Gear composition and the Stable Set Polytope

A. Galluccio, C. Gentile, and P. Ventura
IASI-CNR, viale Manzoni 30, 00185 Rome (Italy)

Abstract

We present a new graph composition that produces a graph $G$ from a given graph $H$ and a fixed graph $B$ called gear and we study its polyhedral properties. This composition yields counterexamples to a conjecture on the facial structure of $STAB(G)$ when $G$ is claw-free.

Key words: stable set polytope, graph composition, polyhedral combinatorics, claw-free graphs.

1. Introduction

Given a graph $G = (V, E)$ and a vector $w \in \mathbb{Q}_+^V$ of node weights, the stable set problem is the problem of finding a set of pairwise nonadjacent nodes (stable set) of maximum weight.

The stable set polytope, denoted by $STAB(G)$, is the convex hull of the incidence vectors of the stable sets of $G$ and it is known to be full-dimensional. A linear system $Ax \leq b$ is said to be defining for $STAB(G)$ if $STAB(G) = \{x : Ax \leq b\}$. The facet defining inequalities for $STAB(G)$ are those inequalities that constitute the unique nonredundant defining linear system of $STAB(G)$.

Clearly, finding a defining linear system for $STAB(G)$ is equivalent to transform the original optimization problem into the linear program

$$\max \{w^T x : Ax \leq b\}$$

and, being the stable set problem $NP$-hard, it is unlikely to find such a system for general graphs.

Nevertheless the facial structure of the stable set polytope has been one of the most studied problems in polyhedral combinatorics. Here is a non-exhaustive list of results related with the study of facets of $STAB(G)$: facet producing graphs [17,20,15], $t$ and $h$-perfectness [11], characterization of $STAB(G)$ when $G$ is series-parallel [13], odd $K_4$-free [9] or quasi-line [6].

Besides the description of new classes of facets, it is of interest to find composition procedures that enable to build new families of facets for $STAB(G)$ starting from facets of a lower dimensional polytope. These compositions are usually based on graph compositions: for example, sequential lifting is based on the extension of a graph with an additional node, the Wolsey’s lifting procedure [21] is based on edge subdivision, and Chvátal’s compositions of polyhedra [3] are based on node substitution and clique identification.

In this paper, we present a new graph composition, named gear composition, which consists of ‘replacing’ an edge of a given graph $H$ with a special graph.
called gear, to obtain the graph $G$. We study the polyhedral properties of this operation and derive sufficient conditions to generate facet defining inequalities of $STAB(G)$ starting from facet defining inequalities of $STAB(H)$. The gear composition can be iteratively applied to generate a very rich family of non-rank facet defining inequalities, that we name geared inequalities.

In the last section, we also show how to use this composition to build counterexamples to a conjecture on the facial structure of the stable set polytope of claw-free graphs.

We denote by $G = (V_G, E_G)$ any graph with node set $V_G$ and edge set $E_G$. Given a vector $\beta \in \mathbb{R}^m$ and a subset $S \subseteq \{1, \ldots, m\}$, define $\beta_S \in \mathbb{R}^{|S|}$ as the subvector of $\beta$ restricted on the indices of $S$ and $\beta(S) = \sum_{i \in S} \beta_i$. Given a subset $S \subseteq \{1, \ldots, m\}$, we denote by $x_S \in \mathbb{R}^m$ the incidence vector of stable edges of these wheels are the only edges of $B$.

In this section we introduce the gear composition. An edge $v_1v_2$ of a graph $H$ is said to be simplicial if $K_1 = N(v_1) \setminus \{v_2\}$ and $K_2 = N(v_2) \setminus \{v_1\}$ are two nonempty cliques of $H$. Notice that the two cliques $K_1$ and $K_2$ might intersect.

**Definition 1** Let $H = (V_H, E_H)$ be a graph with a simplicial edge $v_1v_2$ and let $B = (V_B, E_B)$ be a gear. The gear composition of $H$ and $B$ produces a new graph $G$ such that:

$$V_G = V_H \setminus \{v_1, v_2\} \cup V_B$$

$$E_G = E_H \setminus (\delta(v_1) \cup \delta(v_2)) \cup E_B \cup F_1 \cup F_2,$$

where $F_i = \{d_iu, b_iu | u \in K_i\}$ for $i = 1, 2$.

A graph $G$ resulting from the gear composition of two graphs $H$ and $B$ along the simplicial edge $v_1v_2$ will be denoted by $(H, B, v_1v_2)$. A sketch of how the gear composition works is shown in Fig. 2.

