Ottimizzazione dei Sistemi Complessi

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Introduzione

We consider the problem

$$\begin{aligned} \min_{x} \quad & f(x) \\ c.v. \quad & g(x) \geq 0 \end{aligned}$$

- $\mathcal{F}$ feasible region
- $\overset{\circ}{\mathcal{F}} = Int(\mathcal{F}) = \{x \in \mathbb{R}^n : g_i(x) > 0\}$
- We assume $\overset{\circ}{\mathcal{F}} \neq \emptyset$

Log-barrier function

For the considered problem:

$$
\begin{aligned}
P(x;\mu) &= f(x) - \mu \sum_{i=1}^{m} \log g_i(x), \text{ dove} \\
b(x) &= -\sum_{i=1}^{m} \log g_i(x) \text{ (termine di barriera)} \\
\nabla_x P(x;\mu) &= \nabla f(x) - \sum_{i=1}^{m} \frac{\mu}{g_i(x)} \nabla g_i(x)
\end{aligned}
$$

N.B.

- $b(x)$ (and $P(x;\mu)$) is defined for all $x \in \overset{\circ}{\mathcal{F}}$
- We assume $b(x)$ (and $P(x;\mu)$) $= +\infty$ for all $x \notin \overset{\circ}{\mathcal{F}}$

Example 1

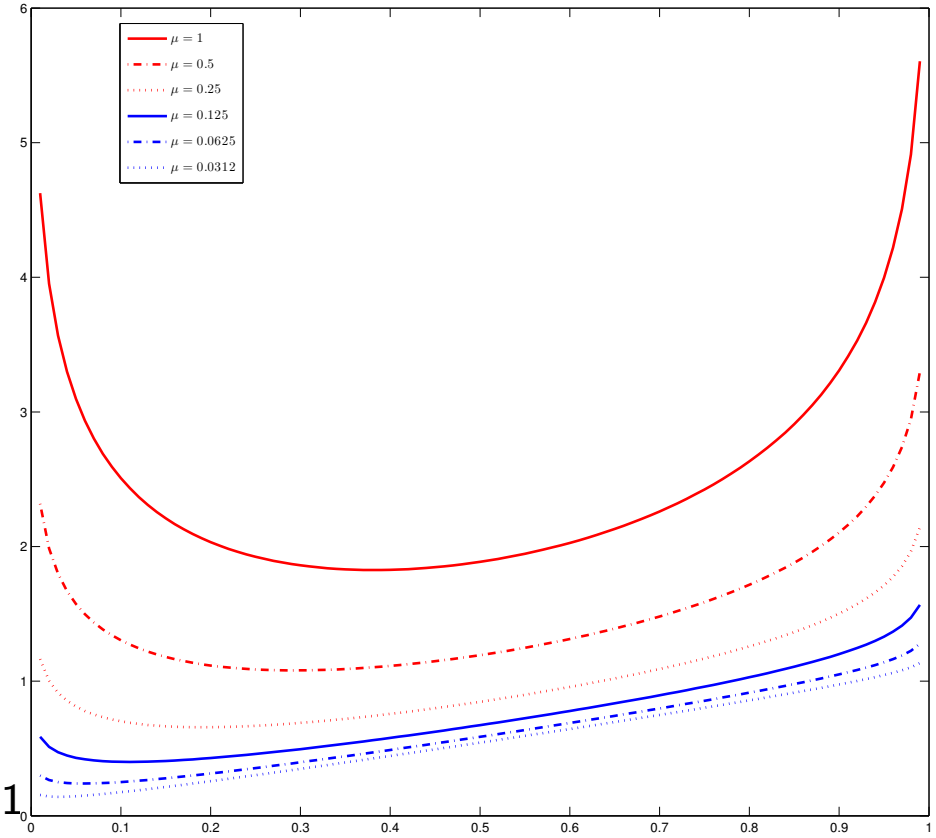
$$\begin{array}{ll} \min\limits_{x} & x \\ c.v. & x \geq 0 \\ & 1 - x \geq 0 \end{array}$$

The log-barrier function for this problem is

$$\begin{aligned} P(x;\mu) &= x - \mu \log x - \mu \log(1-x) \\ \nabla P(x;\mu) &= 1 - \frac{\mu}{x} + \frac{\mu}{1-x} = 0 \end{aligned}$$

$$\frac{x(1-x) - \mu(1-x) + \mu x}{x(1-x)} = 0$$

$$x^2 - (1+2\mu)x + \mu = 0$$



$$\begin{aligned} 0 < x^- &= \frac{1}{2} + \mu - \sqrt{\left(\frac{1}{2} + \mu\right)^2 - \mu} < 1 \\ x^+ &= \frac{1}{2} + \mu + \sqrt{\frac{1}{4} + \mu^2} > 1 \;\Rightarrow\; x^+ \notin \mathcal{F} \end{aligned}$$

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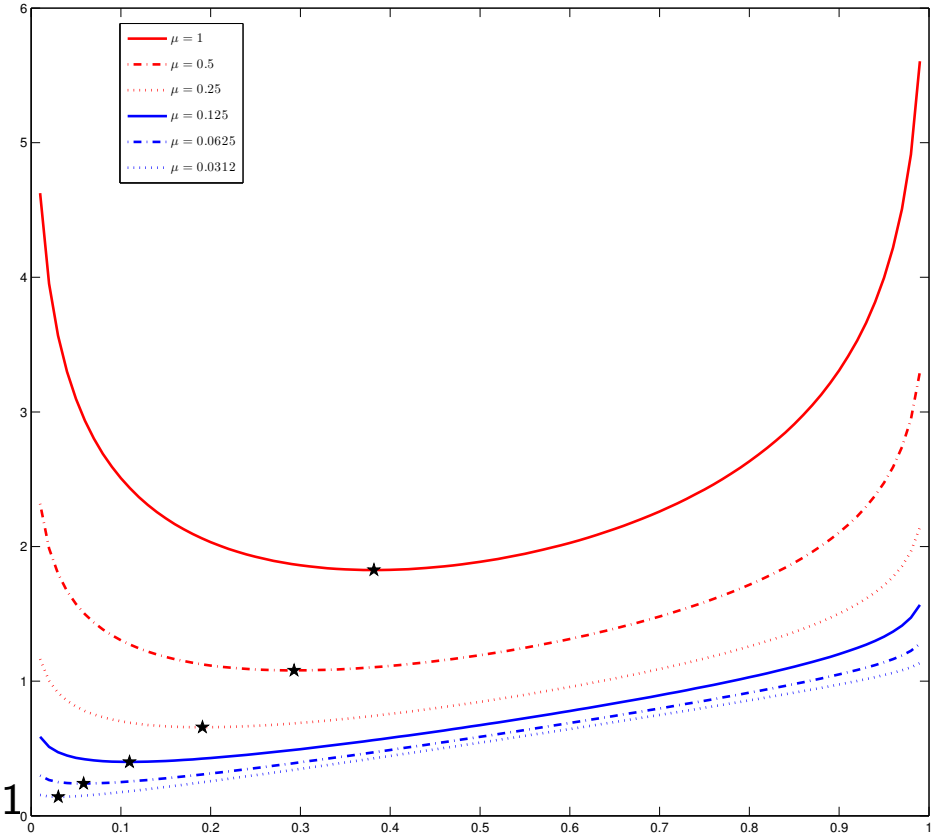

