Ottimizzazione dei Sistemi Complessi

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Giovedì 2 Maggio 2019

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Introduzione

We consider the problem

$$\min_x \quad f(x)$$
$$c.v. \quad g(x) \geq 0$$

- $\mathcal{F}$ feasible region
- $\mathring{\mathcal{F}} = Int(\mathcal{F}) = \{x \in \mathbb{R}^n : g_i(x) > 0\}$
- We assume $\mathring{\mathcal{F}} \neq \emptyset$

Log-barrier function

For the considered problem:

$$
\begin{aligned}
P(x;\mu) &= f(x) - \mu \sum_{i=1}^{m} \log g_i(x), \text{ dove} \\
b(x) &= -\sum_{i=1}^{m} \log g_i(x) \text{ (termine di barriera)} \\
\nabla_x P(x;\mu) &= \nabla f(x) - \sum_{i=1}^{m} \frac{\mu}{g_i(x)} \nabla g_i(x)
\end{aligned}
$$

N.B.

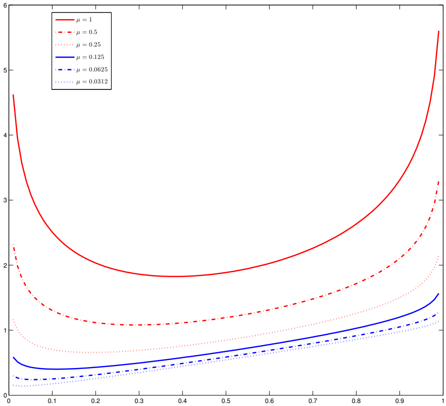
- $b(x)$ (and $P(x;\mu)$) is defined for all $x \in \mathring{\mathcal{F}}$
- We assume $b(x)$ (and $P(x;\mu)$) $= +\infty$ for all $x \notin \mathring{\mathcal{F}}$

Example 1

$$\min_x \quad x$$
$$c.v. \quad x \geq 0$$
$$1 - x \geq 0$$

The log-barrier function for this problem is

$$P(x;\mu) = x - \mu \log x - \mu \log(1-x)$$



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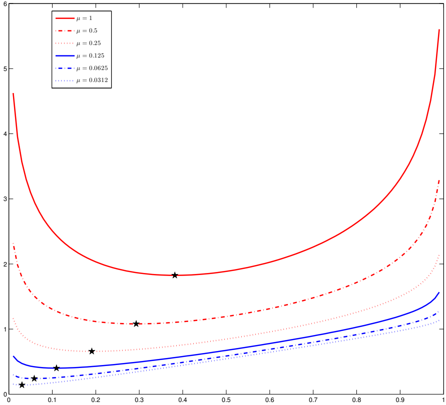

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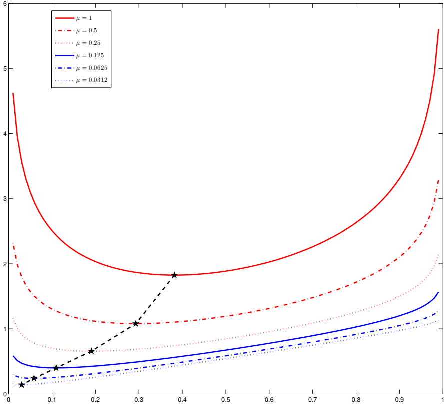


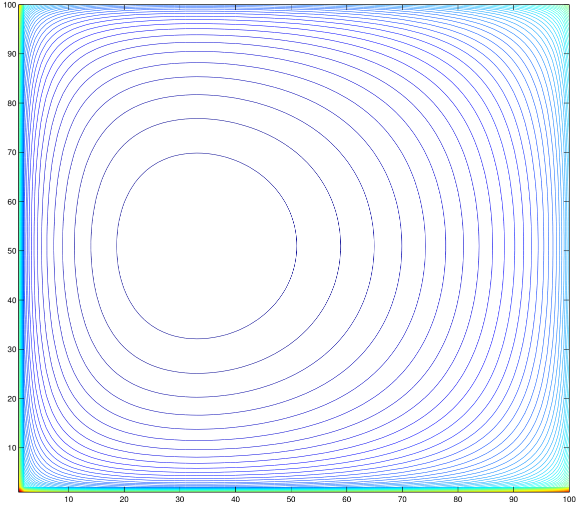
Example 2

$$\begin{array}{ll} \min\limits_{x} & (x+0.5)^2 + (x_2 - 0.5)^2 \\ c.v. & x_1 \in [0,1],\ x_2 \in [0,1] \end{array}$$

