Ottimizzazione dei Sistemi Complessi

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Soli vincoli di uguaglianza

Per il problema

$$\begin{array}{l} \min\ f(x) \\ s.t.\ h_j(x) = 0, \quad j = 1, \ldots, p \end{array}$$

abbiamo introdotto la funzione Lagrangiana “aumentata”

$$L_a(x, \mu; \epsilon) = L(x, \mu) + \frac{1}{\epsilon}\|h(x)\|^2$$

con gradiente

$$\begin{aligned} \nabla_x L_a(x, \mu; \epsilon) &= \nabla f(x) + \nabla h(x)\left(\mu + \frac{2}{\epsilon}h(x)\right) \\ \nabla_\mu L_a(x, \mu; \epsilon) &= \nabla_\mu L(x, \mu) = h(x) \end{aligned}$$

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Metodo di soluzione - Metodo dei Moltiplicatori

Algoritmo `SEQLAGR`

Dati: $\epsilon_\circ > 0$, $\beta > 1$, $\{\tau_k\} \to 0$, $\rho > 0$, $\mu_\circ$, `maxit`

for $k = 0, 1, \ldots,$`maxit`

Calcola x_k t.c. $\|\nabla_x L_a(x_k, \mu_k; \epsilon_k)\| \leq \tau_k$

if $\|\nabla L_a(x_k, \mu_k; \epsilon_k)\| < \rho$ **then**

$x^* \leftarrow x_k$, $\mu^* \leftarrow \mu_k$, STOP

endif

$\epsilon_{k+1} = \epsilon_k / \beta$

$$\boxed{\mu_{k+1} = \mu_k + \frac{2h(x_k)}{\epsilon_k}}$$

endfor

Return: miglior coppia trovata (x^*, μ^*)

Problemi con soli vincoli di disuguaglianza – 1

Consideriamo il problema

$$\begin{array}{ll} \min & f(x) \\ c.v. & g(x) \leq 0 \end{array}$$

Usiamo la trasformazione:

- $g_i(x) \leq 0 \quad \rightsquigarrow \quad g_i(x) + y_i^2 = 0,\ i = 1, \dots, m$

$$L_a(x, y, \lambda; \epsilon) = f(x) + \sum_{i=1}^{m} \lambda_i (g_i(x) + y_i^2) + \frac{1}{\epsilon} \sum_{i=1}^{m} (g_i(x) + y_i^2)^2$$

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$$= f(x) + \lambda^\top g(x) + \frac{1}{\epsilon} \|g(x)\|^2 + \frac{1}{\epsilon} \sum_{i=1}^{m} \left(\epsilon \lambda_i y_i^2 + 2 g_i(x) y_i^2 + y_i^4\right)$$

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Problemi con soli vincoli di disuguaglianza – 2

Quindi

$$\begin{array}{ll} \min & f(x) \\ c.v. & g(x) \leq 0 \end{array} \quad \text{ovvero} \quad \begin{array}{ll} \min & f(x) \\ c.v. & g_i(x) + y_i^2 = 0, \ \ i = 1, \dots, m \end{array}$$

abbiamo

$$L_a(x, y, \lambda; \epsilon) = f(x) + \lambda^\top g(x) + \frac{1}{\epsilon}\|g(x)\|^2 + \frac{1}{\epsilon}\sum_{i=1}^{m}(\epsilon\lambda_i y_i^2 + 2g_i(x)y_i^2 + y_i^4)$$

fissati ϵ e λ, bisogna "risolvere"

$$\min_{x,y} L_a(x, y, \lambda; \epsilon)$$

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$$\begin{array}{ll} \min & f(x) \\ c.v. & g(x) \le 0 \end{array} \qquad \text{ovvero} \qquad \begin{array}{ll} \min & f(x) \\ c.v. & g_i(x) + y_i^2 = 0, \; i = 1, \dots, m \end{array}$$

abbiamo

$$L_a(x, y, \lambda; \epsilon) = f(x) + \lambda^\top g(x) + \frac{1}{\epsilon}\|g(x)\|^2 + \frac{1}{\epsilon}\sum_{i=1}^{m}(\epsilon\lambda_i y_i^2 + 2g_i(x)y_i^2 + y_i^4)$$

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fissati ϵ e λ, bisogna "risolvere"

$$\min_{x,y} L_a(x, y, \lambda; \epsilon)$$

Gradiente della Lagrangiana

$$
\begin{aligned}
\nabla_x L_a &= \nabla f(x) + \nabla g(x)\lambda + \frac{2}{\epsilon}\sum_{i=1}^{m}(g_i(x)+y_i^2)\nabla g_i(x) \\
\nabla_\lambda L_a &= g(x) + y^2 \\
\nabla_{y_i} L_a &= \frac{4}{\epsilon} y_i \left(\epsilon\frac{\lambda_i}{2} + g_i(x) + y_i^2\right)
\end{aligned}
$$

