is a tight configuration for (β, β_0), i.e., there exists a tight stable S for the inequality (β, β_0) such that: $h_1 \in S$, $S \cap K_1 = \emptyset$, and $S \cap K_2 \neq \emptyset$.

We derive the following system $\mathcal{L}_1$ for the components of β_B:

$$\beta_c + \beta_a \le \beta_{h_1} \tag{6}$$
$$\beta_{b_1} + \beta_{h_2} \le \beta_{h_1} \tag{7}$$
$$\beta_{d_1} + \beta_{h_2} \le \beta_{h_1} \tag{8}$$
$$\beta_{b_1} + \beta_a \le \beta_{h_1} \tag{9}$$
$$\beta_{d_1} + \beta_c \le \beta_{h_1}. \tag{10}$$

Inequality (6) follows by observing that if $\beta_c + \beta_a > \beta_{h_1}$, then the stable set $S' = S \setminus \{h_1\} \cup \{a, c\}$ has the property that $\beta(S') > \beta(S)$. Therefore x^S is not a tight solution for (β, β_0), a contradiction. Using similar arguments it is possible to derive inequalities (7)÷(10).

The systems of inequalities $\mathcal{L}_i$ ($i = 2, \ldots, 24$) associated with the other 23 configurations are shown in Appendix C. Each system $\mathcal{L}_i$ describes a cone in $\mathbb{R}^{|V_B|}$ and its solutions represent the coefficients β_B of an inequality (β, β_0) that admits R_i as a tight configuration. Without loss of generality, we add to each system the normalization conditions $\beta_u \le 1$ for each $u \in V_B$. Then we define a vector $y \in \{0,1\}^{24}$ such that

$$y_i = 1 \text{ if and only if } R_i \text{ is a tight configuration of } (\beta, \beta_0).$$

Thus, for each $i = 1, \ldots, 24$, if $y_i = 1$ then the vector β_B must satisfy the linear system $\mathcal{L}_i$. If $A_i\beta_B \le 0$ represents the system $\mathcal{L}_i$, we introduce a big-M representation of the above condition: $A_i\beta_B \le M_i(1 - y_i)$, where M_i is a vector and $(M_i)_j$ is equal to the number of variables in the $j-th$ inequality of system $\mathcal{L}_i$ having positive coefficients in $(A_i)_j$.

Moreover, the vectors in $\mathcal{R}'$ must satisfy the following set $\mathcal{C}$ of conditions:

i) for each $u \in V_B$ there exists a stable set R_i, for some $i \in \{1, \ldots, 24\}$, such that $u \in R_i$ and $x^{R_i} \in \mathcal{R}'$ (Observation 5);

ii) for each maximal clique K of B^*, there exists a stable set R_i, for some $i \in \{1, \ldots, 24\}$, such that $R_i \cap K = \emptyset$ and $x^{R_i} \in \mathcal{R}'$ (Observation 4);

iii) for each $W_j = (h_j : C_j)$ of B, $j = 1, 2$, there exists a stable set R_i, for some $i \in \{1, \ldots, 24\}$, such that $|R_i \cap C_j| < 2$, $h_j \notin R_i$, and $x^{R_i} \in \mathcal{R}'$ (Observation 4);

iv) the rank of the set $\{x^{R_i} \in \mathcal{R}' : R_i$ satisfies (i) ÷ (iii)$\}$ is 10 (Proposition 3.5).

Conditions (i) ÷ (iii) follow from the hypotheses that (β, β_0) has full support on V_B, it is not a clique inequality and it is not a 5-wheel inequality, respectively. Condition (iv) follows from Proposition 3.5. These properties can be translated into a set of constraints on the vector y as follows:

$$\sum_{i: R_i \ni u} y_i \ge 1, \qquad \forall u \in V_B, \tag{11}$$
$$\sum_{i: R_i \cap K = \emptyset} y_i \ge 1, \qquad \forall K \text{clique of } B^*, \tag{12}$$
$$\sum_{i: R_i \not\ni h_j \text{and} |R_i \cap C_j| < 2} y_i \ge 1, \quad \text{for } W_j = (h_j : C_j) \text{ of } B, j = 1, 2, \tag{13}$$
$$\sum_{i=1}^{24} y_i \ge 10. \tag{14}$$

Notice that the last inequality is a relaxation of property (iv).

We define the polyhedron $\mathcal{P}(B)$ as the convex hull of all the vectors (β_B, y) satisfying the following system:

$$\begin{array}{ll} A_i\beta_B \leq M_i(1-y_i) & i=1,\dots,24 \\ \beta_B \leq \mathbb{1} & \\ y \textit{ satisfies } (11),(12),(13),(14) & \\ y \in \{0,1\}^{24} & \end{array} \tag{15}$$

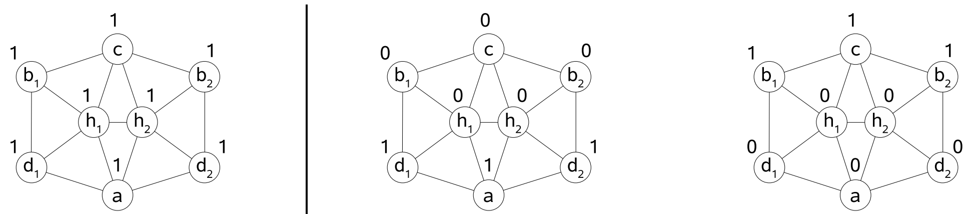
The set of the extreme points of $\mathcal{P}(B)$ was obtained by running the software package PORTA (for details on the procedure see Appendix C). From the list of the extreme points output by PORTA we selected only those satisfying condition (iv), namely for each extreme point (β_B, y) of $\mathcal{P}(B)$ we checked whether the set of vectors $\{x^{R_i} : y_i = 1\}$ has rank 10. We called these extreme points $\mathcal{C}$*-feasible*. The results of these computations are summarized in the following two theorems.

Theorem 3.6. *Let (β_B, y) be a $\mathcal{C}$-feasible extreme point of $\mathcal{P}(B)$. If all the components of β_B are nonzero then β_B equals*

$$\textit{either } (1,1,1,1,1,1,1,1) \textit{ or } (\tfrac{1}{2},\tfrac{1}{2},1,1,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}).$$

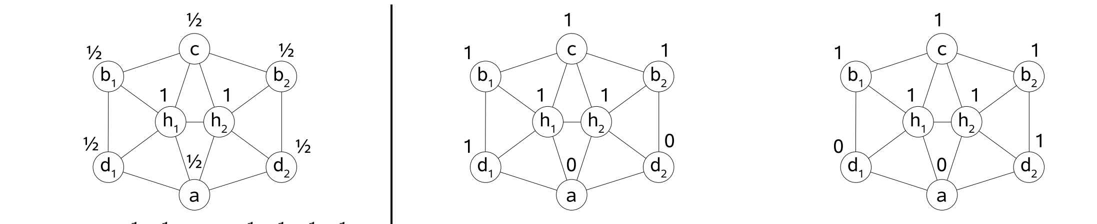
Theorem 3.7. *Let (β'_B, y') and (β''_B, y'') be two $\mathcal{C}$-feasible extreme points of $\mathcal{P}(B)$. If $y' = y''$ then one of the following possibilities occurs:*

a) $\beta'_B = (1,1,1,1,1,1,1,1)$, $\beta''_B \in \{(1,0,0,0,0,1,1,0),(0,1,0,0,1,0,0,1)\}$

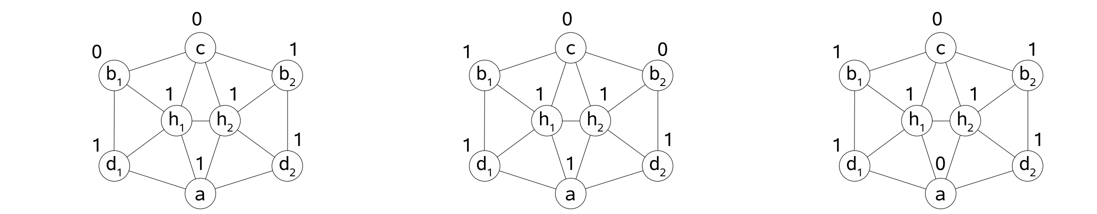


$\beta'_B = (1,1,1,1,1,1,1,1)$ $\beta''_B = (1,0,0,0,0,1,1,0)$ $\beta''_B = (0,1,0,0,1,0,0,1)$

b) $\beta'_B = (\frac{1}{2},\frac{1}{2},1,1,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$,
$\beta''_B \in \{(1,1,1,1,1,0,0,1),(0,1,1,1,1,0,1,1),(1,0,1,1,0,1,1,1),(1,1,1,1,0,1,1,0),$
$(1,1,1,1,0,0,1,1)\}$.