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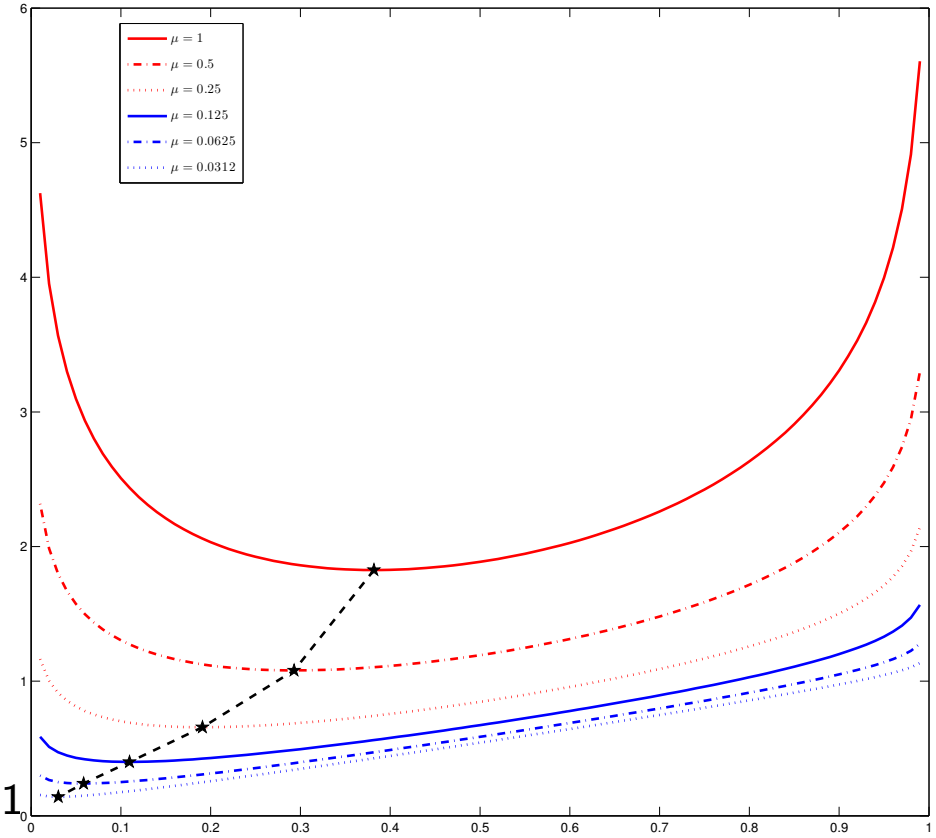


$$0 < x^- = \frac{1}{2} + \mu - \sqrt{\left(\frac{1}{2} + \mu\right)^2 - \mu} < 1$$

$$x^+ = \frac{1}{2} + \mu + \sqrt{\frac{1}{4} + \mu^2} > 1 \Rightarrow x^+ \notin \mathcal{F}$$

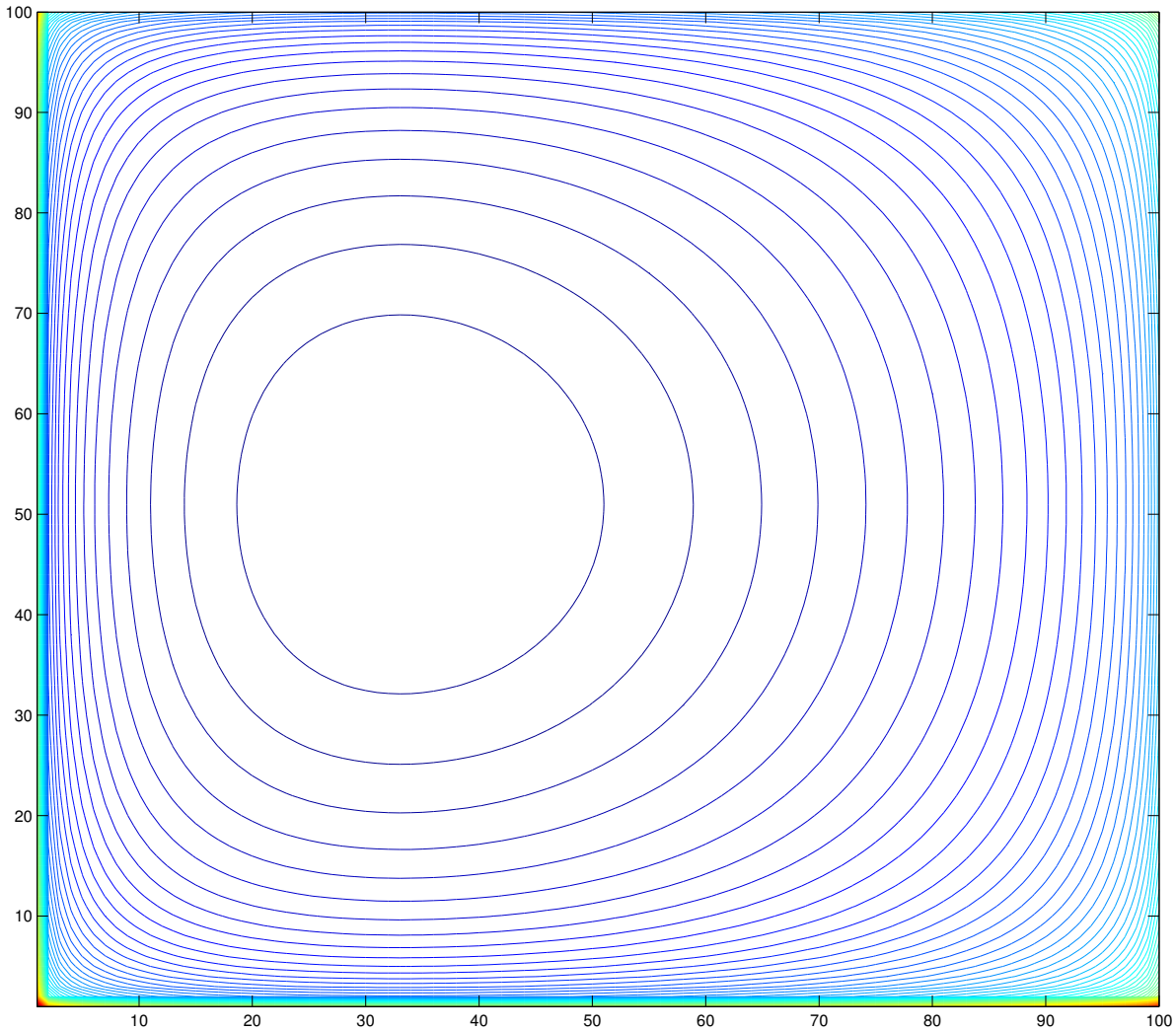
Example 2

$$\begin{aligned} \min_x \quad & (x + 0.5)^2 + (x_2 - 0.5)^2 \\ c.v. \quad & x_1 \in [0, 1],\ x_2 \in [0, 1] \end{aligned}$$