Quindi, ponendo $\nabla_{y_i} L_a = 0$ otteniamo

$$
y_i \left(\epsilon\frac{\lambda_i}{2} + g_i(x) + y_i^2\right) = 0
$$

che è verificata per

$$
y_i^2 = 0, \text{ oppure } y_i^2 = \max\left\{0, -\epsilon\frac{\lambda_i}{2} - g_i(x)\right\}
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Gradiente della Lagrangiana – segue

Calcoliamo $\nabla^2_{y_i} L_a$

$$\nabla^2_{y_i} L_a = \frac{4}{\epsilon}\left(\epsilon\frac{\lambda_i}{2} + g_i(x) + 3y_i^2\right)$$

- Quando $y_i^2 = 0$, $\nabla^2_{y_i} L_a = \frac{4}{\epsilon}\left(\epsilon\frac{\lambda_i}{2} + g_i(x)\right)$
- Quando $y_i^2 = -\epsilon\frac{\lambda_i}{2} - g_i(x) > 0$, $\nabla^2_{y_i} L_a = \frac{8}{\epsilon}\left(-\epsilon\frac{\lambda_i}{2} - g_i(x)\right)$

Imponendo $\nabla^2_{y_i} L_a > 0$, segue che la soluzione ottima è proprio $y_i^2 = \max\left\{0, -\epsilon\frac{\lambda_i}{2} - g_i(x)\right\}$

Espressione per vincoli di disuguaglianza

$$
\begin{aligned}
L_a(x,\lambda;\epsilon) &= f(x) + \lambda^\top \left(g(x) + \max\left\{0, -\epsilon\frac{\lambda}{2} - g(x)\right\}\right) \\
&+ \frac{1}{\epsilon}\left\| g(x) + \max\left\{0, -\epsilon\frac{\lambda}{2} - g(x)\right\}\right\|^2
\end{aligned}
$$

Ovvero

$$
\begin{aligned}
L_a(x,\lambda;\epsilon) &= f(x) + \lambda^\top \max\left\{g(x), -\epsilon\frac{\lambda}{2}\right\} + \frac{1}{\epsilon}\left\|\max\left\{g(x), -\epsilon\frac{\lambda}{2}\right\}\right\|^2 \\
\nabla_x L_a(x,\lambda;\epsilon) &= \nabla f(x) + \nabla g(x)\left(\lambda + \frac{2}{\epsilon}\max\left\{g(x), -\epsilon\frac{\lambda}{2}\right\}\right) \\
\nabla_\lambda L_a(x,\lambda;\epsilon) &= \max\left\{g(x), -\epsilon\frac{\lambda}{2}\right\}
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Espressione per vincoli di disuguaglianza

$$\begin{aligned} L_a(x,\lambda;\epsilon) &= f(x) + \lambda^\top \left(g(x) + \max\left\{0, -\epsilon\frac{\lambda}{2} - g(x)\right\}\right) \\ &+ \frac{1}{\epsilon}\left\| g(x) + \max\left\{0, -\epsilon\frac{\lambda}{2} - g(x)\right\}\right\|^2 \end{aligned}$$

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$$\begin{aligned} L_a(x,\lambda;\epsilon) &= f(x) + \lambda^\top \max\left\{g(x), -\epsilon\frac{\lambda}{2}\right\} + \frac{1}{\epsilon}\left\|\max\left\{g(x), -\epsilon\frac{\lambda}{2}\right\}\right\|^2 \\ \nabla_x L_a(x,\lambda;\epsilon) &= \nabla f(x) + \nabla g(x)\left(\lambda + \frac{2}{\epsilon}\max\left\{g(x), -\epsilon\frac{\lambda}{2}\right\}\right) \\ \nabla_\lambda L_a(x,\lambda;\epsilon) &= \max\left\{g(x), -\epsilon\frac{\lambda}{2}\right\} \end{aligned}$$

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if $\|\nabla L_a(x_k, \mu_k, \lambda_k; \epsilon_k)\| < \rho$ **then**

$x^* \leftarrow x_k$, $\mu^* \leftarrow \mu_k$, $\lambda^* \leftarrow \lambda_k$ STOP

endif

$\epsilon_{k+1} = \epsilon_k / \beta$

$$\boxed{\mu_{k+1} = \mu_k + \frac{2h(x_k)}{\epsilon_k}, \ \lambda_{k+1} = \lambda_k + \frac{2\max\{g(x_k), -\epsilon_k \lambda_k/2\}}{\epsilon_k}}$$

endfor

Return: miglior coppia trovata (x^*, μ^*, λ^*)