$\beta'_B = (\frac{1}{2}, \frac{1}{2}, 1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ $\beta''_B = (1, 1, 1, 1, 1, 0, 0, 1)$ $\beta''_B = (0, 1, 1, 1, 1, 0, 1, 1)$



$\beta''_B = (1, 0, 1, 1, 0, 1, 1, 1)$ $\beta''_B = (1, 1, 1, 1, 0, 1, 1, 0)$ $\beta''_B = (1, 1, 1, 1, 0, 0, 1, 1)$

With each point (β_B, y) of $\mathcal{P}(B)$ such that $y \in \{0,1\}^{24}$ we associate inequalities of the form

$$\beta_{V'} x_{V'} + \beta_B x_B \leq \beta_0, \tag{16}$$

that we denote as $(\beta_{V'}, \beta_B, \beta_0)$, where $V' = V_G \setminus V_B$.

In the following we show that facet defining inequalities of $STAB(G)$ having full support on B are associated only with extreme points of $\mathcal{P}(B)$. To prove this, we first show that any inequality associated with a point of $\mathcal{P}(B)$ that is convex combination of two extreme points (β'_B, y') and (β''_B, y'') of $\mathcal{P}(B)$ is dominated by inequalities of type (16) associated with (β'_B, y') and (β''_B, y'').

Lemma 3.8. *Let (β'_B, y') and (β''_B, y'') be two extreme points of $\mathcal{P}(B)$ with $y', y'' \in \{0,1\}^{24}$. Then no inequality $(\gamma_{V'}, \gamma_B, \gamma_0)$ such that*

$$\gamma_B = \mu \beta'_B + (1-\mu)\beta''_B, \quad 0 < \mu < 1,$$

with $(\gamma_B, \mu y' + (1-\mu) y'') \in \mathcal{P}(B)$ is facet defining for $STAB(G)$.

Proof. First observe that, since $y', y'' \in \{0,1\}^{24}$, then, in order to have $\mu y' + (1-\mu) y'' \in \{0,1\}^{24}$, y' must be equal to y''. Hence, Theorem 3.7 lists all possible pairs of β'_B and β''_B. Suppose now that $(\gamma_{V'}, \gamma_B, \gamma_0)$ is a valid inequality for $STAB(G)$ and consider the following two inequalities:

$$\begin{aligned} \gamma_{V'} x_{V'} + \beta'_B x_B &\leq \gamma_0 + (1-\mu), \\ \gamma_{V'} x_{V'} + \beta''_B x_B &\leq \gamma_0 - \mu. \end{aligned} \tag{17}$$

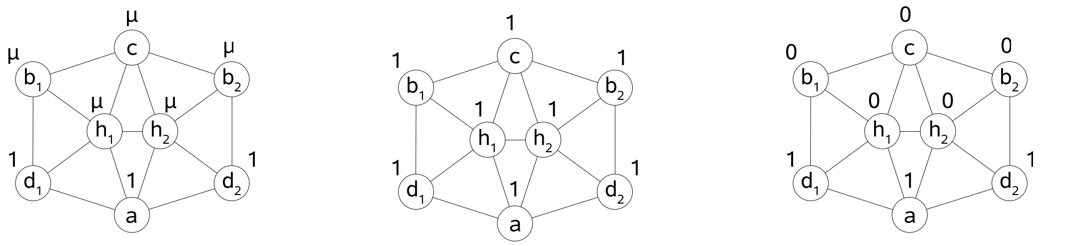
We now prove the lemma only for the case a) of Theorem 3.7 in which we choose β''_B being equal to $(1,0,0,0,0,1,1,0)$: for the other choice of β''_B as for the case b), the proof will follow the same arguments.

Using Fig. 5 and Fig. 7 it is not difficult to check that any tight stable set S for $(\gamma_{V'}, \gamma_B, \gamma_0)$ must satisfy $S \cap V_B \in \{\{a,c\}, \{d_2,h_1\}, \{d_1,h_2\}, \{a,b_1\}, \{d_1,c\}, \{d_2,c\}, \{a,b_2\}, \{d_1,d_2,c\}\}$. Indeed, all other cases lead to stable sets that can be augmented with respect to γ_B (e.g., if $S \cap V_B = \{h_1, b_2\}$, then $\gamma(S \setminus \{h_1\} \cup \{a\}) > \gamma(S)$).

We now show that every tight solution for $(\gamma_{V'}, \gamma_B, \gamma_0)$ is also tight for both inequalities (17) with the help of Fig. 5.

If $S \cap V_B = \{d_1, d_2, c\}$, then $\gamma(S \cap V_B) = 2 + \mu$, while $\beta'_B(S \cap V_B) = 3 = \gamma(S \cap V_B) + (1 - \mu)$ and $\beta''_B(S \cap V_B) = 2 = \gamma(S \cap V_B) - \mu$ (see Fig. 5(b) and Fig. 5(c)), and thus S is tight for both inequalities (17). If $S \cap V_B \in \{\{d_1, h_2\}, \{d_1, c\}, \{d_2, h_1\}, \{d_2, c\}, \{a, b_1\}, \{a, b_2\}, \{a, c\}\}$, then $\gamma(S \cap V_B) = 1 + \mu$, while $\beta'_B(S \cap V_B) = 2 = \gamma(S \cap V_B) + (1 - \mu)$ and $\beta''_B(S \cap V_B) = 1 = \gamma(S \cap V_B) - \mu$, and thus S is tight for both inequalities (17). In a similar way it is possible to asses that if $(\gamma_{V'}, \gamma_B, \gamma_0)$ is valid for $STAB(G)$, then both $(\gamma_{V'}, \beta'_B, \gamma_0 + (1 - \mu))$ and $(\gamma_{V'}, \beta''_B, \gamma_0 - \mu)$ are valid for $STAB(G)$.

Since the inequalities (17) contain all the roots of $(\gamma_{V'}, \gamma_B, \gamma_0)$ and their convex combination yields $(\gamma_{V'}, \gamma_B, \gamma_0)$, it follows that $(\gamma_{V'}, \gamma_B, \gamma_0)$ is not facet defining for $STAB(G)$. ∎



$\gamma_{V'} x_{V'} + \gamma_B x_B \le \gamma_0$ $\gamma_{V'} x_{V'} + \beta'_B x_B \le \gamma_0 + (1 - \mu)$ $\gamma_{V'} x_{V'} + \beta''_B x_B \le \gamma_0 - \mu$

(a) (b) (c)

Figure 5: In (a), (b), and (c) are represented the three inequalities $(\gamma_{V'}, \gamma_B, \gamma_0)$, $(\gamma_{V'}, \beta'_B, \gamma_0 + (1 - \mu))$, and $(\gamma_{V'}, \beta''_B, \gamma_0 - \mu)$, respectively.

Finally, we prove that facet defining inequalities for $STAB(G)$ having full rank on V_B are associated only with the extreme points of $\mathcal{P}(B)$ identified in Theorem 3.6.

Theorem 3.9. *Let $G = (H, B, v_1 v_2)$ be a geared graph and let $V' = V_H \setminus \{v_1, v_2\}$ and $B' = B \setminus \{h_1, h_2\}$. Then each facet defining inequality (β, β_0) of $STAB(G)$ having full support on V_B has one of the following forms:*

a) $\beta^T_{V'} x_{V'} + \lambda x_{B'} + 2\lambda(x_{h_1} + x_{h_2}) \le \beta_0$,

b) $\beta^T_{V'} x_{V'} + \lambda x_B \le \beta_0$.

Proof. With every point (β_B, y) of $\mathcal{P}(B)$ such that $y \in \{0, 1\}^{24}$ it is associated an inequality of the form (16). If $(\beta, \beta_0) = (\beta_{V'}, \beta_B, \beta_0)$ is associated with the extreme points $(\frac{1}{2}, \frac{1}{2}, 1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, y)$ or $(1, 1, 1, 1, 1, 1, 1, 1, y)$ of $\mathcal{P}(B)$ then it is an inequality of type a) or b).

Lemma 3.8 shows that no facet defining inequality of $STAB(G)$ is associated with a point of $\mathcal{P}(B)$ which is not an extreme point of $\mathcal{P}(B)$.

Finally, by Theorem 3.6 the only $\mathcal{C}$-feasible extreme points of $\mathcal{P}(B)$ having all components of β_B different from zero have either $\beta_B = (1, 1, 1, 1, 1, 1, 1, 1)$ or $\beta_B = (\frac{1}{2}, \frac{1}{2}, 1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$: thus the theorem follows. ∎

3.3. The stable set polytope of a geared graph

In this section we are given a geared graph G that is generated by H and B along e and we consider a facet defining inequality (β, β_0) of $STAB(G)$ that has nonzero coefficients on V_B and that is neither a clique inequality nor a lifted 5-wheel inequality. We prove that (β, β_0) is either a geared inequality associated with a facet defining inequality for $STAB(H)$ or a g-lifted inequality associated with a facet defining inequality for $STAB(H^e)$.

