Ottimizzazione dei Sistemi Complessi

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Ordinamento di Pareto

Dati due vettori z^1 e z^2 in $\Re^k$ diciamo che:

z^1 *domina* (secondo Pareto) z^2 ($z^1 \leq_P z^2$) se

$z_i^1 \leq z_i^2$ per **ogni** $i = 1, \dots, k$, e

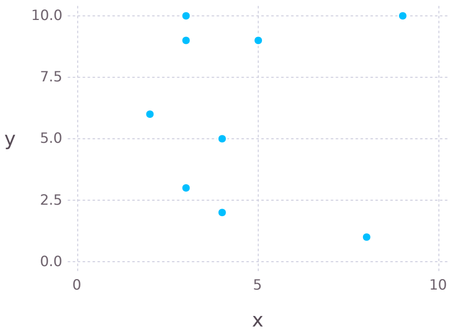
$z_j^1 < z_j^2$ per **qualche** $j \in \{1, \dots, k\}$.

Esempio (1)

Determinare l'insieme dei punti **non dominati** tra i seguenti

	a	b	c	d	e	f	g	h	i	j
x	4	9	4	3	2	3	5	8	3	5
y	2	10	5	10	6	3	9	1	9	9

Esempio (1)



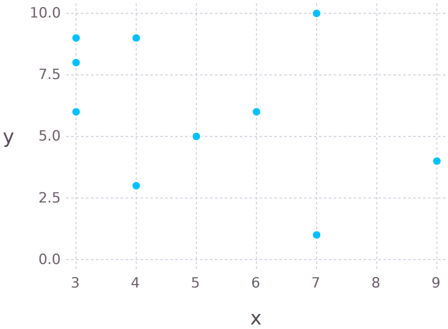
	a	e	f	h
x	4	2	3	8
y	2	6	3	1

Esempio (2)

Determinare l'insieme dei punti **non dominati** tra i seguenti

	a	b	c	d	e	f	g	h	i	j
x	3	4	4	5	7	6	3	7	3	9
y	6	9	3	5	1	6	8	10	9	4

Esempio (2)



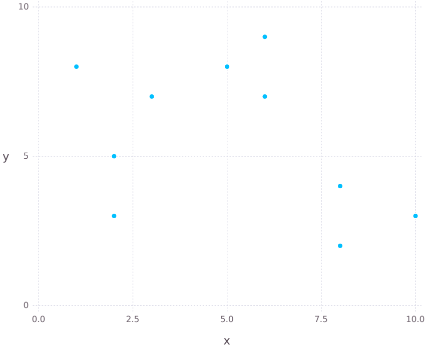
	a	c	e
x	3	4	7
y	6	3	1

Esempio (3)

Determinare l'insieme dei punti **non dominati** tra i seguenti

	a	b	c	d	e	f	g	h	i	j
x	1	3	5	8	10	6	2	2	8	6
y	8	7	8	2	3	7	3	5	4	9

Esempio (3)



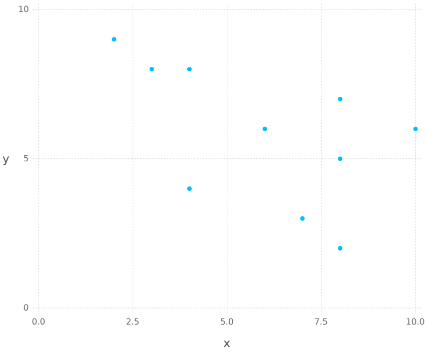
	a	d	g
x	1	8	2
y	8	2	3

Esempio (4)

Determinare l'insieme dei punti **non dominati** tra i seguenti

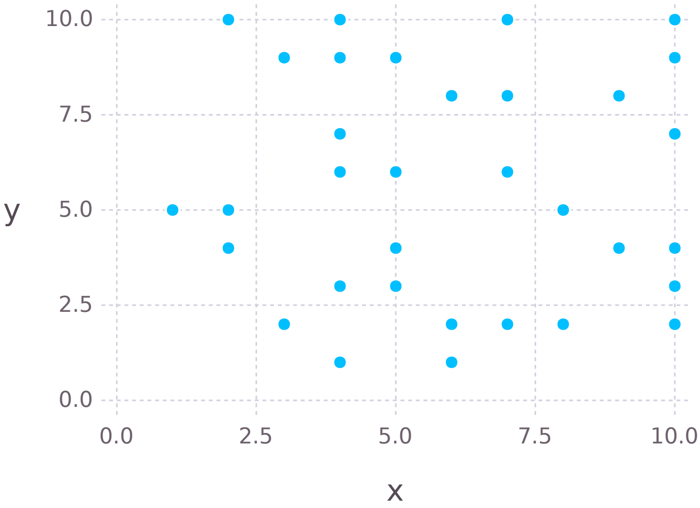
	a	b	c	d	e	f	g	h	i	j
x	4	3	8	8	6	2	4	7	10	8
y	8	8	5	2	6	9	4	3	6	7

Esempio (4)