The log-barrier function for this problem is

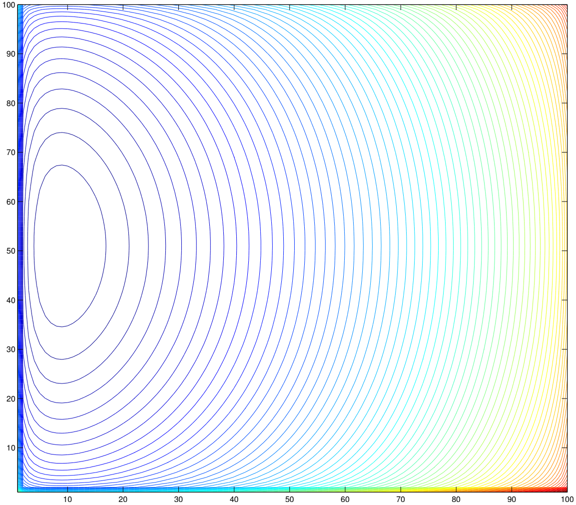
$$\begin{aligned} P(x;\mu) = (x+0.5)^2 + (x_2-0.5)^2 \\ -\mu[\log x_1 + \log(1-x_1)] \\ -\mu[\log x_2 + \log(1-x_2)] \end{aligned}$$

Example 2



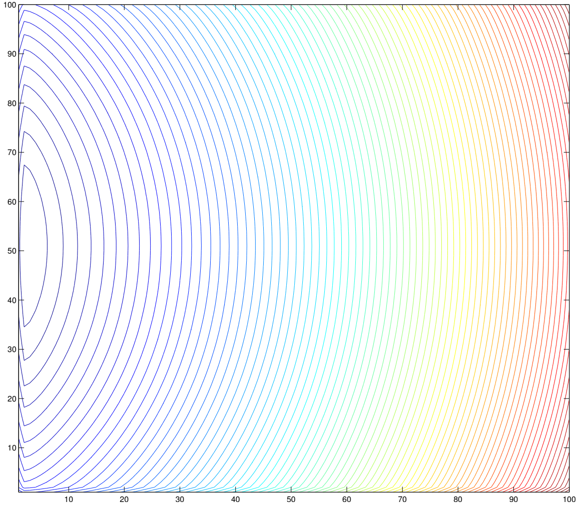
$\mu = 1$

Example 2



$\mu = 0.1$

Example 2



$\mu = 0.01$

Example 2

When $\mu = 1 \rightarrow 0.0085$ we have

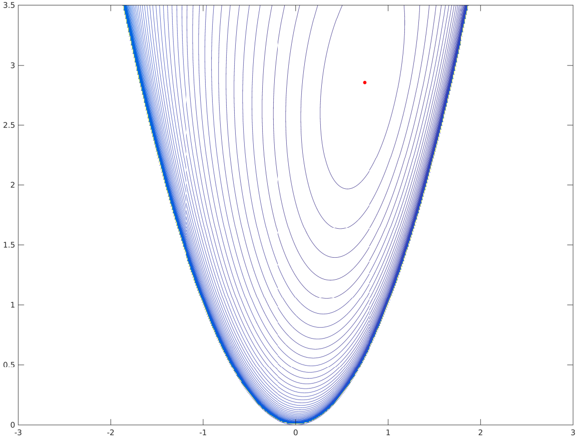
Example 3 (HS 11)