From the results in Subsection 3.1, we know that each facet defining inequality of $STAB(G)$ that does not have full support on V_B is of the form described in Theorem 3.4. From the results in Subsection 3.2, we know that each facet defining inequality of $STAB(G)$ with full support on V_B is either of type a) or of type b) described in Theorem 3.9.

The next theorem shows that the inequalities of type a) are proper geared inequalities associated with facet defining inequalities of $STAB(H)$.

Theorem 3.10. *Let $G = (H, B, v_1v_2)$ be a geared graph and let $V' = V_H \setminus \{v_1, v_2\}$ and $B' = B \setminus \{h_1, h_2\}$. If (β, β_0) is a facet defining inequality for $STAB(G)$ of type*

$$\beta_{V'}^T x_{V'} + \lambda x_{B'} + 2\lambda(x_{h_1} + x_{h_2}) \leq \beta_0,$$

with $\lambda > 0$. Then $(\beta_H, \beta_0 - 2\lambda)$ with $\beta_{v_1} = \beta_{v_2} = \lambda$ is a facet defining inequality for $STAB(H)$.

Proof. Suppose conversely that $(\beta_H, \beta_0 - 2\lambda)$ is not facet defining for $STAB(H)$. Then there exists an inequality (π, π_0) that is facet defining for $STAB(H)$ and such that all the roots of $(\beta_H, \beta_0 - 2\lambda)$ are roots of (π, π_0). By Proposition 2.1, $\pi_{v_1} = \pi_{v_2}$. If $\pi_{v_1} = 0$ then (π, π_0) can be lifted to a facet defining inequality for $STAB(G)$ that contains all the roots of (β, β_0) and has $\pi_v = 0$ for each $v \in V_B$, a contradiction. If $\pi_{v_1} > 0$ then we assume without loss of generality that $\pi_{v_1} = \lambda$ and consider the following proper geared inequality:

$$\sum_{i \in V_H \setminus \{v_1, v_2\}} \pi_i x_i + \lambda \sum_{i \in V_B \setminus \{h_1, h_2\}} x_i + 2\lambda(x_{h_1} + x_{h_2}) \leq \pi_0 + 2\lambda. \tag{18}$$

Since (π, π_0) is g-extendable and facet defining for $STAB(H)$, it follows, by Theorem 2.5, that (18) is facet defining for $STAB(G)$.

Let x^S be a root of (β, β_0). Notice that $\beta(S \cap V_B)$ equals either 2λ or 3λ; hence, every tight solution x^S of (β, β_0) can be reduced to a tight solution x^{S_H} of $(\beta_H, \beta_0 - 2\lambda)$ by removing from S an appropriate stable set T of weight 2λ contained in B: if $\beta(S \cap \{a, c, h_1, h_2\}) = 2\lambda$ we remove this subset, i.e., we define $S' = S \setminus \{a, c, h_1, h_2\}$; otherwise $S \cap V_B \in \{\{d_1, d_2, c\}, \{b_1, b_2, a\}\}$ and we define $S' = S \setminus \{a, c, b_2, d_2\}$; finally S_H is built from S' and $v_i \in S_H$ if and only if $S' \cap \{b_i, d_i\} \neq \emptyset$, for $i = 1, 2$, while $S_H \setminus \{v_1, v_2\} = S \setminus V_B$.

By assumption x^{S_H} is also a tight solution for (π, π_0). Thus x^S is also a root of (18) once we reintroduce the previously removed stable set T. Therefore, (β, β_0) and (18) are equivalent. As $STAB(G)$ is full dimensional, the two inequalities only differ by a positive scalar factor. Hence, (π, π_0) is equivalent to $(\beta_H, \beta_0 - 2\lambda)$, contradicting the assumption. ∎

A similar result holds for facet defining inequalities (β, β_0) not having full support on V_B. In fact, by Theorem 3.4, all the nonzero components of β_B have the same value, say λ, and so, the above proof can be repeated almost literally (using Theorem 2.6 and replacing 3λ and 2λ with 2λ and λ, respectively) to show that

Theorem 3.11. *Let $G = (H, B, v_1v_2)$ be a geared graph and let $V' = V_H \setminus \{v_1, v_2\}$ and $A \in \{\{b_1, c\}, \{b_2, c\}, \{d_1, a\}, \{d_2, a\}, \{a, c\}\}$. If (β, β_0) is a facet defining inequality for $STAB(G)$ of type*

$$\beta_{V'}^T x_{V'} + \lambda x_{B \setminus A} \leq \beta_0,$$

with $\lambda > 0$. Then $(\beta_H, \beta_0 - \lambda)$ with $\beta_{v_1} = \beta_{v_2} = \lambda$ is a facet defining inequality for $STAB(H)$.

The next theorem shows that inequalities of type b) in Theorem 3.9 are proper g-lifted inequalities associated with facet defining inequalities of $STAB(H^e)$.

Theorem 3.12. *Let $G = (H, B, e)$ be a geared graph and let H^e be the graph obtained from H by subdividing the edge $e = v_1v_2$ with the new node t. Let $V' = V_H \setminus \{v_1, v_2\}$ and $B' = B \setminus \{h_1, h_2\}$. If (β, β_0) is a facet defining inequality of $STAB(G)$ of type*

$$\beta^T_{V'} x_{V'} + \lambda x_B \leq \beta_0,$$

with $\lambda > 0$. Then $(\beta_{H^e}, \beta_0 - \lambda)$ with $\beta_{v_1} = \beta_{v_2} = \beta_t = \lambda$ is a facet defining inequality for $STAB(H^e)$.

Proof. Suppose conversely that $(\beta_{H^e}, \beta_0 - \lambda)$ is not facet defining for $STAB(H^e)$. Then there exists an inequality (π, π_0) that is facet defining for $STAB(H^e)$ and such that all the roots of $(\beta_{H^e}, \beta_0 - \lambda)$ are roots of (π, π_0). By Proposition 2.1, $\pi_{v_1} = \pi_{v_2} = \pi_t$. If $\pi_{v_1} = 0$ then (π, π_0) can be lifted to a facet defining inequality for $STAB(G)$ that contains all the roots of (β, β_0) and has $\pi_v = 0$ for each $v \in V_B$, a contradiction. If $\pi_{v_1} > 0$ then we assume without loss of generality that $\pi_{v_1} = \lambda$ and consider the following g-lifted inequality:

$$\sum_{i \in V_H \setminus \{v_1, v_2\}} \pi_i x_i + \lambda \sum_{i \in V_B} x_i \leq \pi_0 + \lambda. \tag{19}$$

By Theorem 2.9, the inequality (19) is facet defining for $STAB(G)$.

Let x^S be a root of (β, β_0). Notice that $\beta(S \cap B) = 3\lambda$ if $S \cap B$ equals $\{d_1, c, d_2\}$ or $\{b_1, a, b_2\}$. In the remaining cases $\beta(S \cap B) = 2\lambda$. It follows that every root x^S of (β, β_0) can be reduced to a root $x^{S'}$ of $(\beta_{H^e}, \beta_0 - \lambda)$ by removing from S an appropriate stable set T of weight λ contained in B. By assumption $x^{S'}$ is also a tight solution of (π, π_0). Hence x^S is also a root of (19) once we reintroduce the stable set T previously removed. Therefore, (β, β_0) and (19) are equivalent. As $STAB(G)$ is full dimensional, the two inequalities only differ by a positive scalar factor. Hence, (π, π_0) is equivalent to $(\beta_{H^e}, \beta_0 - \lambda)$, contradicting the assumption. ∎

Finally, observe that the g-lifted inequalities (4) are isomorphic to the original facet defining inequality (π, π_0) of $STAB(H^e)$.

Summing up, theorems 3.4, 3.9, 3.10,3.11, and 3.12 prove Theorem 3.1 as explained in the outline of the proof given at the beginning of Section 3.

4. $\mathcal{G}$-perfect graphs

Up to this point we have considered only graphs that are obtained by performing a single gear composition on a given graph H. In this section we focus on graphs obtained by repeated applications of the gear composition and we generalize to these graphs the results obtained so far.

We start by extending the definition of geared graphs.