**Definition 2** Let $H = (V_H, E_H)$ be a graph containing the simplicial edge $v_1v_2$. The inequality $(\pi, \pi_0)$ is said to be $g$-extendable with respect to $v_1v_2$ if it is valid for $STAB(H)$, it has $\pi_v = \pi_m = \lambda > 0$ and it is not the clique inequality $x_{v_1} + x_{v_2} \leq 1$. If $B = (V_B, E_B)$ is a gear, the following inequality

$$\sum_{i \in V'} \pi_i x_i + \lambda \sum_{i \in V' \setminus \{h_1, h_2\}} x_i + 2\lambda(x_{h_1} + x_{h_2}) \leq \pi_0 + 2\lambda \ (1)$$

where $V' = V_H \setminus \{v_1, v_2\}$, is called the geared inequality associated with $(\pi, \pi_0)$ and will be denoted as $(\bar{\pi}, \bar{\pi}_0)$.

In the following we show that geared inequalities are essential in the linear description of the stable set polytope of geared graphs. We first prove that they are valid inequalities for $STAB(G)$.
**Lemma 1** If $G$ is a geared graph, then the geared inequality (1) is valid for $\text{STAB}(G)$.

**Proof:** Let $S$ be a stable set of $G$. Since each non trivial facet defining inequality of $\text{STAB}(G)$ has non negative coefficients, we can assume that $S$ is maximal. To prove the lemma we distinguish two cases depending on the intersection of $S$ with the subset $\{b_1, b_2, d_1, d_2\}$ of $V_B$.

If $|S \cap \{b_1, b_2, d_1, d_2\}| \geq 1$, then we can suppose without loss of generality that $b_1 \in S$. Then $(S \setminus V_B) \cup \{v_1\}$ is a stable set of $H$ and therefore $\pi(S \setminus V_B) = \pi(S \setminus V_B) = 0$. Moreover, it is not difficult to check that $\pi(S \setminus V_B) \leq 3\lambda$ and thus, $\pi(S \setminus V_B) = 0$, $\pi(S \setminus V_B) \leq 0$, $\lambda + 3\lambda = \pi_0 + 2\lambda$. If $|S \cap \{b_1, b_2, d_1, d_2\}| = 0$ then $S \setminus V_B$ is a stable set in $H$. By the maximality of $S$, exactly one among the sets $\{h_1\}$, $\{h_2\}$, and $\{a, c\}$, is contained in $S$, thus implying that $\pi(S \setminus V_B) = 2\lambda$. Hence, $\pi(S \setminus V_B) + \pi(S \setminus V_B) \leq \pi_0 + 2\lambda$ and the thesis follows. \(\square\)

**Theorem 1** Let $(\pi, \pi_0)$ be a g-extendible inequality. If $(\pi, \pi_0)$ is facet defining for $\text{STAB}(H)$, then the associated geared inequality (1) is facet defining for $\text{STAB}(G)$.

**Proof:** Suppose $\beta^T x \leq \beta_0$ is facet defining for $\text{STAB}(G)$ and contains all the roots of (1): we show below that such inequality is equivalent to (1).

We start with the following three observations.

i) Let $x^{S_1}$ be a root of $(\pi, \pi_0)$ such that $v_2 \in S_1$.

Consider the sets:

$$S_1^1 = S_1 \setminus \{v_2\} \cup \{h_1, d_2\}$$
$$S_1^2 = S_1 \setminus \{v_2\} \cup \{h_1, b_2\}.$$  

They are stable sets of $G$ and their incidence vectors $x^{S_1^1}$ and $x^{S_1^2}$ are roots of (1); consequently, they are roots of $(\beta, \beta_0)$. As $\beta(S_1^1) = \beta(S_1^2) = \beta_0$, we have that $\beta_{h_1} = \beta_{b_2}$. Symmetrically, we prove that $\beta_{h_1} = \beta_{d_1}$.

ii) Let $x^{S_2}$ be a root of $(\pi, \pi_0)$ such that $v_1, v_2 \notin S_2$.

The existence of such a root is guaranteed by the fact that $(\pi, \pi_0)$ is not the clique inequality $x_{v_1} + x_{v_2} \leq 1$.