The log-barrier function for this problem is

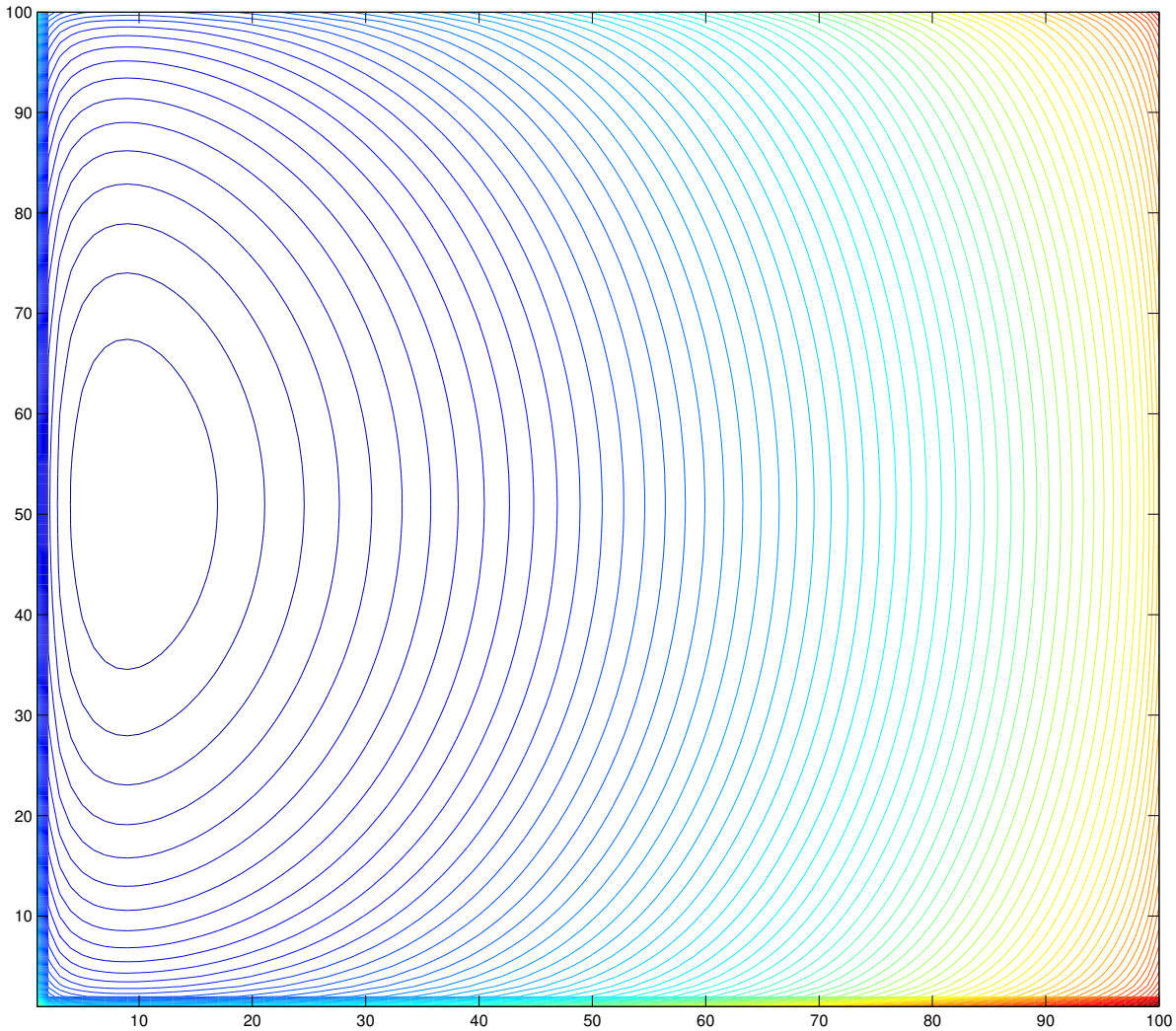
$$\begin{aligned} P(x; \mu) = & (x + 0.5)^2 + (x_2 - 0.5)^2 \\ & -\mu[\log x_1 + \log(1 - x_1)] \\ & -\mu[\log x_2 + \log(1 - x_2)] \end{aligned}$$