Definition 4.1. *Given a graph H which is not a clique, let Γ_H be the set of the simplicial edges of H and let a* g-operation *on $e \in \Gamma_H$ be either a gear composition or an edge subdivision applied along e.*
A graph $G \in \mathcal{G}_H$ if and only if

either $G = H$,

or $G = (L, B, e)$, *where $L \in \mathcal{G}_H$, B is an extended gear, and $e \in \Gamma_H \cap E_L$ (i.e., e is a simplicial edge of both L and H),*

or $G = L^e$, *where $L \in \mathcal{G}_H$ and $e \in \Gamma_H \cap E_L$.*

We call $\mathcal{G}_H$ the class of multiple geared graphs *generated by H.*

Notice that in Definition 4.1 the g-operations, namely gear compositions and edge subdivisions, are performed only along simplicial edges of L that are also simplicial in the given graph H. This implies that in order to generate graphs in $\mathcal{G}_H$ we are not allowed to use any of the edges created by an earlier g-operation: in particular, the edges v_1t and tv_2, created by an edge subdivision of $e = v_1v_2 \in \Gamma_H$, cannot be used to perform any g-operation. In fact, these two edges do not belong to Γ_H even though they have the property of being super simplicial. It follows that any graph in $\mathcal{G}_H$ is obtained by performing at most $|\Gamma_H|$ g-operations, thus implying that, for any fixed graph H, the family $\mathcal{G}_H$ contains a finite number of graphs.

Accordingly with Definition 4.1 we need to define a larger family of inequalities that contains the geared and the g-lifted inequalities obtained by repeated applications of the gear composition.

Definition 4.2. *A facet defining inequality $(\gamma, \gamma_0) \in \mathcal{G}$ if and only if it is (the sequential lifting of)*

either a rank inequality,

or a 5-wheel inequality,

or a geared or a g-lifted inequality associated with an inequality in $\mathcal{G}$.

Consider now the polyhedron

$$\mathcal{G}STAB(G) = \{x \in \mathbb{R}^V_+ |\ x \text{ satisfies } \mathcal{G}\}. \tag{20}$$

Since geared and g-lifted inequalities are valid for $STAB(G)$, it follows that $STAB(G) \subseteq \mathcal{G}STAB(G)$ if G is a geared graph. A graph G is said to be $\mathcal{G}$-perfect if and only if $STAB(G) = \mathcal{G}STAB(G)$. The results of the previous section state that a defining linear system for $STAB(G)$ can be easily provided once defining linear systems for $STAB(H)$ and $STAB(H^e)$ are known. So, an immediate consequence of Theorem 3.1 is the following:

Corollary 4.3. *Let $G = (H, B, e)$ be a geared graph generated by H and B along e. If H and H^e are $\mathcal{G}$-perfect then G is $\mathcal{G}$-perfect.*

In the following we denote by H^F the graph obtained from H by subdividing all the edges in $F \subseteq \Gamma_H$.

Theorem 4.4. *Let H be a graph, $F^* = \{e_1, e_2, \ldots, e_k\} \subseteq \Gamma_H$. If H and H^F are $\mathcal{G}$-perfect for any $F \subseteq F^*$, and $G \in \mathcal{G}_H$ is obtained from H by a sequence of k g-operations along the edges in F^*, then G is $\mathcal{G}$-perfect.*

Proof. Let G_i denote the graph obtained from H by performing the first i g-operations on the edges e_j for $j = 1, \ldots, i$. Then $G = G_k$ by hypothesis. We prove the theorem by induction on the number k of g-operations. If $k = 1$ the theorem is true by Corollary 4.3. If $k > 1$, then, by induction, the theorem holds for every graph $L \in \mathcal{G}_H$ obtained by performing at most $k-1$ g-operations. Suppose by contradiction that G_k is not $\mathcal{G}$-perfect. If G_k is obtained as the gear composition of a graph G_{k-1} and a gear B along a simplicial edge e_k, namely $G_k = (G_{k-1}, B, e_k)$, then, by Corollary 4.3, at least one between G_{k-1} and $G^{e_k}_{k-1}$ is not $\mathcal{G}$-perfect. Since, by induction, G_{k-1} is $\mathcal{G}$-perfect, it follows that $G^{e_k}_{k-1}$ is not. If G_k is obtained from a graph G_{k-1} by subdividing the edge e_k, we again have that $G^{e_k}_{k-1}$ is not $\mathcal{G}$-perfect. Now, by applying the same reasoning to $G^{e_k}_{k-1}$, we obtain that $G^{e_k}_{k-1}$ is not $\mathcal{G}$-perfect only if $G^{\{e_k, e_{k-1}\}}_{k-2}$ is not $\mathcal{G}$-perfect. Thus, iteratively, if G_k is not $\mathcal{G}$-perfect, then $G^{\{e_1, e_2, \ldots, e_k\}}_0 = H^{\{e_1, e_2, \ldots, e_k\}}$ is not $\mathcal{G}$-perfect, a contradiction. ∎

An immediate consequence of Theorem 4.4 is the following:

Corollary 4.5. *Let H be a graph and Γ_H be the set of its simplicial edges. If H and H^F are $\mathcal{G}$-perfect for any $F \subseteq \Gamma_H$, then every graph $G \in \mathcal{G}_H$ is $\mathcal{G}$-perfect.*

In the following we exhibit a significant class of graphs that is $\mathcal{G}$-perfect. This class properly contains the class of line graphs and it is contained in the class of claw-free graphs. To prove this result we need to restrict the application of the g-operations to simplicial edges having the further property that: $N(K_1 \cap K_2) \subseteq N(v_1) \cup N(v_2)$. We call these edges *super simplicial* edges.

Theorem 4.6. *Let H be a line graph that is not a clique. Then the graphs belonging to the subfamily of $\mathcal{G}_H$ obtained by performing g-operations only along super simplicial edges are $\mathcal{G}$-perfect.*

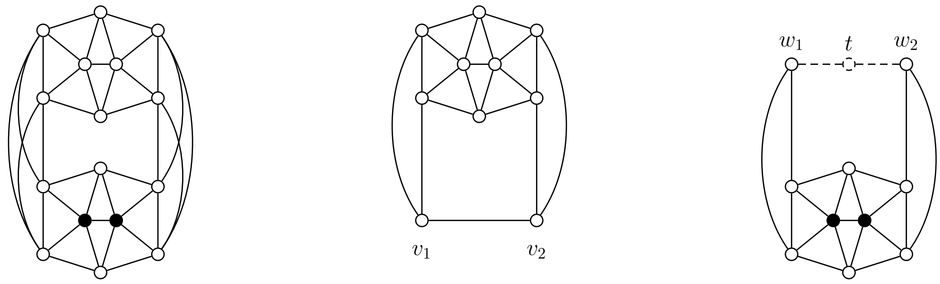
Proof. By the results of Chvátal on composition of polyhedra [4], we may assume without loss of generality that H does not contain a clique-cutset. This implies that $K_1 \setminus K_2$ and $K_2 \setminus K_1$ are both nonempty.

It is well known that $STAB(H)$ is described only by nonnegativity and rank inequalities [5]; thus, H is $\mathcal{G}$-perfect. In order to apply Corollary 4.5 to the line graph H it suffices to guarantee that H^F is a line graph for any subset $F \subseteq \Gamma_H$ of super simplicial edges of H. Let $e = v_1v_2$ be a super simplicial edge of H. The root graph $R(H)$ of H contains two edges $f_{v_1} = \{w_e, s_1\}$ and $f_{v_2} = \{w_e, s_2\}$ sharing the common node w_e. Each node in $K_1 \setminus K_2$ corresponds to an edge of $R(H)$ adjacent to f_{v_1} and not to f_{v_2}. Symmetrically each node in $K_2 \setminus K_1$ corresponds to an edge of $R(H)$ adjacent to f_{v_2} and not to f_{v_1}. Finally, since e is super simplicial, it follows that every node in $N(K_1 \cap K_2)$ is completely adjacent to $(K_2 \setminus K_1) \cup (K_1 \setminus K_2) \cup \{v_1, v_2\}$; therefore, each node in $K_1 \cap K_2$ (if any) is associated with an edge of $R(H)$ joining the nodes s_1 and s_2. Consider now the graph obtained from $R(H)$ by splitting the node w_e into two nodes w_1, w_2 joined by the edge w_1w_2 and such that w_i corresponds to the endnode of the edge f_{v_i} for $i = 1, 2$. This graph is the root graph of H^e and so H^e is a line graph. By iteratively applying the above argument, we prove that H^F is a line graph for any subset $F \subseteq \Gamma_H$ of super simplicial edges of H. Thus, H^F is $\mathcal{G}$-perfect [5] and Corollary 4.5 holds for the subfamily of $\mathcal{G}_H$ obtained from a line graph H by performing g-operations only along super simplicial edges. Therefore the graphs belonging to this subfamily are $\mathcal{G}$-perfect. ∎

In the remaining of this section we explain in a less formal way how $\mathcal{G}STAB(G)$ looks like when $G \in \mathcal{G}_H$ (obtained by performing the g-operations only along super simplicial edges) and H is a line graph. Since H is a line graph, a single application of the gear composition to H produces geared inequalities and g-lifted inequalities associated only with rank inequalities. By definitions 2.4 and 2.7, the proper geared inequalities (when associated with rank inequalities) contain at least a pair of coefficients equal to 2 corresponding to the hubs of a gear while the g-lifted and the non-proper geared inequalities (when associated with rank inequalities) have all coefficients equal to 1. By applying the gear composition several times, it is possible to produce g-lifted inequalities associated with geared inequalities; so, it is not true that every g-lifted inequality in $\mathcal{G}$ is a rank inequality. Nevertheless, we can say that the inequalities in $\mathcal{G}$, which are not 5-wheel inequalities, are only of two types: either they contain pairs of hubs of a gear with coefficients 2 and have all the remaining coefficients equal to 1, or they have all the coefficients equal to 1. We call the former inequalities *multiple geared rank inequalities* and we refer to the others simply as rank inequalities.