Consider now the sets

$$S_2^1 = S_2 \cup \{h_1\}$$
$$S_2^2 = S_2 \cup \{a, c\}.$$  

They are stable sets of $G$ and their incidence vectors $x^{S_2^1}$ and $x^{S_2^2}$ are roots of (1) and, hence, of $(\beta, \beta_0)$. This implies that $\beta_a + \beta_d = \beta_0$. Replacing $S_2^1$ with $S_2 \cup \{h_2\}$, we get $\beta_a + \beta_d = \beta_{b_2}$ and then $\beta_{h_1} = \beta_{b_2}$.

iii) Let $x^{S'}$ be a root of $(\pi, \pi_0)$ such that $(K_2 \cup \{v_2\}) \cap S' = \emptyset$. This root always exists because $(\pi, \pi_0)$ is not the clique inequality defined by $K_2 \cup \{v_2\}$ (since by hypothesis $\pi_{v_1} = \pi_{v_2} = \lambda > 0$). Then $v_1 \in S'$, since otherwise $S' \cup \{v_2\}$ would be a stable set violating $(\pi, \pi_0)$. Let $S_3 = S' \setminus \{v_1\}$: we have that $\pi(S_3) = \pi_0 - \lambda$, as $(\pi, \pi_0)$ is g-extendible. Finally, consider the following stable sets whose incidence vectors are roots of (1):

$$S_3^1 = S_3 \cup \{d_1, d_2, c\}$$
$$S_3^2 = S_3 \cup \{b_1, b_2, a\}$$
$$S_3^3 = S_3 \cup \{b_2, h_1\}.$$  

From $\beta(S_3^1) = \beta(S_3^2)$ and (i) it follows that $\beta_a = \beta_c$ and so, by (ii), $\beta_{h_1} = 2\beta_a$. From $\beta(S_3^3) = \beta(S_3^2)$ it follows that $\beta_{b_1} + \beta_{b_2} = \beta_{h_1}$, that is $\beta_{b_1} = \beta_{b_2}$. Replacing $S_3^3$ with $S_3 \cup \{h_2\}$, we get $\beta_{b_2} = \beta_{b_3}$.

So, by (i)-(iii), we have that $\beta_v = \beta_{d_1}$ for each $v \in V_B \setminus \{h_1, h_2\}$ and $\beta_{h_1} = \beta_{b_3} = 2\beta_{d_1}$.

Let $M$ be a matrix whose rows are $|V_H|$ incidence vectors of stable sets of $H$ which are linearly independent roots of $(\pi, \pi_0)$, i.e.,

$$M \pi = \pi_0 1.$$  

Any stable set $S$ of $H$ can be transformed into a stable set $S$ of $G$ as follows: let $S = S \setminus \{v_1, v_2\} \cup S_B$, where $S_B$ is a stable set of $B$ such that $d_i \in S_B$ if
and only if \( v_i \in \tilde{S} \) for \( i = 1, 2 \). It is not difficult to verify that if \( x^S \) defines a root of \( (\pi, \pi_0) \) then \( S_B \) can be chosen so that \( x^S \) defines a root of (1) such that \( \beta(S \cap \{h_1, h_2, a, c\}) = 2\beta_{d_1} \). By replacing \( V_H \) with \( V' = V_H \setminus \{v_1, v_2\} \cup \{d_1, d_2\} \), we have \( M^{\beta_{V'}} = (\beta_0 - 2\beta_{d_1})I \) and by (2),

\[
\beta_{V'} = (\beta_0 - 2\beta_{d_1})M^{-1}I = \beta_0 - 2\beta_{d_1}\pi_0.
\]

In particular, we have

\[
\beta_{d_1} = \frac{\beta_0 - 2\beta_{d_1} \pi_{v_1}}{\pi_0} = \frac{\beta_0 - 2\beta_{d_1} \lambda}{\pi_0}.
\]

Then \( \beta_{d_1} > 0 \) and, without loss of generality, we can fix \( \beta_{d_1} = \pi_{v_1} = \lambda \); consequently, we have that

\[
\beta_0 = \pi_0 + 2\lambda,
\]

\[
\beta_v = \pi_v \quad \text{for each} \ v \in V_H \setminus \{v_1, v_2\},
\]

\[
\beta_v = \lambda \quad \text{for each} \ v \in V_B \setminus \{h_1, h_2\},
\]

\[
\beta_{h_1} = \beta_{h_2} = 2\lambda,
\]

and the theorem follows. \( \square \)

The following example shows a geared graph obtained by two applications of the gear composition to a 5-hole and the corresponding geared inequalities.

**Example 1** Consider the 5-hole \( C_5 = (v_1, v_2, u, v_2', v_1') \) and the geared 5-hole \( H_1 = (C_5, B^1, v_1'v_2') \) in Fig. 3. Two simplicial edges are emphasized as thick lines.