$$\min_{x} \quad (x-5)^2 + y^2$$
$$s.t. \quad -x^2 + y \geq 0$$

The log-barrier function for this problem is

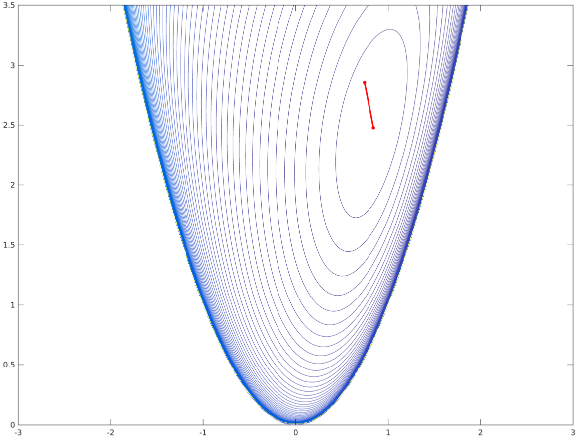
$$P(x;\mu) = (x-5)^2 + y^2 - \mu[\log(y - x^2)]$$

Example 3



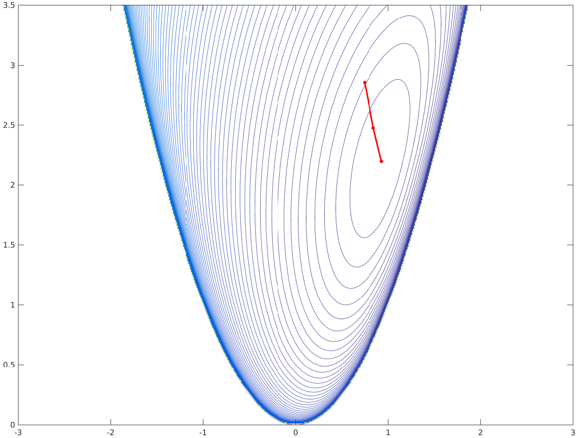
$\mu = \texttt{1.317e+1}$

Example 3



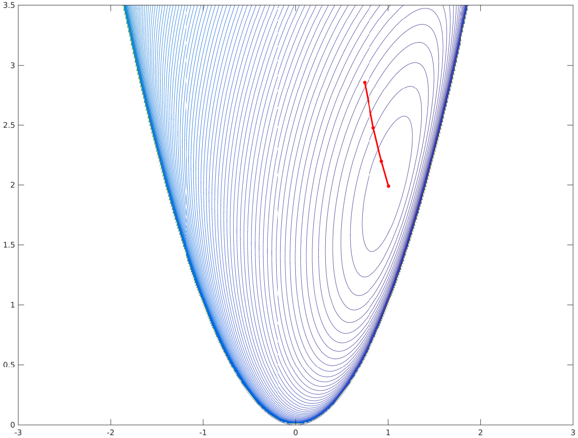
$\mu = 8.779$

Example 3



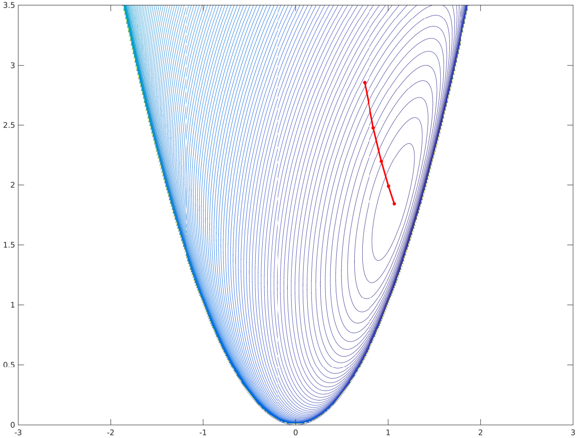
$\mu = 5.853$

Example 3



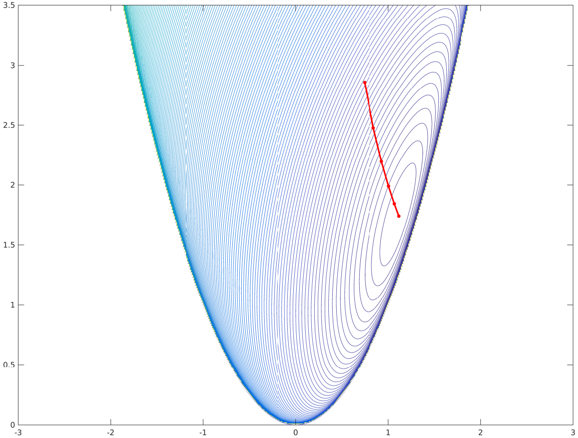
μ =3.902

Example 3



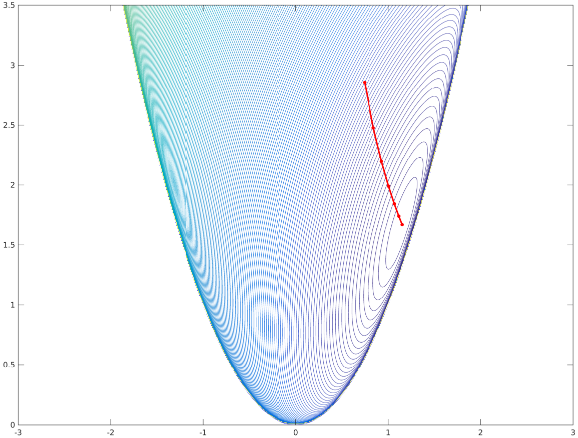
$\mu = 2.601$

Example 3



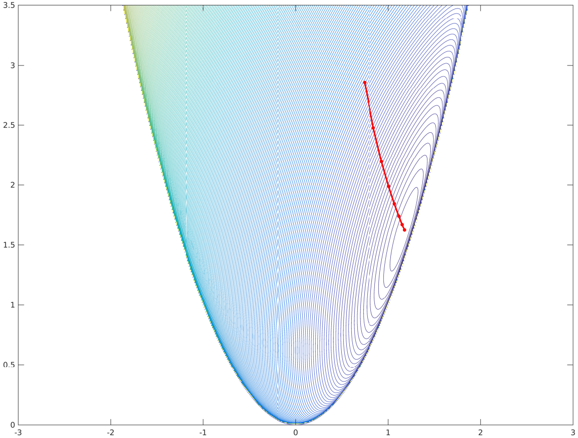
$\mu = 1.156$

Example 3



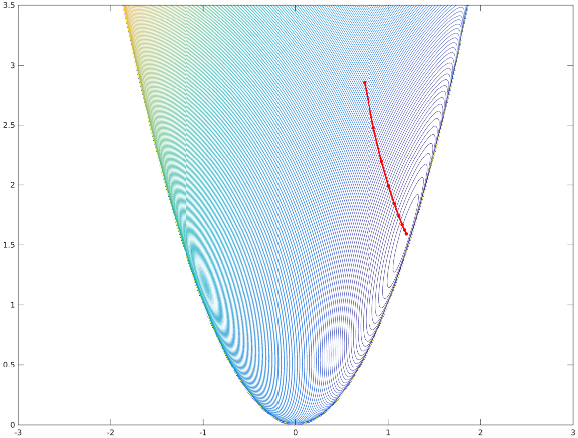
$\mu =$7.707e-1

Example 3



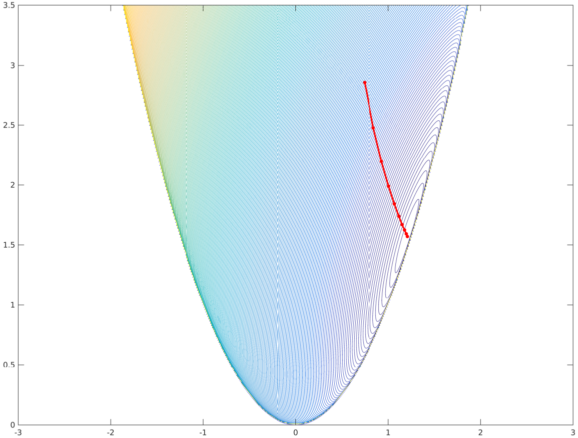
$\mu = 5.138\text{e-}1$

Example 3



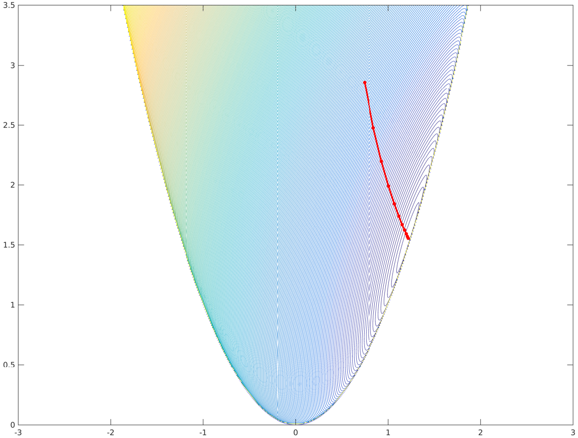
$\mu =$3.425e-1

Example 3



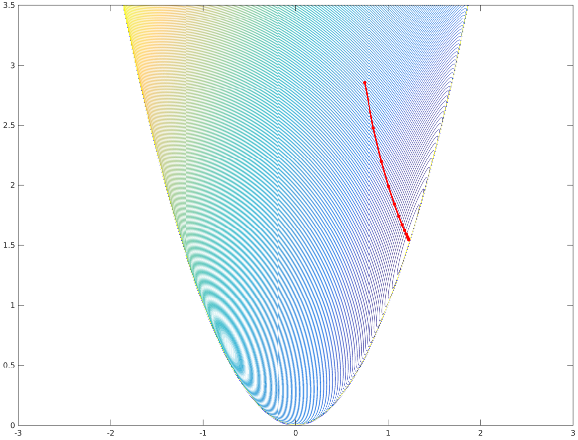
μ =2.284e-1