The iterative application of the gear composition yields some complications; in fact, the same inequality can be seen both as a geared inequality and as a g-lifted inequality depending on the order the gear compositions have been performed. To see an example consider the graph G depicted in Fig. 6 (a) obtained by applying twice the gear composition to the 4-hole $C_4 = (v_1, v_2, w_2, w_1)$. Indeed, there are two ways to generate G:

1. Apply the gear composition to C_4 and a gear B_1 along w_1w_2 to generate the graph H_1 in Fig. 6 (b). Then apply to H_1 another gear composition with a gear B_2 along the edge v_1v_2 to obtain the



(a) $\sum_{v\in\circ} x_v + 2\sum_{v\in\bullet} \leq 5$ (b) $\sum_{v\in\circ} x_v \leq 3$ (c) $\sum_{v\in\circ} x_v + 2\sum_{v\in\bullet} \leq 4$

Figure 6: (a) is both the proper geared inequality associated with (b) and the proper g-lifted inequality associated wuith (c).

graph G. Since the inequality (b) is g-extendable with respect to v_1v_2, the inequality (a) is a proper geared inequality associated with the inequality (b) (see Definition 2.4). The inequality (a) is also facet defining for $STAB(G)$ by Theorem 2.5.

2. Apply the gear composition to C_4 and a gear B_2 along v_1v_2 to obtain the graph $H_2 = (C_4, B_2, v_1v_2)$. Then apply to H_2 another gear composition with a gear B_1 along the edge w_1w_2 to obtain the graph G. Since the inequality (c) is g-liftable with respect to w_1w_2, the inequality (a) is a proper g-lifted inequality associated with the inequality (c) (see Definition 2.7). The inequality (a) is also facet defining for $STAB(G)$ by Theorem 2.9.

As a consequence, the inequalities in $\mathcal{G}$ are (the sequential liftings of) either multiple geared rank inequalities or rank inequalities or 5-wheel inequalities.

If H is a line graph then the graphs in $\mathcal{G}_H$ (obtained by performing the g-operations only along super simplicial edges) are not quasi-line since they contain 5-wheels, but they are claw-free. To see this, suppose by contradiction that a graph $L \in \mathcal{G}_H$ contains a claw C. Since the gear B is claw-free and the only edges that were removed from the original line graph H were super simplicial edges, we have that C must contain at least two nodes, say v_1 and v_2, corresponding to the endnodes of a super simplicial edge e of H. So, $C = (y : v_1, v_2, w)$ where y is the center of the claw. Clearly either $y \in V_B$ or $y \in K_1 \cap K_2$. In both cases we have that $N(y) \subseteq N(v_1) \cup N(v_2)$, and so w is adjacent to v_1 or v_2, contradicting the hypothesis that C was a claw.

The problem of finding a linear description for $STAB(G)$ when G is claw-free is an open problem which has been studied for decades [8, 15, 19, 13, 22] and for which many conjectures have been stated and disproved [10, 7]. The case when G has stability number 2 has been solved by Cook (see [21]) while for the case $\alpha(G) = 3$ there exists a characterization of the roots of the facet defining inequalities of $STAB(G)$ [17]. The recent decomposition theorem for claw-free graphs of Chudnovsky and Seymour [3] offers new perspectives to face the problem of finding a linear description of the stable set polytope of a claw-free graph. Indeed they identify subclasses of claw-free graphs which might be easier to treat from the polyhedral point of view. For instance, their decomposition theorem restricted to quasi-line [3] graphs led to the settlement of the Ben Rebea's conjecture [6].

Chudnovsky and Seymour also pointed out in [2] that, when dealing with claw-free graphs with stability number at least 4, it is convenient to assume that they do not admit a 1-join (a graph G admits a 1-join if V_G can be partitioned into four sets A_1, B_1, A_2, B_2 such that $A_1 \cup A_2$ is a clique, B_1 and B_2 are nonempty, and the only edges between $A_1 \cup B_1$ and $A_2 \cup B_2$ are those between A_1 and A_2). Indeed, this assumption is very convenient also from the polyhedral point of view. In fact, if G admits a 1-join then G has a clique-cutset and so, by Theorem 3.2, it does not support a facet defining inequality of $STAB(G)$.

So, when looking for facet defining inequalities for the stable set polytope, it is quite natural to assume that the graph that supports the inequality does not admit 1-joins.

Hence, a subclass of claw-free graphs that is likely to investigate is that of: claw-free graphs which are not quasi-line, have $\alpha(G) \geq 4$ and admit no 1-join. Following [2], these graphs are built from certain quasi-line graphs using only two composition operations which we believe have a polyhedral counterpart. This led us to conjecture that:

Conjecture 4.1. *The stable set polytope of a claw-free graph G which is not quasi-line, admits no 1-join and has $\alpha(G) \geq 4$ is described by (sequential liftings of):*

- *nonnegativity inequalities*
- *rank inequalities*
- 5*-wheel inequalities*
- *multiple geared rank inequalities.*

An earlier version of this conjecture already appeared in [7] but it was not precisely stated since it did not contain explicitly the hypothesis that G does not admit 1-joins. This was pointed out to us by Pietropaoli and Wagler [18] who observed that it is possible to compose with a 1-join two claw-free graphs with stability number less than or equal to 3 to obtain a claw-free, not quasi-line graph G with stability number 4 such that the inequalities listed in the conjecture are not sufficient to describe $STAB(G)$.

As a final remark notice that the results in this paper supports the validity of Conjecture 4.1 since the graphs considered in Theorem 4.6 form a large subclass of claw-free, not quasi-line graphs with stability number at least 4.

We end the paper by observing that Theorem 4.4 also applies to graphs that are not claw-free. As an example, consider a 5-wheel W. Since $STAB(W)$ and $STAB(W^F)$ (for any subdivision of a subset F of simplicial edges of the rim) are described by nonnegativity constraints and inequalities in $\mathcal{G}$, we have that any graph $G \in \mathcal{G}_W$ is $\mathcal{G}$-perfect, but it is easy to see that a single application of the gear composition to W creates a claw.

A. Details in the proof of Lemma 3.3

In Lemma 3.3 we prove that there are only 7 possible supporting subgraphs of B for a facet defining inequality $(\beta_{V\setminus B}, \beta_B, \beta_0)$ of $STAB(G)$.

The proof is by enumeration of all the possible 2^8 supports and shows that all the supports that are different from the ones listed in the thesis cannot be associated with a facet defining inequality.

Here we examine in detail the case $|A| = 4$, where A is the subset of nodes in V_B that are not included in the support.

First observe that any supporting graph of a facet defining inequality that is neither a clique inequality nor a 5-wheel inequality must contain a path between K_1 and K_2 whose internal nodes are contained in B, otherwise these cliques are clique-cutset and by Theorem 1 $G \setminus A$ is not the supporting graph of a facet defining inequality. This means that A cannot separate K_1 from K_2.

In particular A contains neither $\{b_1, d_1\}$ nor $\{b_2, d_2\}$, therefore $A \cap \{b_1, d_1, b_2, d_2\}$ is one of the following sets:

a) $\{b_1, b_2\}$,
b) $\{b_1, d_2\}$,
c) $\{d_1, d_2\}$,
d) $\{d_1, b_2\}$,
e) $\{b_1\}$,
f) $\{d_1\}$,
g) $\{b_2\}$,
h) $\{d_2\}$,
j) $\emptyset$.

It is easy to see that the gear B is a highly symmetric graph: if we reverse B upside-down we again obtain a gear with a different order of the nodes, and the same if we reverse B from left to right. This means that if the supporting graph of a facet defining inequality has a nonempty intersection with B, there exists a symmetric facet defining inequality with a symmetric supporting graph. Therefore we list the cases up to symmetry.

Clearly, case c) is symmetric to case a) and case d) is symmetric to case b); cases f), g), and h) are symmetric to case e) (with a upside-down and/or left-to-right reversal); finally, case j) implies $A = \{h_1, h_2, a, c\}$ which separates K_1 and K_2. Thus we are left with only three cases a), b), and e).