As the 5-hole inequality \( x(V_{C_5}) \leq 2 \) is facet defining for \( STAB(C_5) \) and \( g \)-extendable, we have, by Theorem 1, that

\[
x(V_{H_1} \setminus \{h_1', h_2'\}) + 2x_{h_1} + 2x_{h_2} \leq 4
\]

is facet defining for \( STAB(H_1) \).

Observe that the gear composition can be applied iteratively provided that the graph involved in the operation at the \( i \)-th step has a simplicial edge. For instance, the graph \( H_1 \) in the Example 1 contains \( v_1'v_2' \) and thus it can be composed with another gear \( B^2 \) to obtain the graph \( G = (H_1, B^2, v_1'v_2') \) shown in Fig. 4. A further application of Theorem 1 yields the following “double” geared facet defining inequality

\[
x(V_G \setminus T) + 2x(T) \leq 6,
\]

where \( T = \{h_1', h_2', h_1, h_2\} \).

**Figure 4. A double geared graph**

Notice that other geared inequalities appear in the linear description of \( STAB(G) \). In fact, the following inequalities:

\[
x(V_{H_1} \setminus A) \leq 3
\]

where \( A \in \{\{d_1', a_1\}, \{d_2', a_2\}, \{b_1, c_1\}, \{b_1', c_1\}, \{a_1', c_1\}\} \), are rank facet defining for \( STAB(H_1) \) and they are also \( g \)-extendable with respect to \( v_1'v_2' \).

Hence, by Theorem 1, each of the inequalities (6) generates a geared inequality which is facet defining for \( STAB(G) \) and different from (5).

Simmetrically, other geared inequalities are generated by performing gear compositions in a different order: first build \( H_2 = (C_5, B^2, v_1'v_2') \), and then \( G = (H_2, B^1, v_1'v_2') \) as the gear composition of \( H_2 \) and \( B^1 \). As above, the first gear composition generates five rank inequalities (similar to (6)) which are facet defining for \( STAB(H_2) \) and \( g \)-extendable while the second gear composition generates their associated geared inequalities. All the inequalities mentioned so far are different and, by Theorem 1, they are all facet defining for \( STAB(G) \). It follows that two applications of the gear composition to a 5-hole have produced 11 geared inequalities which are facet defining for the stable set polytope of \( G \). \( \square \)
The situation illustrated above may be generalized to the case when $G$ contains $k$ gears: in this case, any subset of gears may be possibly involved in a facet defining inequality, thus producing an exponential number of geared inequalities. To see this we need a preliminary result:

**Theorem 2** Let $(\pi, \pi_0)$ be a $g$-extendable inequality. If $(\pi, \pi_0)$ is facet defining for $STAB(H)$, then the inequality

$$\sum_{i \in V_G \setminus \{v_1, v_2\}} \pi_i x_i + \pi_{v_1} \sum_{i \in V_G \setminus \{a, c\}} x_i \leq \pi_0 + \pi_{v_1}$$

(7)

is facet defining for $STAB(G')$.

**Proof:** Consider the graph $G'$ obtained from $H$ by subdividing the edge $e = v_1v_2$ with two nodes $h_1$ and $h_2$ and renaming $v_i$ as $d_i$, $i = 1, 2$. Clearly $G'$ is a subgraph of $G$ and, by a result of Wolsey [21] on edge subdivisions, the following inequality

$$\sum_{i \in V_G \setminus \{v_1, v_2\}} \pi_i x_i + \pi_{v_1} \sum_{i \in \{d_1, h_1, h_2, d_2\}} x_i \leq \pi_0 + \pi_{v_1}$$

is facet defining for $STAB(G')$. This inequality can be lifted to yield a facet defining inequality of $STAB(G)$ by observing that $b_1$ and $b_2$ can be lifted with coefficient $\pi_{v_1}$, and then $a$ and $c$ can be lifted with coefficient zero. This completes the proof. \(\Box\)

We now show an example where a linear number of gear compositions yields an exponential number of facet defining geared inequalities. Consider the graph $H$ as a $(2k + 1)$-hole $(v_1, v_2, \ldots, v_{2k+1})$ and the following set $F = \{e_i = v_{2i+1} : i = 1, \ldots, k\}$ of disjoint simplicial edges of $H$. Let $F'$ be such that $i_1 < i_2 < \cdots < i_k < k$ and let $G_{F'}$ denote the graph obtained from $H$ by iteratively applying the gear composition on the edge $e_i$, for $j = 1, 2, \ldots, h$ (notice that, since the edges in $F$ are disjoint, the edges in $F \setminus F'$ remain simplicial in $G_{F'}$). Denote by $T$ the set of hubs’ pairs belonging to the $h$ gears of $G_{F'}$. As $|V_H| \leq k$ is facet defining for $STAB(H)$ and $g$-extendable with respect to each edge of $F$, by iteratively applying Theorem 1, we have that the geared inequality