Example 2



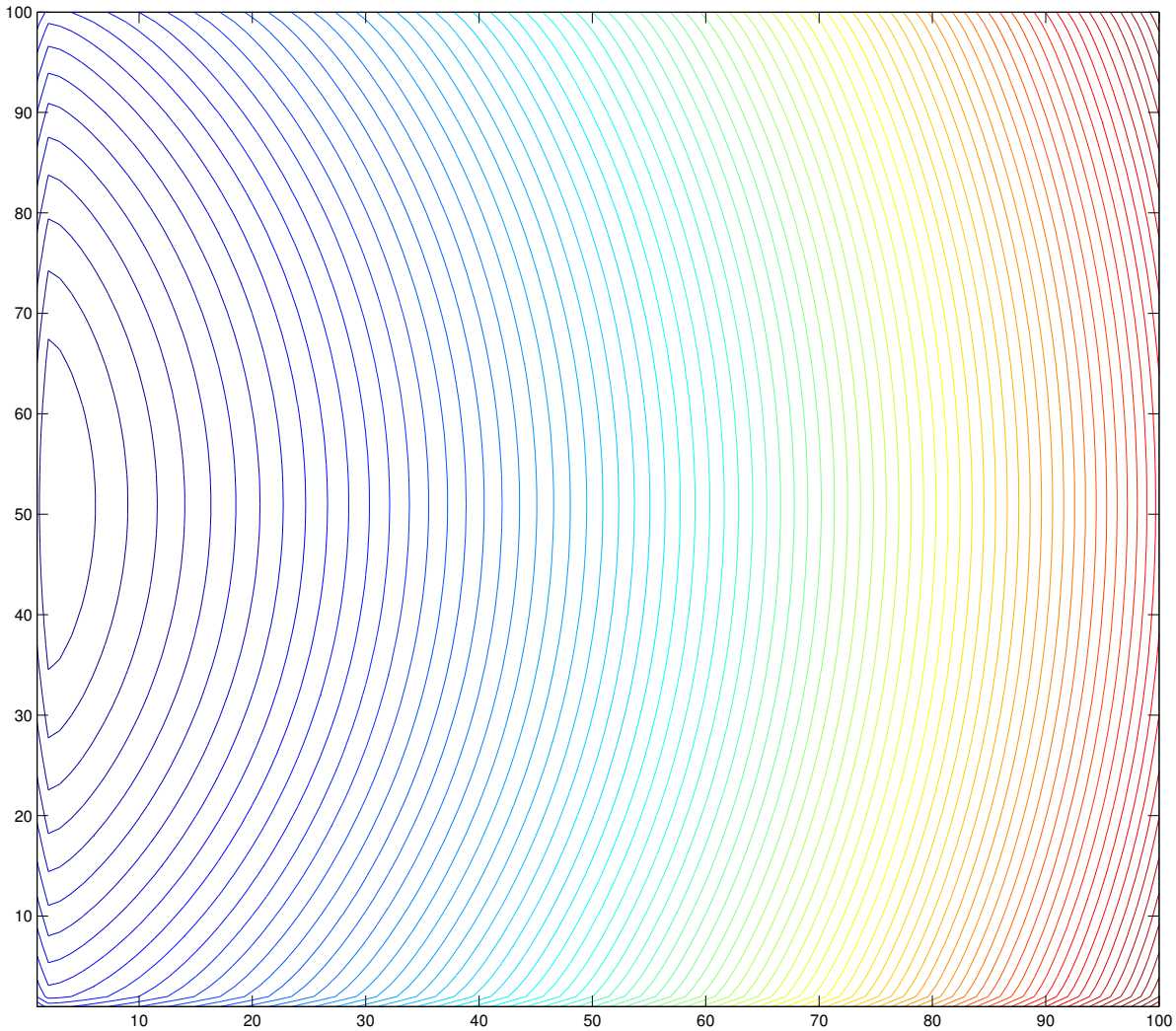
$\mu = 1$

Example 2



$\mu = 0.1$

Example 2



$\mu = 0.01$

Example 2

When $\mu = 1 \rightarrow 0.0085$ we have

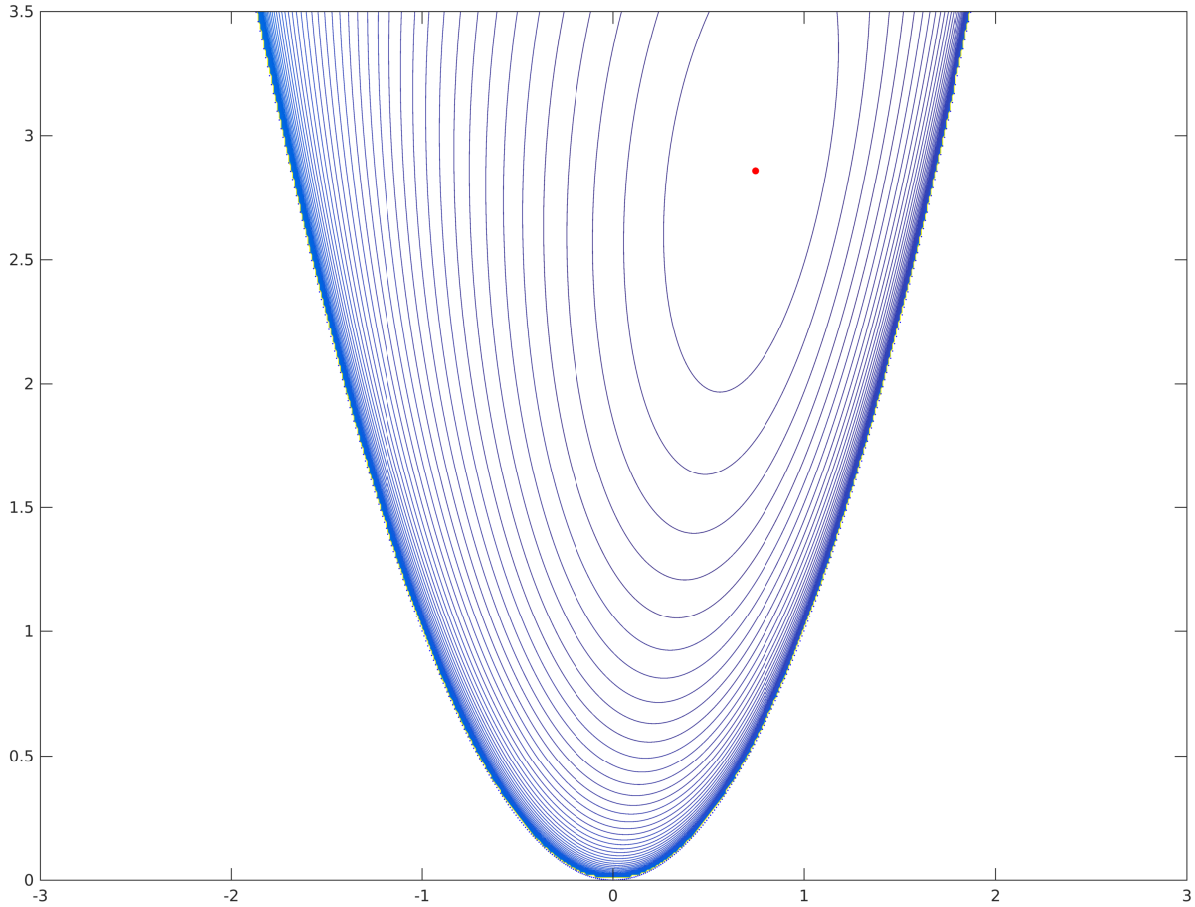
Example 3 (HS 11)

$$\begin{aligned} \min_{x} \quad & (x-5)^2 + y^2 \\ s.t. \quad & -x^2 + y \geq 0 \end{aligned}$$

The log-barrier function for this problem is

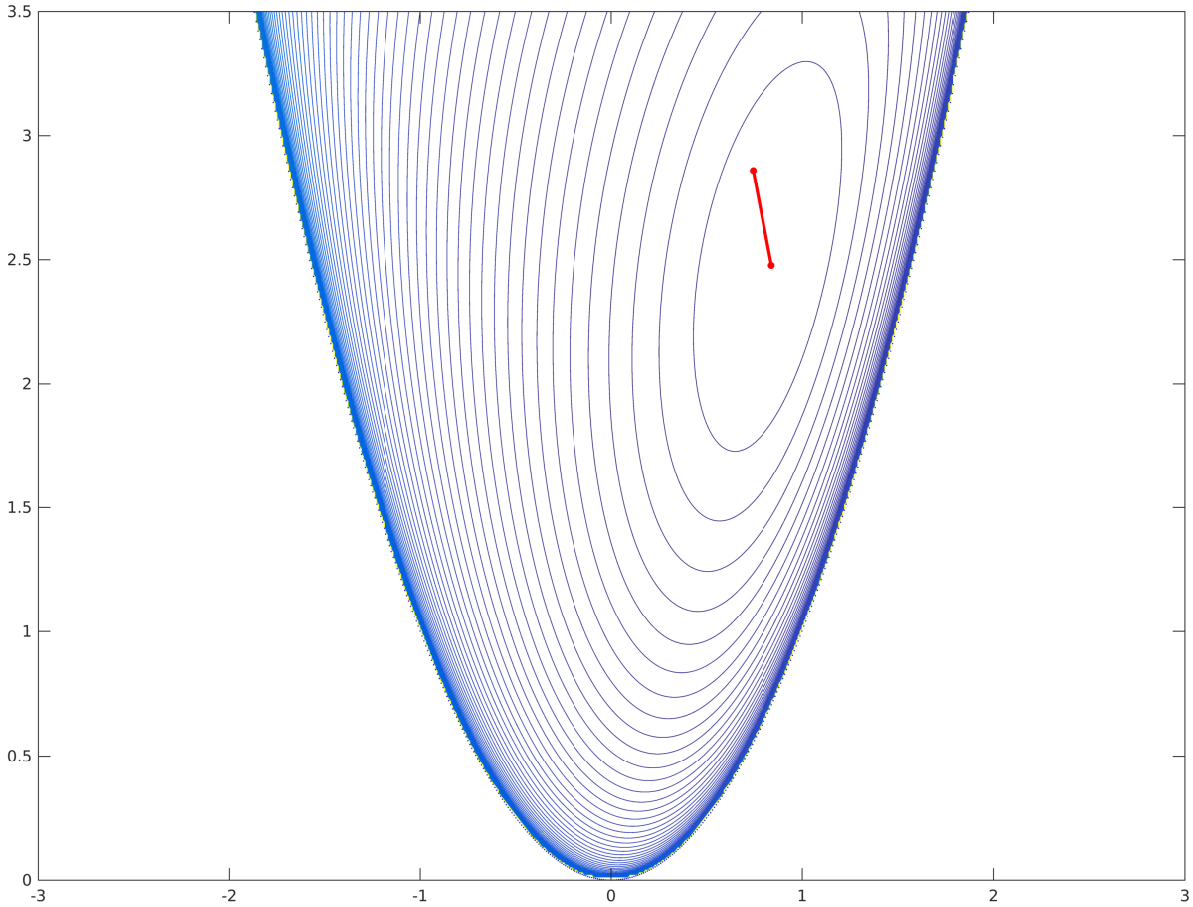
$$P(x;\mu) = (x-5)^2 + y^2 - \mu[\log(y-x^2)]$$