Example 3



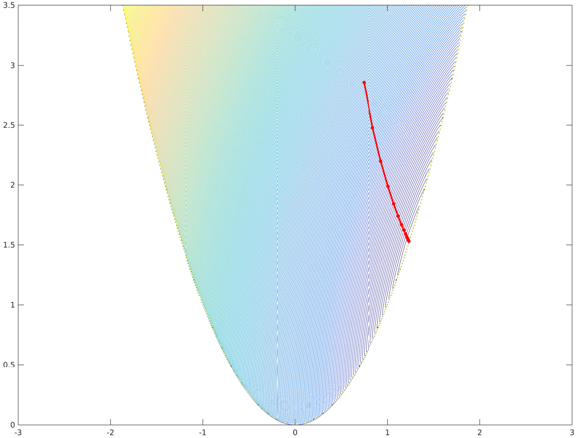
$\mu = 1.522\text{e-}1$

Example 3



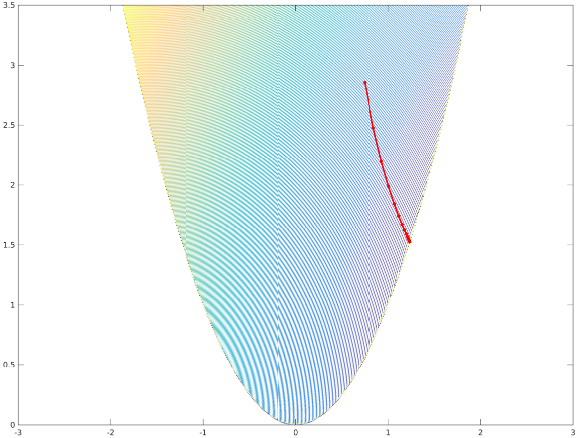
μ =1.015e-1

Example 3



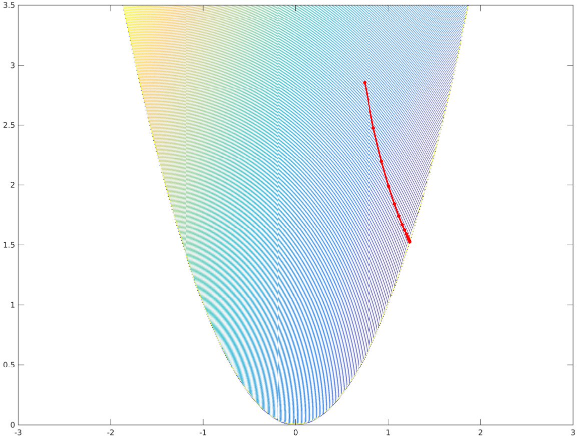
$\mu =$ 3.007e-2

Example 3



μ =8.910e-3 KKT error = 8.910e-3

Example 3



μ =8.910e-3 KKT error = 8.910e-3

KKT conditions – stationary points of $P(x;\mu)$

KKT point

Let (x,λ) be a KKT pair

$$\begin{aligned} \nabla f(x) - \lambda^\top \nabla g(x) &= 0 \\ g(x) &\geq 0 \\ \lambda &\geq 0 \\ \lambda^\top g(x) &= 0 \end{aligned}$$

Stationary point of $P(x;\mu)$

Let $x(\mu)$ be such that

$$\nabla f(x(\mu)) - \sum_{i=1}^{m} \lambda_i(\mu) \nabla g_i(x(\mu)) = 0$$

$$\lambda_i(\mu) = \frac{\mu}{g_i(x(\mu))}$$

The pair $(x(\mu),\lambda(\mu))$

- defines the so-called **central-path** or **interior-path**
- it is "almost" a KKT pair; complementary slackness $\lambda^\top g(x) = 0$ is violated