$$\sum_{v \in V_{G_{F'}} \setminus T} x_v + 2 \sum_{v \in F} x_v \leq k + 2h$$

is facet defining for $STAB(G_{F'})$. Moreover, this inequality may be extended to a facet defining inequality for $STAB(G_F)$ by applying Theorem 2 to the $k - h$ edges of $F \setminus F'$. Since this procedure can be applied to any subset $F'$ of $F$, we have that an exponential number of geared inequalities appear in the linear description of $STAB(G_F)$.

### 3. Geared rank inequalities

In this section we show how to use the gear composition to build a new class of inequalities that naturally extend the inequalities supported by the line graph of hypomatchable graphs [5].

It is well-known that the stable set polytope $STAB(L(G))$ of a line graph $L(G)$ is equivalent to the matching polytope $M(G)$ of $G$. Since the only nontrivial inequalities describing $M(G)$ are rank inequalities [4], we have that the same holds for $STAB(L(G))$.

However these inequalities are not sufficient to describe $STAB(G)$ as long as $G$ is not a line graph and the structure of $STAB(G)$ becomes quite complex even for those graphs that are natural generalizations of line graphs as the claw-free graphs, i.e., graphs such that the neighborhood of each node has no stable set of size three. For claw-free graphs, as for the line graphs, the optimization problem over the stable set polytope is polynomial time solvable [14] and, by a well-known result of Grötschel, Lovász and Schrijver (see [11]), the same holds for the separation problem. Thus, it is expected that $STAB(G)$ has a nice linear description when $G$ is claw-free. But up to now no explicit set of facet defining inequalities is known despite many research efforts [8,10,12,16,18] and several disproved conjectures [10].

A complete linear description of $STAB(G)$ was given by Eisenbrand et al. [6] for a subclass of claw-free graphs: the quasi-line graphs (a graph is quasi-line if the neighborhood of each node can be partitioned into two cliques). These graphs generalize the line graphs and their stable set polytope is completely described by the clique family inequalities [16] which are a generalization of the Edmonds’ inequalities [2].

It remains open the problem of finding the linear description of $STAB(G)$ when $G$ is claw-free and not quasi-line. It is well-known [7] that any claw-free graph $G$ which is not quasi-line and has $\alpha(G) \geq 4$, contains at least one 5-wheel and no odd antihole $C_{2p+1}$ with $p \geq 3$. Recently, Stauffer [19] conjectured that:
Conjecture 1 The stable set polytope of a claw-free graph $G$ which is not quasi-line and has $\alpha(G) \geq 4$ is described by: non-negativity inequalities, rank inequalities and (lifted) 5-wheel inequalities.

To build counterexamples to the above conjecture it suffices to observe that each geared inequality is

- supported by a graph $G$ that is not quasi-line (since it contains a 5-wheel) and moreover, is a
- non-rank valid inequality for $STAB(G)$ with rhs greater than 2.

Thus, any geared inequality that is facet defining for $STAB(G)$, when $G$ is claw-free with $\alpha(G) \geq 4$, is a counterexample to Conjecture 1 because $G$ is not quasi-line and the rhs of the geared inequality is greater than the rhs of a (lifted) 5-wheel inequality which is 2. Instances of such inequalities are provided in Example 1.

We define recursively the family $G_R$ of geared rank inequalities as the family of geared inequalities associated with inequalities that: either are rank inequalities or belong to $G_R$. By repeated applications of Definition 2, we have that the coefficients of geared rank inequalities are all 1’s apart from some pairs of gears’ hubs which have coefficient 2; moreover, their rhs is greater than 2.

Geared rank inequalities play a role in the study of $STAB(G)$ when $G$ is claw-free. The results of this paper imply that the geared rank inequalities have to be necessarily added to the defining linear system of $STAB(G)$. Moreover, the recent decomposition theorem for claw-free graphs of Chudnovsky and Seymour [1] made us quite confident that geared rank inequalities are also sufficient to give a linear description of $STAB(G)$ when $G$ is claw-free, not quasi-line and has stability number greater than 3. This led us to conjecture that

Conjecture 2 The stable set polytope of a claw-free graph $G$ which is not quasi-line and has $\alpha(G) \geq 4$ is described by:

- non-negativity inequalities
- rank inequalities
- (lifted) 5-wheel inequalities
- (lifted) geared rank inequalities.

References