Example 3



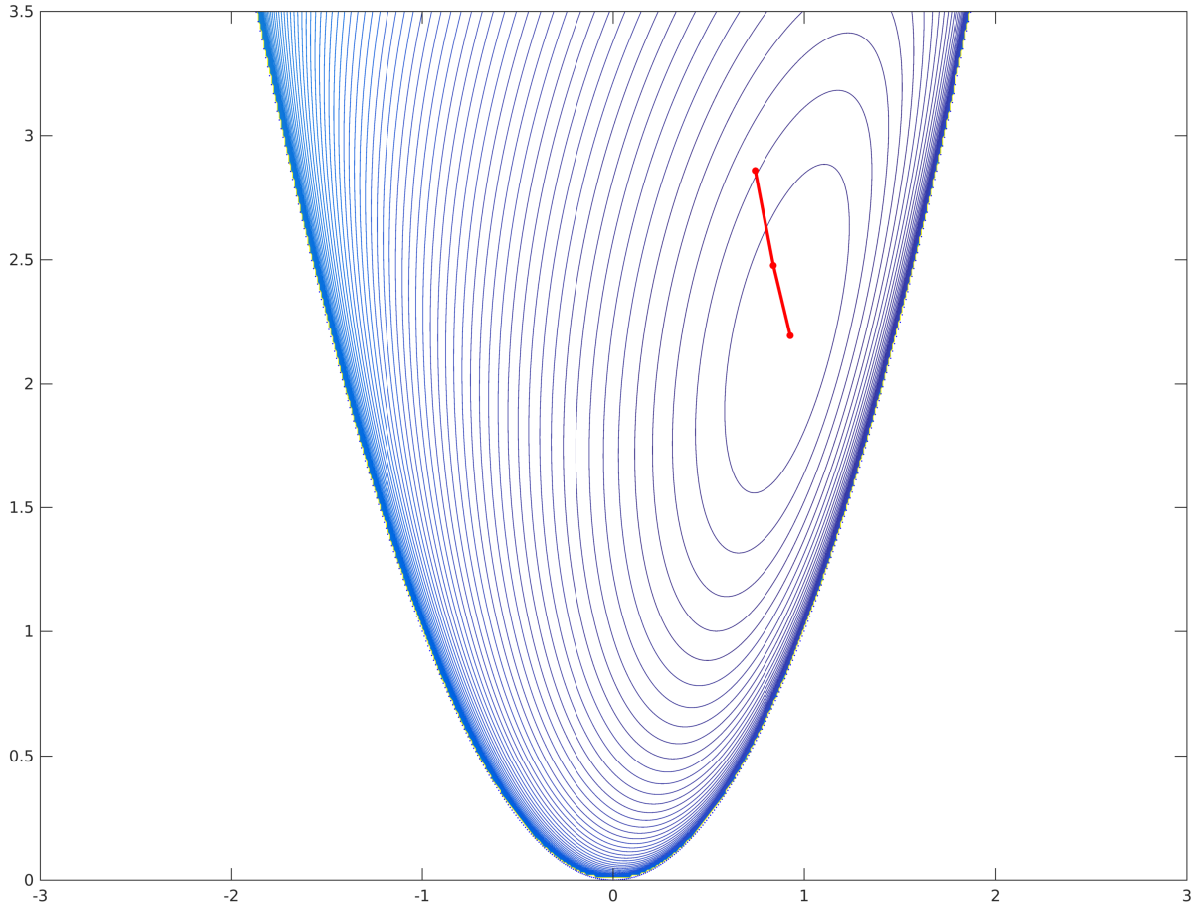
$\mu =$1.317e+1

Example 3



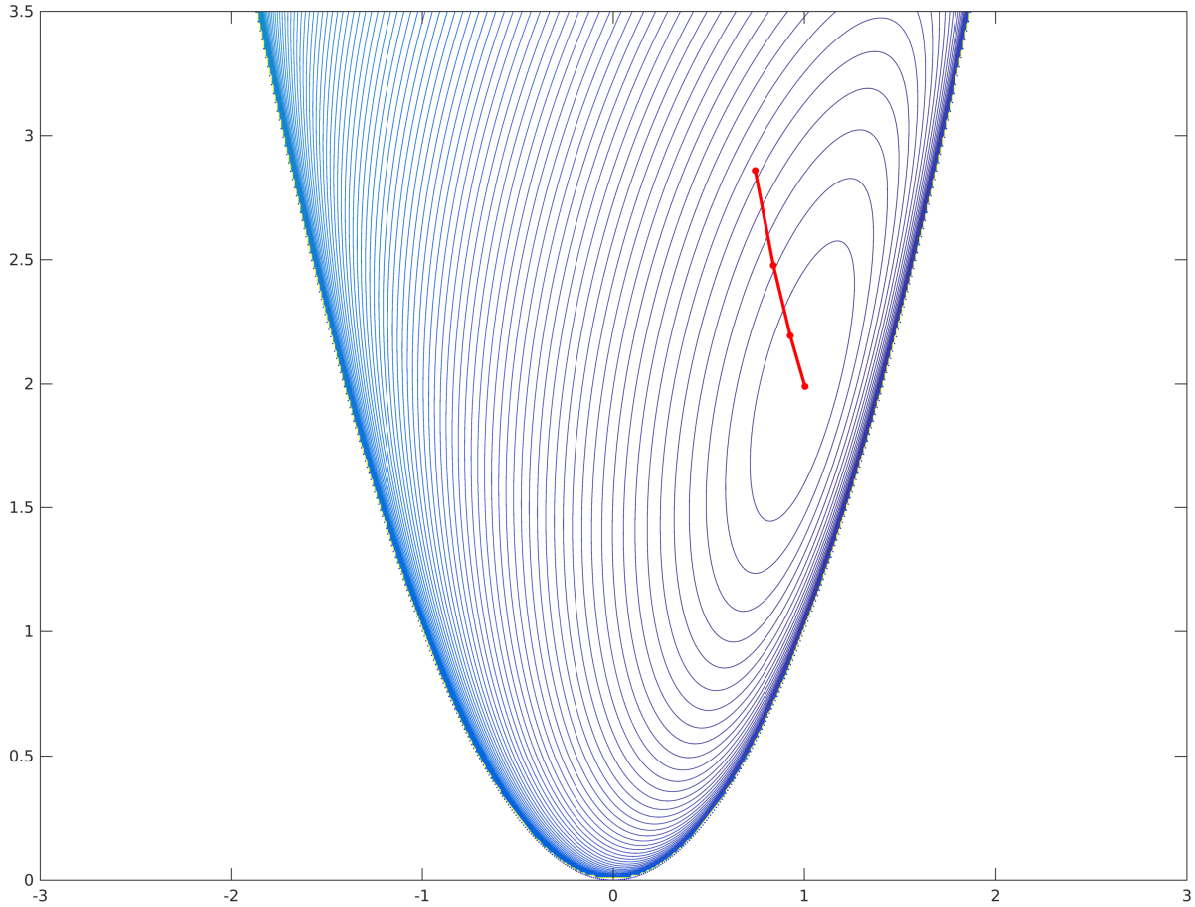
$\mu = 8.779$

Example 3