Case a) ($A \cap \{b_1, d_1, b_2, d_2\} = \{b_1, b_2\}$) produces the following subcases by considering all the possibile subsets of 2 nodes in $\{h_1, h_2, a, c\}$:

a1) $A = \{b_1, b_2, a, c\}$,
a2) $A = \{b_1, b_2, h_1, h_2\}$,
a3) $A = \{b_1, b_2, a, h_1\}$,
a4) $A = \{b_1, b_2, a, h_2\}$,
a5) $A = \{b_1, b_2, c, h_1\}$,
a6) $A = \{b_1, b_2, c, h_2\}$.

Case a1) matches case i) in the proof of Lemma 3.3 (case $|A| = 4$). In case a2) node c is isolated, i.e., $G \setminus A$ is not admissible. In all other cases $G \setminus A$ contains a clique-cutset, i.e., it is not admissible: K_1 in a3) and a4), $\{d_2, a\}$ in a5), $\{d_1, a\}$ in a6).

Case b) ($A \cap \{b_1, d_1, b_2, d_2\} = \{b_1, d_2\}$) produces the following subcases by considering all the possibile subsets of 2 nodes in $\{h_1, h_2, a, c\}$:

b1) $A = \{b_1, d_2, a, c\}$,

b2) $A = \{b_1, d_2, c, h_1\}$,
b3) $A = \{b_1, d_2, a, h_2\}$,
b4) $A = \{b_1, d_2, h_1, h_2\}$,
b5) $A = \{b_1, d_2, a, h_1\}$,
b6) $A = \{b_1, d_2, c, h_2\}$.

Cases b1) and b2) match cases ii) and iii) of the proof of Lemma 3.3 (case $|A| = 4$), respectively. Case b3) is symmetric to case b2) (take the subgraph associated with case b2, first reverse it upside-down and then reverse the resulting graph from left to right and you will obtain a graph isomorphic to case b3). In all other cases K_2 always defines a clique-cutset of $G \setminus A$.

Case e) ($A \cap \{b_1, d_1, b_2, d_2\} = \{b_1\}$) produces the following subcases by considering all the possibile subsets of 3 nodes in $\{h_1, h_2, a, c\}$:

e1) $A = \{b_1, a, c, h_1\}$,
e2) $A = \{b_1, a, c, h_2\}$,
e3) $A = \{b_1, a, h_1, h_2\}$,
e4) $A = \{b_1, c, h_1, h_2\}$.

Is it easy to check that K_1 defines a clique-cutsets set for the cases e1), e2), and e3), and $K_2 \cup \{d_2\}$ is a clique-cutset for case e4).

B. List of possible tight solutions

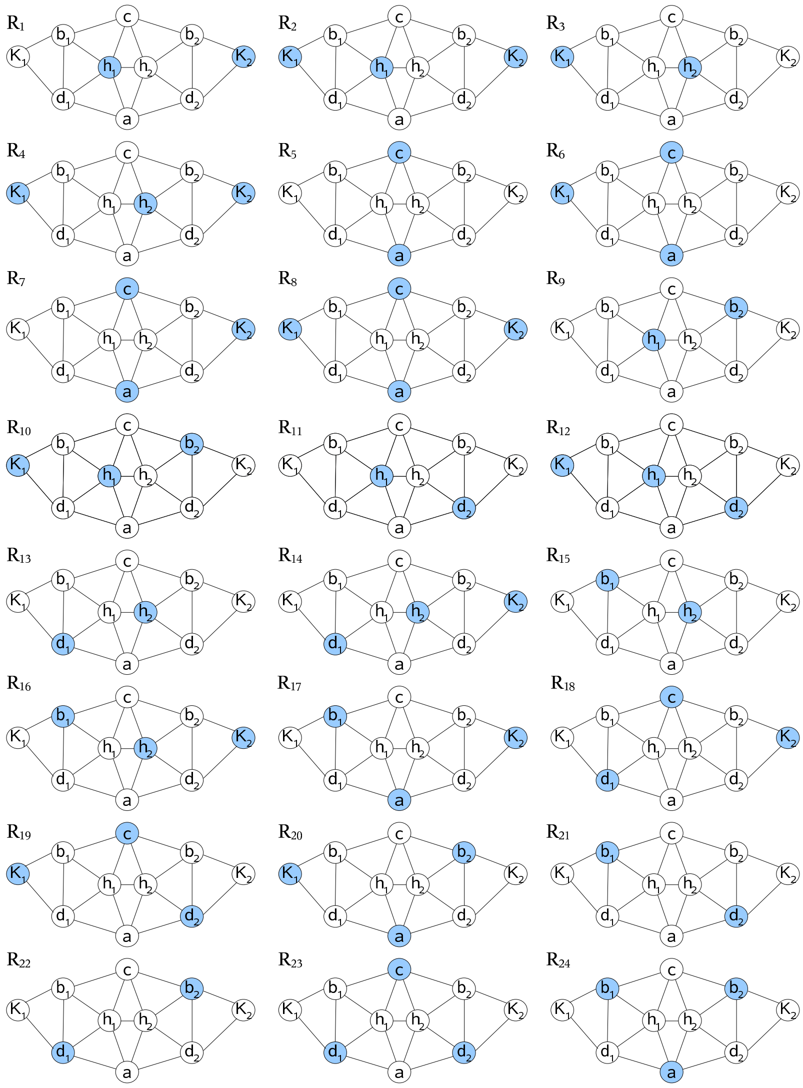


Figure 7: The maximal stable sets of $\mathscr{S}(B^*)$

30.

C. Generation of extreme points of $\mathcal{P}(B)$

In Subsection 3.2 we stated that each configuration R_i produces a linear system of inequalities $\mathcal{L}_i$ on β_B by simply considering maximality conditions. In particular, we presented the system $\mathcal{L}_1$ together with the rules used to generate it. Similar arguments allow us to derive the systems of inequalities $\mathcal{L}_i$ ($i = 2, \dots, 24$) associated with the other 23 tight configurations, which are listed in Fig 7. The complete list of systems is presented below:

$$\mathcal{L}_1 \begin{cases} \beta_c + \beta_a & \leq \beta_{h_1} \\ \beta_{b_1} + \beta_{h_2} & \leq \beta_{h_1} \\ \beta_{d_1} + \beta_{h_2} & \leq \beta_{h_1} \\ \beta_{b_1} + \beta_a & \leq \beta_{h_1} \\ \beta_{d_1} + \beta_c & \leq \beta_{h_1} \end{cases} \qquad \mathcal{L}_2 \begin{cases} \beta_c + \beta_a & \leq \beta_{h_1} \\ \beta_{h_2} & \leq \beta_{h_1} \end{cases} \qquad \mathcal{L}_3 \begin{cases} \beta_c + \beta_a & \leq \beta_{h_2} \\ \beta_{b_2} + \beta_{h_1} & \leq \beta_{h_2} \\ \beta_{d_2} + \beta_{h_1} & \leq \beta_{h_2} \\ \beta_{b_2} + \beta_a & \leq \beta_{h_2} \\ \beta_{d_2} + \beta_c & \leq \beta_{h_2} \end{cases}$$

$$\mathcal{L}_4 \begin{cases} \beta_c + \beta_a & \leq \beta_{h_2} \\ \beta_{h_1} & \leq \beta_{h_2} \end{cases} \qquad \mathcal{L}_5 \begin{cases} \beta_{d_1} + \beta_{d_2} & \leq \beta_a \\ \beta_{b_1} + \beta_{b_2} & \leq \beta_c \\ \beta_{b_2} + \beta_{h_1} & \leq \beta_a + \beta_c \\ \beta_{d_2} + \beta_{h_1} & \leq \beta_a + \beta_c \\ \beta_{d_1} + \beta_{h_2} & \leq \beta_a + \beta_c \\ \beta_{b_1} + \beta_{h_2} & \leq \beta_a + \beta_c \\ \beta_{b_1} + \beta_{d_2} & \leq \beta_a + \beta_c \\ \beta_{d_1} + \beta_{b_2} & \leq \beta_a + \beta_c \end{cases} \qquad \mathcal{L}_6 \begin{cases} \beta_{d_2} & \leq \beta_a \\ \beta_{b_2} & \leq \beta_c \\ \beta_{h_1} + \beta_{b_2} & \leq \beta_a + \beta_c \\ \beta_{h_1} + \beta_{d_2} & \leq \beta_a + \beta_c \\ \beta_{h_2} & \leq \beta_a + \beta_c \end{cases}$$