- but we have:

$$\lambda_i(\mu) g_i(x(\mu)) = \mu$$

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Theorically ...

Proposizione

Let $\mathring{\mathcal{F}} \neq \emptyset$, (x^*, λ^*) be a local solution for the constrained problem that satisfies LICQ[a], strict complementarity and SOSC[b]. Then, when μ is sufficiently small:

- $\nabla^2_{xx} P(x; \mu)$ *è definita positiva*
- *it exists a unique and smooth function $x(\mu)$ (local minimum of $P(x, \mu)$ in a neighborhood of x^*)*
- $\lim_{\mu \downarrow 0} x(\mu) = x^*$
- $\lim_{\mu \downarrow 0} \lambda(\mu) = \lambda^*$

[a] Linear Independence Constraint Qualification
[b] Second Order Sufficient Condition

An inexact solution method

Algorithm `LOG-BARRIER`

Data: $\mu_\circ > \mu_{tol} > 0$, `maxit`, x_0 s.t. $g(x_0) > 0$

for $k = 0, 1, \dots,$`maxit`

Compute $x_k = x(\mu_k)$ t.c. $\|\nabla P(x_k; \mu_k)\| \leq \mu_k$

if $\mu_k < \mu_{tol}$ **then**

$x^* \leftarrow x_k$, $\lambda^* \leftarrow \mu_k / g(x_k)$, STOP

endif

endfor

Return: (x^*, λ^*)

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Choose $\mu_{k+1} \in (0, \mu_k)$

endfor

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Equality constraints

Let us consider the problem

$$
\begin{array}{ll}
\min_x & f(x) \\
c.v. & g(x) \geq 0 \\
 & h(x) = 0
\end{array}
$$

The log-barrier function is

$$
\begin{aligned}
P(x;\mu) &= f(x) - \mu \sum_{i=1}^{m} \log g_i(x) + \frac{1}{2\mu}\|h(x)\|^2 \\
\nabla_x P(x;\mu) &= \nabla f(x) - \sum_{i=1}^{m} \frac{\mu}{g_i(x)} \nabla g_i(x) + \frac{1}{\mu} \sum_{i=1}^{p} h_i(x) \nabla h_i(x)
\end{aligned}
$$

Initial point

since,

$$P(x;\mu) < +\infty \text{ iff } g(x) > 0,$$

then ...

in order to use $P(x;\mu)$ within a practical solution method a strictly feasible point x_0, i.e. $g(x_0) > 0$, must be known (see LOG-BARRIER).