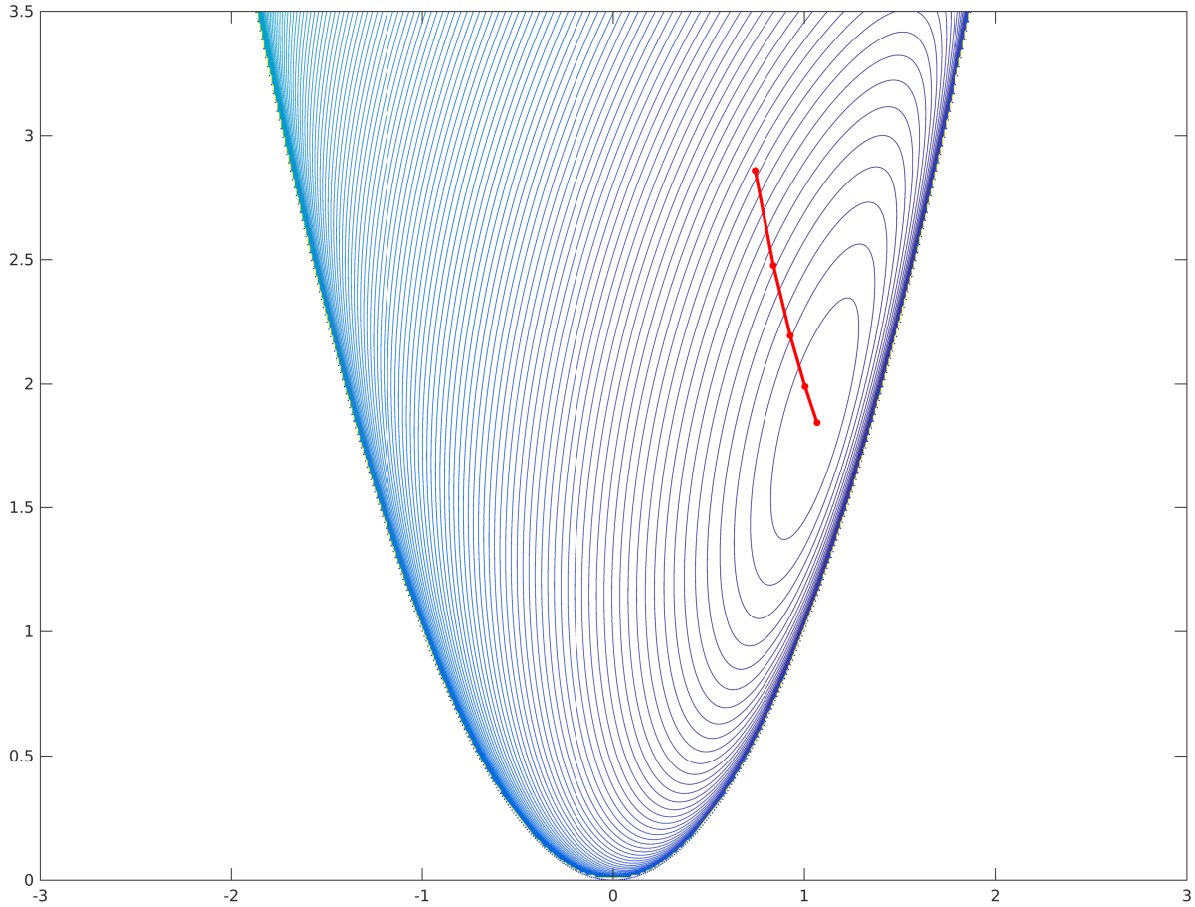
μ =5.853

Example 3



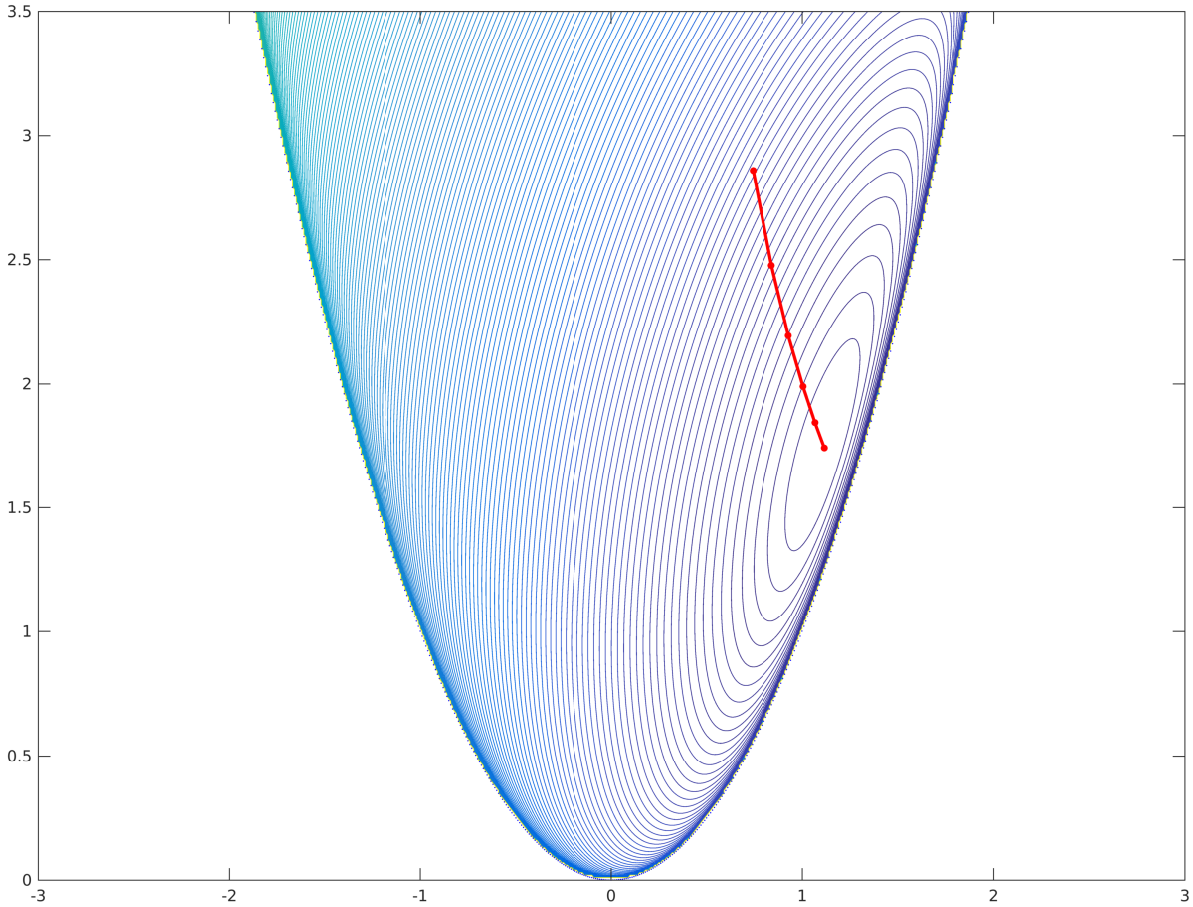
$\mu = 3.902$

Example 3



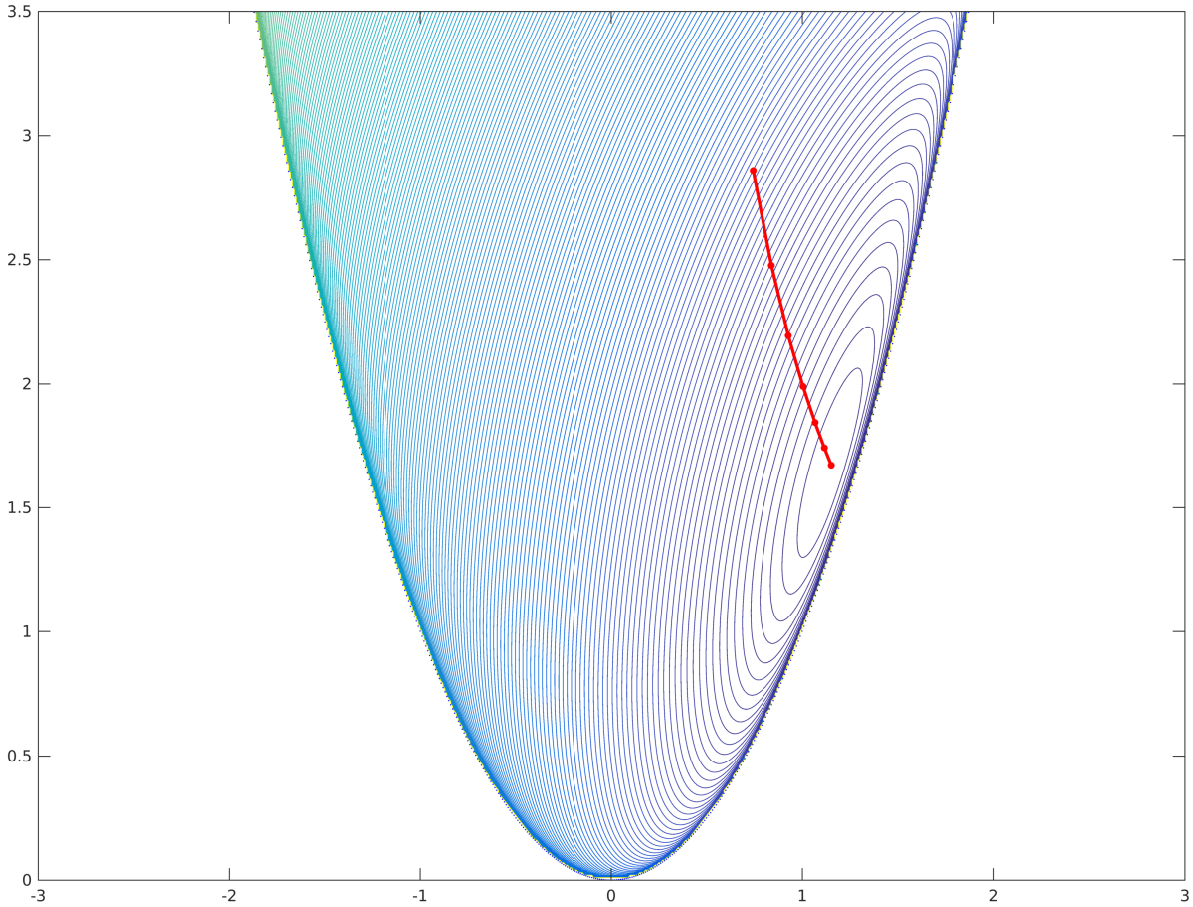