$$\mathcal{L}_7 \begin{cases} \beta_{d_1} & \leq \beta_a \\ \beta_{b_1} & \leq \beta_c \\ \beta_{h_2} + \beta_{d_1} & \leq \beta_a + \beta_c \\ \beta_{h_2} + \beta_{b_1} & \leq \beta_a + \beta_c \\ \beta_{h_1} & \leq \beta_a + \beta_c \end{cases} \qquad \mathcal{L}_8 \begin{cases} \beta_{h_1} & \leq \beta_a + \beta_c \\ \beta_{h_2} & \leq \beta_a + \beta_c \end{cases} \qquad \mathcal{L}_9 \begin{cases} \beta_{d_2} & \leq \beta_{b_2} \\ \beta_{d_1} & \leq \beta_{h_1} \\ \beta_a + \beta_{b_1} & \leq \beta_{h_1} \\ \beta_a + \beta_c & \leq \beta_{h_1} + \beta_{b_2} \\ \beta_{d_1} + \beta_{h_2} & \leq \beta_{h_1} + \beta_{b_2} \\ \beta_{b_1} + \beta_{h_2} & \leq \beta_{h_1} + \beta_{b_2} \\ \beta_{b_1} + \beta_{d_2} & \leq \beta_{h_1} + \beta_{b_2} \\ \beta_c + \beta_{d_1} + \beta_{d_2} & \leq \beta_{h_1} + \beta_{b_2} \end{cases}$$

$$\mathcal{L}_{10} \begin{cases} \beta_{d_2} & \leq \beta_{b_2} \\ \beta_a & \leq \beta_{h_1} \\ \beta_a + \beta_c & \leq \beta_{h_1} + \beta_{b_2} \\ \beta_c + \beta_{d_2} & \leq \beta_{h_1} + \beta_{b_2} \\ \beta_{h_2} & \leq \beta_{h_1} + \beta_{b_2} \end{cases} \qquad \mathcal{L}_{11} \begin{cases} \beta_{b_2} & \leq \beta_{d_2} \\ \beta_{b_1} & \leq \beta_{h_1} \\ \beta_c + \beta_{d_1} & \leq \beta_{h_1} \\ \beta_a + \beta_c & \leq \beta_{h_1} + \beta_{d_2} \\ \beta_{b_1} + \beta_{h_2} & \leq \beta_{h_1} + \beta_{d_2} \\ \beta_{d_1} + \beta_{h_2} & \leq \beta_{h_1} + \beta_{d_2} \\ \beta_{d_1} + \beta_{b_2} & \leq \beta_{h_1} + \beta_{d_2} \\ \beta_a + \beta_{b_1} + \beta_{b_2} & \leq \beta_{h_1} + \beta_{d_2} \end{cases} \qquad \mathcal{L}_{12} \begin{cases} \beta_{b_2} & \leq \beta_{d_2} \\ \beta_c & \leq \beta_{h_1} \\ \beta_a + \beta_c & \leq \beta_{h_1} + \beta_{d_2} \\ \beta_a + \beta_{b_2} & \leq \beta_{h_1} + \beta_{d_2} \\ \beta_{h_2} & \leq \beta_{h_1} + \beta_{d_2} \end{cases}$$

$$\mathcal{L}_{13} \begin{cases} \beta_{b_1} & \leq \beta_{d_1} \\ \beta_{b_2} & \leq \beta_{h_2} \\ \beta_c + \beta_{d_2} & \leq \beta_{h_2} \\ \beta_a + \beta_c & \leq \beta_{h_2} + \beta_{d_1} \\ \beta_{b_2} + \beta_{h_1} & \leq \beta_{h_2} + \beta_{d_1} \\ \beta_{d_2} + \beta_{h_1} & \leq \beta_{h_2} + \beta_{d_1} \\ \beta_{d_2} + \beta_{b_1} & \leq \beta_{h_2} + \beta_{d_1} \\ \beta_a + \beta_{b_2} + \beta_{b_1} & \leq \beta_{h_2} + \beta_{d_1} \end{cases} \qquad \mathcal{L}_{14} \begin{cases} \beta_{b_1} & \leq \beta_{d_1} \\ \beta_c & \leq \beta_{h_2} \\ \beta_a + \beta_c & \leq \beta_{h_2} + \beta_{d_1} \\ \beta_a + \beta_{b_1} & \leq \beta_{h_2} + \beta_{d_1} \\ \beta_{h_1} & \leq \beta_{h_2} + \beta_{d_1} \end{cases} \qquad \mathcal{L}_{15} \begin{cases} \beta_{d_1} & \leq \beta_{b_1} \\ \beta_{d_2} & \leq \beta_{h_2} \\ \beta_a + \beta_{b_2} & \leq \beta_{h_2} \\ \beta_a + \beta_c & \leq \beta_{h_2} + \beta_{b_1} \\ \beta_{d_2} + \beta_{h_1} & \leq \beta_{h_2} + \beta_{b_1} \\ \beta_{b_2} + \beta_{h_1} & \leq \beta_{h_2} + \beta_{b_1} \\ \beta_{b_2} + \beta_{d_1} & \leq \beta_{h_2} + \beta_{b_1} \\ \beta_c + \beta_{d_2} + \beta_{d_1} & \leq \beta_{h_2} + \beta_{b_1} \end{cases}$$

$$\mathcal{L}_{16} \begin{cases} \beta_{d_1} & \leq \beta_{b_1} \\ \beta_a & \leq \beta_{h_2} \\ \beta_a + \beta_c & \leq \beta_{h_2} + \beta_{b_1} \\ \beta_c + \beta_{d_1} & \leq \beta_{h_2} + \beta_{b_1} \\ \beta_{h_1} & \leq \beta_{h_2} + \beta_{b_1} \end{cases} \qquad \mathcal{L}_{17} \begin{cases} \beta_{h_2} & \leq \beta_a \\ \beta_c & \leq \beta_{b_1} \\ \beta_{d_1} + \beta_c & \leq \beta_{b_1} + \beta_a \\ \beta_{d_1} + \beta_{h_2} & \leq \beta_{b_1} + \beta_a \\ \beta_{h_1} & \leq \beta_{b_1} + \beta_a \end{cases} \qquad \mathcal{L}_{18} \begin{cases} \beta_{h_2} & \leq \beta_c \\ \beta_a & \leq \beta_{d_1} \\ \beta_{b_1} + \beta_a & \leq \beta_{d_1} + \beta_c \\ \beta_{b_1} + \beta_{h_2} & \leq \beta_{d_1} + \beta_c \\ \beta_{h_1} & \leq \beta_{d_1} + \beta_c \end{cases}$$

$$\mathcal{L}_{19} \begin{cases} \beta_{h_1} & \leq \beta_c \\ \beta_a & \leq \beta_{d_2} \\ \beta_{b_2} + \beta_a & \leq \beta_{d_2} + \beta_c \\ \beta_{b_2} + \beta_{h_1} & \leq \beta_{d_2} + \beta_c \\ \beta_{h_2} & \leq \beta_{d_2} + \beta_c \end{cases} \qquad \mathcal{L}_{20} \begin{cases} \beta_{h_1} & \leq \beta_a \\ \beta_c & \leq \beta_{b_2} \\ \beta_{d_2} + \beta_c & \leq \beta_{b_2} + \beta_a \\ \beta_{d_2} + \beta_{h_1} & \leq \beta_{b_2} + \beta_a \\ \beta_{h_2} & \leq \beta_{b_2} + \beta_a \end{cases} \qquad \mathcal{L}_{21} \begin{cases} \beta_{d_1} + \beta_c & \leq \beta_{b_1} \\ \beta_{h_1} & \leq \beta_{b_1} \\ \beta_{h_2} & \leq \beta_{d_2} \\ \beta_{b_2} + \beta_a & \leq \beta_{d_2} \\ \beta_{d_1} + \beta_{b_2} & \leq \beta_{d_2} + \beta_{b_1} \\ \beta_{d_1} + \beta_{h_2} & \leq \beta_{d_2} + \beta_{b_1} \\ \beta_{h_1} + \beta_{b_2} & \leq \beta_{d_2} + \beta_{b_1} \\ \beta_a + \beta_c & \leq \beta_{d_2} + \beta_{b_1} \end{cases}$$