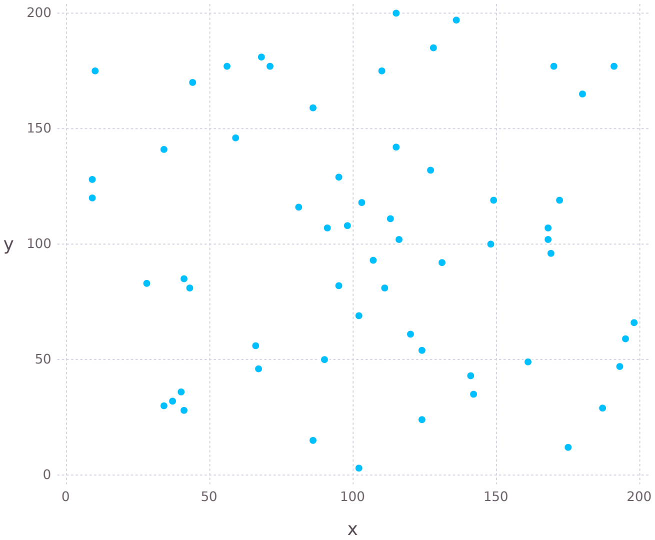


	b	d	f	g	h
x	3	8	2	4	7
y	8	2	9	4	3

Esempio (5)



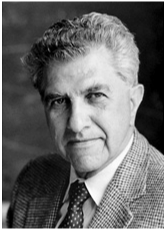
Esempio (6)



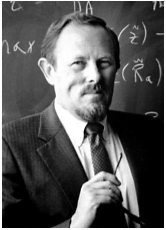
Introduction

Nel 1990 il premio in memoria di A. Nobel viene conferito congiuntamente a:

Harry M. Markowitz



Merton H. Miller



William F. Sharpe

The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 1990
Harry M. Markowitz, Merton H. Miller, William F. Sharpe

KUNGL. VETENSKAPSAKADEMIEN
THE ROYAL SWEDISH ACADEMY OF SCIENCES

Press Release

Portfolio selection

PORTFOLIO SELECTION*

HARRY MARKOWITZ
The Rand Corporation

THE PROCESS OF SELECTING a portfolio may be divided into two stages. The first stage starts with observation and experience and ends with beliefs about the future performances of available securities. The second stage starts with the relevant beliefs about future performances and ends with the choice of portfolio. This paper is concerned with the second stage. We first consider the rule that the investor does (or should) maximize discounted expected, or anticipated, returns. This rule is rejected both as a hypothesis to explain, and as a maximum to guide investment behavior. We next consider the rule that the investor does (or should) consider expected return a desirable thing *and* variance of return an undesirable thing. This rule has many sound points, both as a maxim for, and hypothesis about, investment behavior. We illustrate geometrically relations between beliefs and choice of portfolio according to the "expected returns—variance of returns" rule.

H.M. Markowitz, "Portfolio selection", The Journal of Finance, Vol. 7, No. 1. (Mar., 1952), pp. 77-91.

Motivazione del premio

"Before the 1950s, there was hardly any theory whatsoever of financial markets. A first pioneering contribution in the field was made by **Harry Markowitz**, who **developed a theory of portfolio decisions of households and firms under conditions of uncertainty**. The theory shows how the multidimensional problem of investing under conditions of uncertainty in a large number of assets, each with different characteristics, may be reduced to the issue of a trade-off between only two dimensions, namely the expected return and the variance of the return of the portfolio." "Professors Markowitz, Miller and Sharpe,

You have by your research established the foundation for the field *Financial Economics and Corporate Finance*. The impressive development of this field of research in economics in recent years is largely based on your achievements. It is a pleasure to convey to you the warmest congratulations from the Royal Academy of Sciences and to ask you to receive from the hands of His Majesty

Calcolo delle Probabilità – richiami

Indichiamo con **x** una variabile aleatoria (v.a.) a valori reali.

Conoscendo la funzione $p(\mathbf{x})$ di densità di probabilità della v.a. **x**, possiamo calcolare:

- il momento del primo ordine (valore atteso) di **x**:

$$E[\mathbf{x}] = \bar{\mu}_{\mathbf{x}} = \int_{-\infty}^{+\infty} x p(x) dx;$$

- il momento del secondo ordine (varianza) di **x**:

$$E[(\mathbf{x} - \bar{\mu}_{\mathbf{x}})^2] = \bar{\sigma}_{\mathbf{x}}^2 = \int_{-\infty}^{+\infty} (x - \bar{\mu}_{\mathbf{x}})^2 p(x) dx;$$

- ...

Combinazione lineare di due v.a.

Indichiamo con $\mathbf{x}$ e $\mathbf{y}$ due v.a. a valori reali.