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Slack variables

$$\begin{array}{ll} \min\limits_{x} & f(x) \\ c.v. & g(x) \geq 0 \end{array} \quad \rightsquigarrow \quad \begin{array}{ll} \min\limits_{x,s} & f(x) \\ c.v. & g_i(x) - s_i = 0 \\ & \boxed{s_i \geq 0} \end{array}$$

$$P(x,s;\mu) = f(x) - \mu \sum_{i=1}^{m} \log s_i + \frac{1}{2\mu} \sum_{i=1}^{m} (g_i(x) - s_i)^2$$

Infeasible Interior Point Methods

N.B.: $P(x;\mu)$ defined even when $g(x) \not\geq 0$

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Infeasible Interior Point Methods

N.B. $P(x; \mu)$ defined even when $g(x) \not\geq 0$

Relation with KKT

$$\begin{array}{ll} \min\limits_{x,s} & f(x) \\ c.v. & g_i(x) - s_i = 0 \\ & s_i \geq 0 \end{array}$$

Lagrangian function: $L(x,\rho,\lambda) = f(x) + \rho^\top(g-s) - \lambda^\top s$

KKT conditions:

$$\begin{array}{rcl} \nabla_x L & = & \nabla f(x) + \sum_{i=1}^m \rho_i \nabla g_i(x) = 0 \\ \nabla_s L & = & -\rho - \lambda = 0 \Rightarrow \rho = -\lambda \\ g - s & = & 0 \\ \lambda & \geq & 0, \ s \geq 0, \ \lambda^\top s = 0 \end{array}$$

Relation with KKT

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Relation with KKT

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\nabla_s L & = & -\rho - \lambda = 0 \Rightarrow \rho = -\lambda \\
g - s & = & 0 \\
\lambda & \geq & 0,\ s \geq 0,\ \lambda^\top s = 0
\end{array}
$$

Relation with KKT

$$\min_{x,s} \quad P(x,s;\mu)$$

$$\begin{aligned}
P(x,s;\mu) &= f(x) - \mu\sum_{i=1}^m \log s_i + \frac{1}{2\mu}\sum_{i=1}^m (g_i(x)-s_i)^2 \\
\nabla_x P &= \nabla f(x) + \sum_{i=1}^m \frac{g_i(x)-s_i}{\mu}\nabla g_i(x) = 0 \\
\nabla_{s_i} P &= -\frac{\mu}{s_i} - \frac{g_i(x)-s_i}{\mu} = 0 \Rightarrow \frac{g_i(x)-s_i}{\mu} = -\frac{\mu}{s_i} = -\lambda_i \\
\lambda_i s_i &= \mu, \quad g_i(x) - s_i = -\frac{\mu^2}{s_i} = -\lambda_i\mu
\end{aligned}$$

Relation with KKT

$$\min_{x,s} \quad P(x,s;\mu)$$

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P(x,s;\mu) &= f(x) - \mu\sum_{i=1}^{m}\log s_i + \frac{1}{2\mu}\sum_{i=1}^{m}(g_i(x)-s_i)^2 \\
\nabla_x P &= \nabla f(x) + \sum_{i=1}^{m}\frac{g_i(x)-s_i}{\mu}\nabla g_i(x) = 0 \\
\nabla_{s_i} P &= -\frac{\mu}{s_i} - \frac{g_i(x)-s_i}{\mu} = 0 \;\Rightarrow\; \frac{g_i(x)-s_i}{\mu} = -\frac{\mu}{s_i} = -\lambda_i \\
\lambda_i s_i &= \mu, \quad g_i(x)-s_i = -\frac{\mu^2}{s_i} = -\lambda_i\mu
\end{aligned}$$

Relation with KKT

$$\min_{x,s} \quad P(x, s; \mu)$$

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P(x, s; \mu) &= f(x) - \mu \sum_{i=1}^{m} \log s_i + \frac{1}{2\mu} \sum_{i=1}^{m} (g_i(x) - s_i)^2 \\
\nabla_x P &= \nabla f(x) - \sum_{i=1}^{m} \lambda_i \nabla g_i(x) = 0 \\
\nabla_{s_i} P &= -\frac{\mu}{s_i} - \frac{g_i(x) - s_i}{\mu} = 0 \Rightarrow \frac{g_i(x) - s_i}{\mu} = -\frac{\mu}{s_i} = -\lambda_i \\
\lambda_i s_i &= \mu, \quad g_i(x) - s_i = -\frac{\mu^2}{s_i} = -\lambda_i \mu
\end{aligned}
$$