$\mu = 2.601$

Example 3



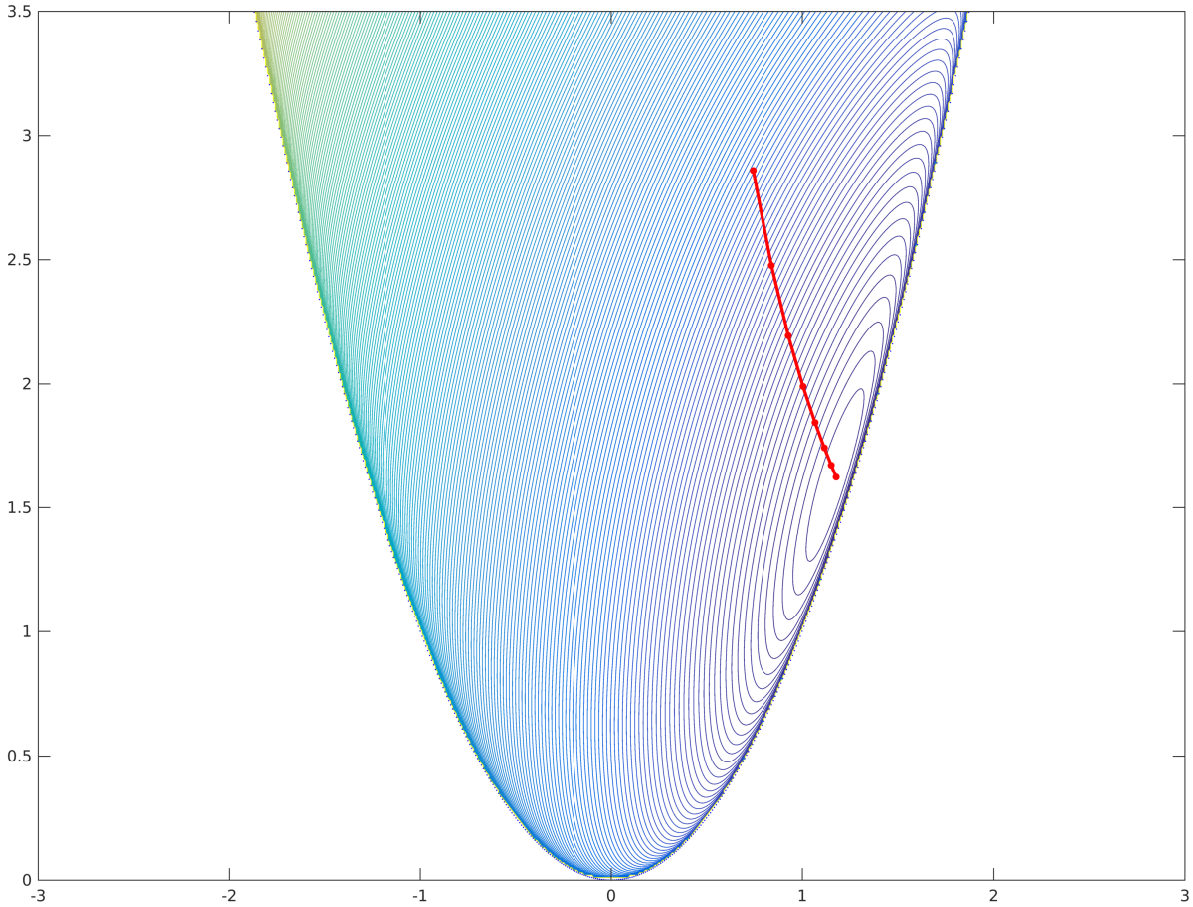
$\mu = 1.156$

Example 3



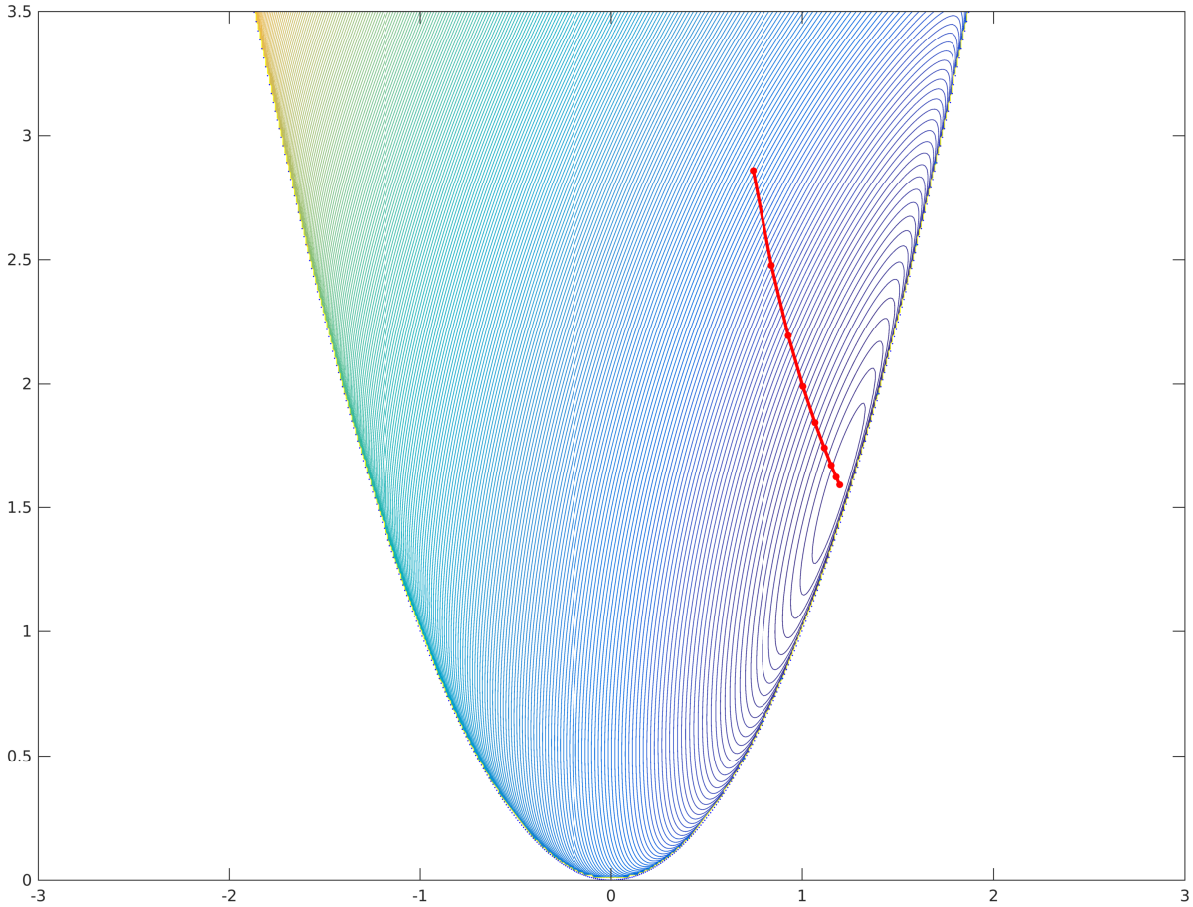