$$\mathcal{L}_{22}\begin{cases}\beta_{d_2}+\beta_c & \leq \beta_{b_2}\\ \beta_{h_2} & \leq \beta_{b_2}\\ \beta_{h_1} & \leq \beta_{d_1}\\ \beta_{b_1}+\beta_a & \leq \beta_{d_1}\\ \beta_{d_2}+\beta_{b_1} & \leq \beta_{d_1}+\beta_{b_2}\\ \beta_{d_2}+\beta_{h_1} & \leq \beta_{d_1}+\beta_{b_2}\\ \beta_{h_2}+\beta_{b_1} & \leq \beta_{d_1}+\beta_{b_2}\\ \beta_a+\beta_c & \leq \beta_{d_1}+\beta_{b_2}\end{cases} \quad \mathcal{L}_{23}\begin{cases}\beta_a & \leq \beta_{d_1}+\beta_{d_2}\\ \beta_{b_2} & \leq \beta_c+\beta_{d_2}\\ \beta_{h_2} & \leq \beta_c+\beta_{d_2}\\ \beta_{b_1} & \leq \beta_c+\beta_{d_1}\\ \beta_{h_1} & \leq \beta_c+\beta_{d_1}\\ \beta_a+\beta_{b_1}+\beta_{b_2} & \leq \beta_c+\beta_{d_1}+\beta_{d_2}\\ \beta_{b_1}+\beta_{h_2} & \leq \beta_c+\beta_{d_1}+\beta_{d_2}\\ \beta_{h_1}+\beta_{b_2} & \leq \beta_c+\beta_{d_1}+\beta_{d_2}\end{cases} \quad \mathcal{L}_{24}\begin{cases}\beta_c & \leq \beta_{b_1}+\beta_{b_2}\\ \beta_{d_2} & \leq \beta_a+\beta_{b_2}\\ \beta_{h_2} & \leq \beta_a+\beta_{b_2}\\ \beta_{d_1} & \leq \beta_a+\beta_{b_1}\\ \beta_{h_1} & \leq \beta_a+\beta_{b_1}\\ \beta_c+\beta_{d_1}+\beta_{d_2} & \leq \beta_a+\beta_{b_1}+\beta_{b_2}\\ \beta_{d_1}+\beta_{h_2} & \leq \beta_a+\beta_{b_1}+\beta_{b_2}\\ \beta_{h_1}+\beta_{d_2} & \leq \beta_a+\beta_{b_1}+\beta_{b_2}\end{cases}$$

As explained in Subsection 3.2, we considered the polyhedron $\mathcal{P}(B)$ that is the convex hull of the feasible solutions of system (15). Theorems 3.6 and 3.7 follow by exhibiting the set of all the extreme points of $\mathcal{P}(B)$. This was done with the help of the software package PORTA [1]. This software receives as an input a system of linear inequalities and returns the list of the extreme points of the polyhedron described by the given system.

In our case the system is:

$$\begin{array}{ll} A_i\beta_B \leq M_i(1-y_i) & i=1,\ldots,24\\ \beta_B \leq \mathbb{1} & \\ \textit{y satisfies } (11),(12),(13),(14) & \\ 0\leq y_i\leq 1 & i=1,\ldots,24. \end{array} \tag{21}$$

Unfortunately, PORTA could not run on the whole system (21) in a reasonable amount of time. So, we subdivided the problem in 2^{16} subproblems by fixing y_i to zero or to one for $i=9,\ldots,24$ as follows. Let $\mathcal{Y}=\{\bar{y}^0,\bar{y}^1,\ldots,\bar{y}^k\}$ with $k=2^{16}-1$ denote the set consisting of the vectors $\bar{y}^j\in\{0,1\}^{16}$ that are binary encoding of j for $j=0,\ldots,k$.

We split the vector y into two parts $y=(y_1,\ldots,y_8|\bar{y})$ where y_i, $i=1,\ldots,8$, are variables and $\bar{y}$ is some vector in $\mathcal{Y}$. Then we ran PORTA on the 2^{16} polyhedra $\mathcal{P}^j(B)$ obtained from system (21) by fixing $\bar{y}$ to each vector $\bar{y}^j$ for $j=0,\ldots,k$. Namely we applied PORTA to the following 2^{16} linear systems (each system is associated with a different vector $\bar{y}^j$):

$$\begin{array}{ll} A_i\beta_B \leq M_i(1-y_i) & i=1,\ldots,8\\ A_i\beta_B \leq 0 & \forall i\in\{9,\ldots,24\} \text{ such that } \bar{y}^j_{i-8}=1\\ \beta_B \leq \mathbb{1} & \\ \textit{y satisfies } (11),(12),(13),(14) & \\ 0\leq y_i\leq 1 & i=1,\ldots,8. \end{array} \tag{22}$$

Let $\mathcal{E}$ be the union of the extreme points of $\mathcal{P}^j(B)$ for $j=0,\ldots,k$ output by PORTA. As a final step, we defined $\mathcal{E}'$ as the set of points of $\mathcal{E}$ such that: i) $y_i\in\{0,1\}$ for $i=1,\ldots,8$; ii) $rank(x^{R_i}:y_i=1)=10$ (this check was carried out using the free software Octave [20]). Finally, $\mathcal{E}'$ corresponds to the set of $\mathcal{C}$-feasible of extreme points of $\mathcal{P}(B)$ as defined in Subsection 3.2. The codes to replicate the whole computation can be found at the web page: http://www.iasi.cnr.it/∼gentile/ClaudioGentileFiles/papers/G-perfect.html.

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References

[1] T. Christof and A. Löbel, "Porta - polyhedron representation transformation algorithm," 2004.

[2] M. Chudnovsky and P. Seymour, "The structure of claw-free graphs," in *Surveys in Combinatorics*, vol. 327 of *London Math. Soc. Lecture Notes*, 2005.

[3] M. Chudnovsky and P. Seymour, "Claw-free graphs IV: Decomposition theorem," *J. Comb. Th. B*, vol. 98, no. 5, pp. 839–938, 2008.

[4] V. Chvátal, "On certain polytopes associated with graphs," *J. Comb. Th. B*, vol. 18, pp. 138–154, 1975.

[5] J. Edmonds, "Maximum matching and a polyhedron with $0, 1$ vertices," *J. Res. of Nat. Bureau of Stand. B*, vol. 69B, pp. 125–130, 1965.

[6] F. Eisenbrand, G. Oriolo, G. Stauffer, and P. Ventura, "The stable set polytope of quasi-line graphs," *Combinatorica*, vol. 28, no. 1, pp. 45–67, 2008.

[7] A. Galluccio, C. Gentile, and P. Ventura, "Gear composition and the stable set polytope," *Operations Research Letters*, vol. 36, pp. 419–423, 2008.

[8] A. Galluccio and A. Sassano, "The rank facets of the stable set polytope for claw-free graphs," *J. Comb. Th. B*, vol. 69, pp. 1–38, 1997.

[9] A. Gerards, "A min-max relation for stable sets in graphs with no odd-K_4," *J. Comb. Th. B*, vol. 47, pp. 330–348, 1989.

[10] R. Giles and L. Trotter, "On stable set polyhedra for $K_{1,3}$-free graphs," *J. Comb. Th. B*, vol. 31, pp. 313–326, 1981.

[11] M. Grötschel, L. Lovász, and A. Schrijver, "The ellipsoid method and its consequences in combinatorial optimization," *Combinatorica*, vol. 1, no. 2, pp. 169–197, 1981.

[12] M. Grötschel, L. Lovász, and A. Schrijver, *Geometric algorithms and combinatorial optimization*. Springer Verlag, Berlin, 1988.

[13] T. Liebling, G. Oriolo, B. Spille, and G. Stauffer, "On non-rank facets of the stable set polytope of claw-free graphs and circulant graphs," *Math. Methods of Oper. Research*, 2004.

[14] A. Mahjoub, "On the stable set polytope of a series-parallel graph," *Mathematical Programming*, vol. 40, pp. 53–57, 1988.

[15] G. Oriolo, "Clique family inequalities for the stable set polytope for quasi-line graphs," *Discrete Applied Mathematics*, vol. 132, pp. 185–201, 2003.

[16] M. W. Padberg, "On the facial structure of vertex packing polytope," *Mathematical Programming*, vol. 5, pp. 199–215, 1973.

[17] A. Pêcher and A. Wagler, "Results and conjectures on the stable set polytope of claw-free graphs," Tech. Rep. 11, Otto-von-Guericke-Universität Magdeburg, Faculty of Mathematics/IMO, 2006.

[18] U. Pietropaoli and A. Wagler, "Personal communication," 2008.

[19] W. Pulleyblank and B. Shepherd, "Formulations of the stable set polytope," in *Proceedings Third IPCO Conference* (G. Rinaldi and L. Wolsey, eds.), pp. 267–279, 1993.

[20] J. B. Rawlings, J. G. Ekerdt, *et al.*, "Octave version 2.9.12," 2007.

[21] B. Shepherd, "Near-perfect matrices," *Mathematical Programming*, vol. 64, pp. 295–323, 1994.

[22] G. Stauffer, *On the stable set polytope of claw-free graphs*. PhD thesis, EPF Lausanne, 2005.

[23] A. Wagler, "Rank-perfect and weakly rank-perfect graphs," *Mathematical Methods of Operations Research*, vol. 95, pp. 127–149, 2002.

[24] L. Wolsey, "Further facet generating procedures for vertex packing polytopes," *Mathematical Programming*, vol. 11, pp. 158–163, 1976.