Vogliamo calcolare valore atteso e varianza della v.a.

$$\mathbf{z} = \alpha\mathbf{x} + \beta\mathbf{y}$$

- Valore atteso $E[\mathbf{z}]$:

$$E[\mathbf{z}] = \alpha E[\mathbf{x}] + \beta E[\mathbf{y}];$$

- Varianza di $\mathbf{z}$:

$$\begin{aligned} E[(\mathbf{z} - E[\mathbf{z}])^2] &= E[(\alpha\mathbf{x} + \beta\mathbf{y} - \alpha E[\mathbf{x}] - \beta E[\mathbf{y}])^2] \\ &= E[(\alpha(\mathbf{x} - E[\mathbf{x}]) + \beta(\mathbf{y} - E[\mathbf{y}]))^2] \end{aligned}$$

Combinazione lineare di due v.a.

$$
\begin{aligned}
E[(\mathbf{z} - E[\mathbf{z}])^2] &= E[\alpha^2(\mathbf{x} - E[\mathbf{x}])^2 + \beta^2(\mathbf{y} - E[\mathbf{y}])^2 + \\
&\quad 2\alpha\beta(\mathbf{x} - E[\mathbf{x}])(\mathbf{y} - E[\mathbf{y}])] \\
&= \alpha^2 E[(\mathbf{x} - E[\mathbf{x}])^2] + \beta^2 E[(\mathbf{y} - E[\mathbf{y}])^2] + \\
&\quad 2\alpha\beta E[(\mathbf{x} - E[\mathbf{x}])(\mathbf{y} - E[\mathbf{y}])]
\end{aligned}
$$

$$
\mathrm{cov}[\mathbf{x}, \mathbf{y}] = E[(\mathbf{x} - E[\mathbf{x}])(\mathbf{y} - E[\mathbf{y}])]
$$

$$
E[(\mathbf{z} - E[\mathbf{z}])^2] = \bar{\sigma}_{\mathbf{z}}^2 = \alpha^2 \bar{\sigma}_{\mathbf{x}}^2 + \beta^2 \bar{\sigma}_{\mathbf{y}}^2 + 2\alpha\beta \mathrm{cov}[\mathbf{x}, \mathbf{y}]
$$

Combinazione lineare di v.a.

Consideriamo ora n v.a. $\mathbf{x_1}, \dots, \mathbf{x_n}$ e n scalari $\alpha_1, \dots, \alpha_n$.

Sia $\mathbf{z} = \sum_{i=1}^n \alpha_i \mathbf{x_i}$.

Risulta:

- Valore atteso:

$$E[\mathbf{z}] = \sum_{i=1}^n \alpha_i E[\mathbf{x_i}];$$

- Varianza:

$$\begin{aligned} E[(\mathbf{z} - E[\mathbf{z}])^2] &= \sum_{i=1}^n \alpha_i^2 \bar{\sigma}_{\mathbf{x_i}}^2 + 2 \sum_{i=1}^n \sum_{j=i+1}^n \alpha_i \alpha_j \mathrm{cov}[\mathbf{x_i}, \mathbf{x_j}] \\ &= \sum_{i=1}^n \sum_{j=i}^n \alpha_i \alpha_j \mathrm{cov}[\mathbf{x_i}, \mathbf{x_j}] \end{aligned}$$

Se indichiamo con $\mathrm{cov}[\mathbf{x}, \mathbf{x}] = \bar{\sigma}_{\mathbf{x}}^2$ e con $Q = (q_{ij})$ la *matrice di covarianza* dove

Portfolio selection

Markowitz osservò che il comportamento tipico di un investitore finanziario è quello di *diversificare* il proprio portafoglio di titoli.

"The hypothesis that the investor does maximize discounted return must be rejected [...] the foregoing rule never implies that there is a diversified portfolio which is preferable to all non-diversified portfolios"

"There is a rule which implies both that the investor should diversify and that he should maximize expected return. [...] This rule is a special case of the expected returns-variance of returns rule" Consideriamo n titoli sul mercato finanziario e sia R_i la v.a. che rappresenta il rendimento dell' i-esimo titolo.
Sia μ_i il rendimento atteso di R_i e σ_{ij} la covarianza tra R_i e R_j (quindi $\sigma_{ii} = \sigma_i^2$ è la varianza di R_i).

Indichiamo con x_i la frazione di capitale investita nel titolo i-esimo e supponiamo di voler investire tutto il capitale, per cui è

Portfolio selection

Quindi, come abbiamo visto in precedenza:

$$\mu_R(x) = \sum_{i=1}^{n} x_i \mu_{R_i},$$

$$\sigma_R^2(x) = x^T Q x,$$

dove, $q_{ij} = \sigma_{ij}$.

Portfolio selection - formulazione multiobiettivo

Markowitz dimostrò che il comportamento osservato di un investitore poteva essere spiegato considerando il seguente problema *multiobiettivo*

$$\max \mu_R(x), \min \sigma_R^2(x)$$

$$s.t. \sum_{i=1}^{n} x_i = 1$$

$$x_i \geq 0, \;\; i = 1, \dots, n.$$

in cui si vuole *contemporaneamente* massimizzare il rendimento atteso e minimizzare la varianza. Per questo, il modello è anche noto con il nome di "*mean-variance*"