μ =7.707e-1

Example 3



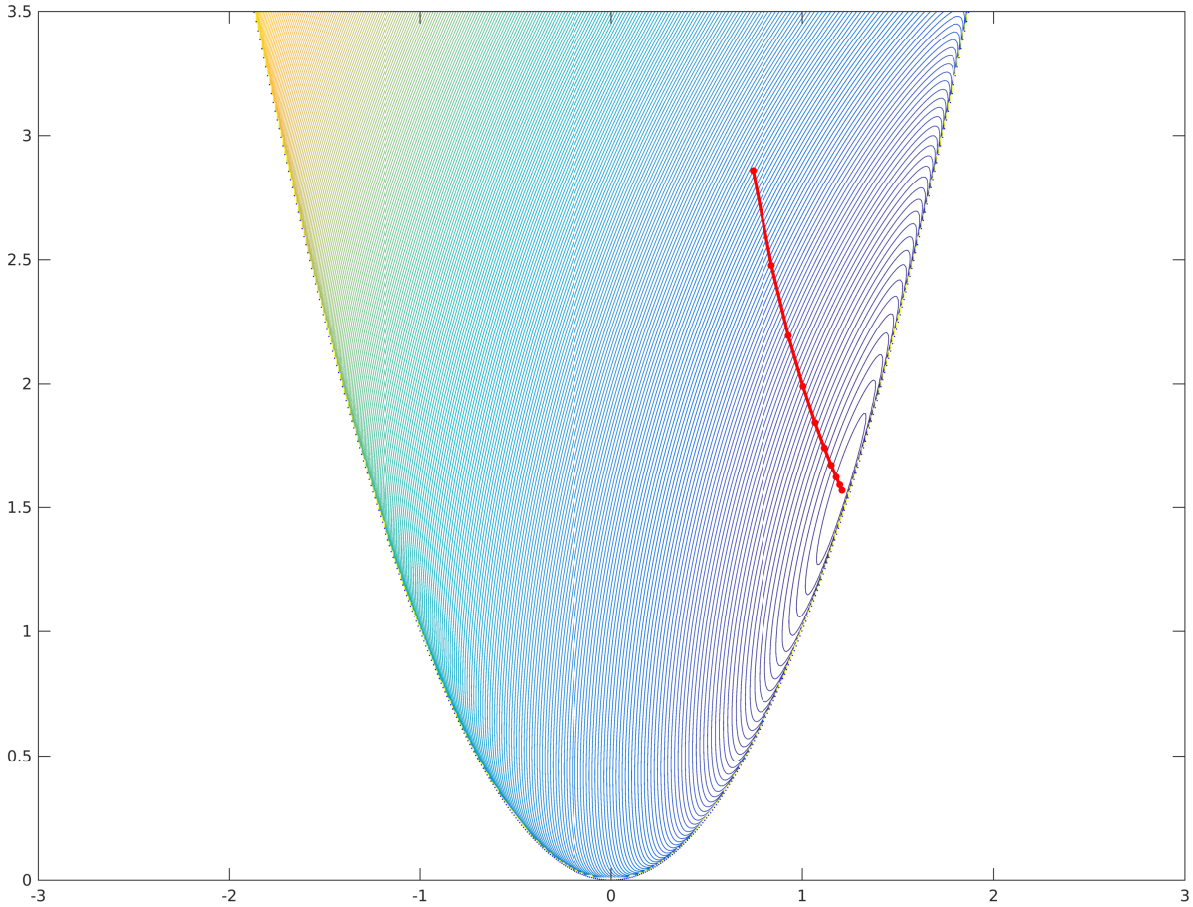
μ =5.138e-1

Example 3



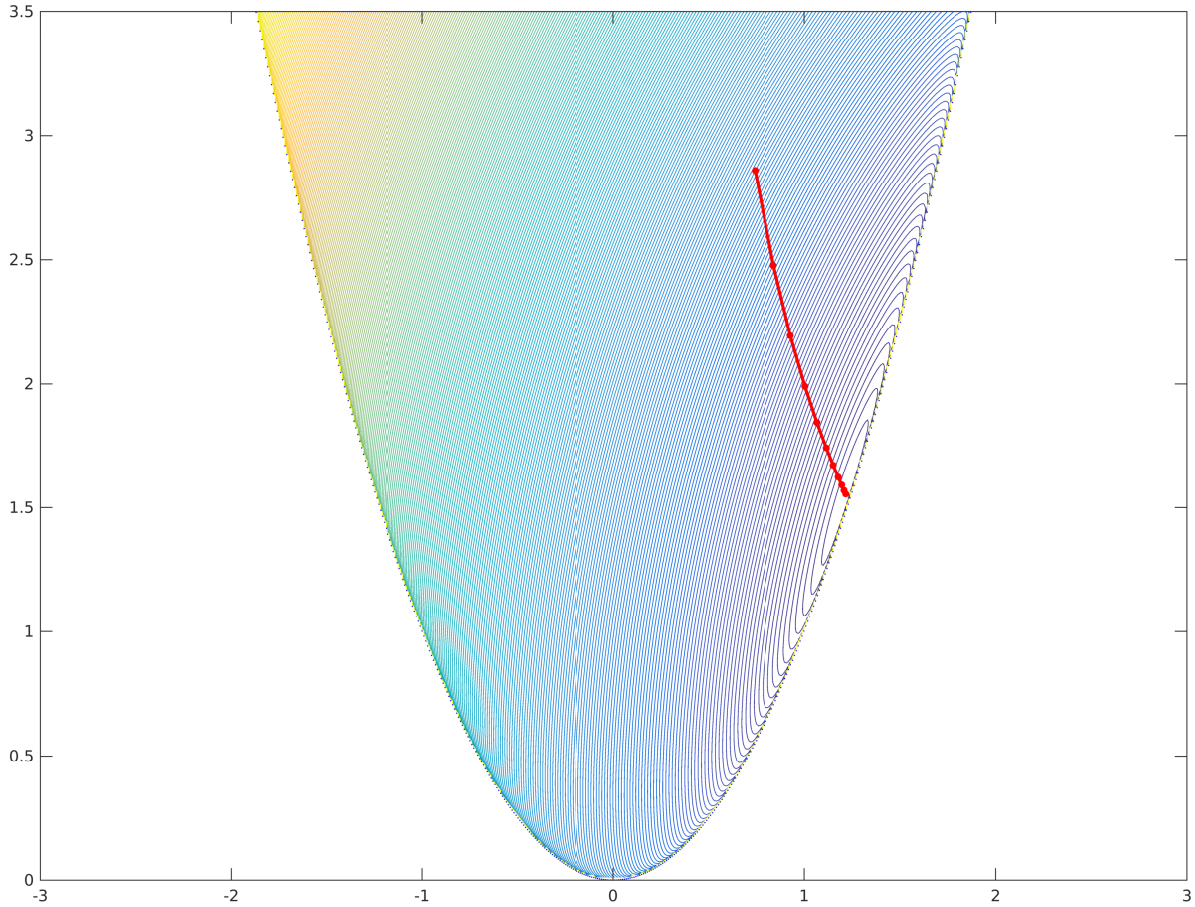
μ =3.425e-1

Example 3



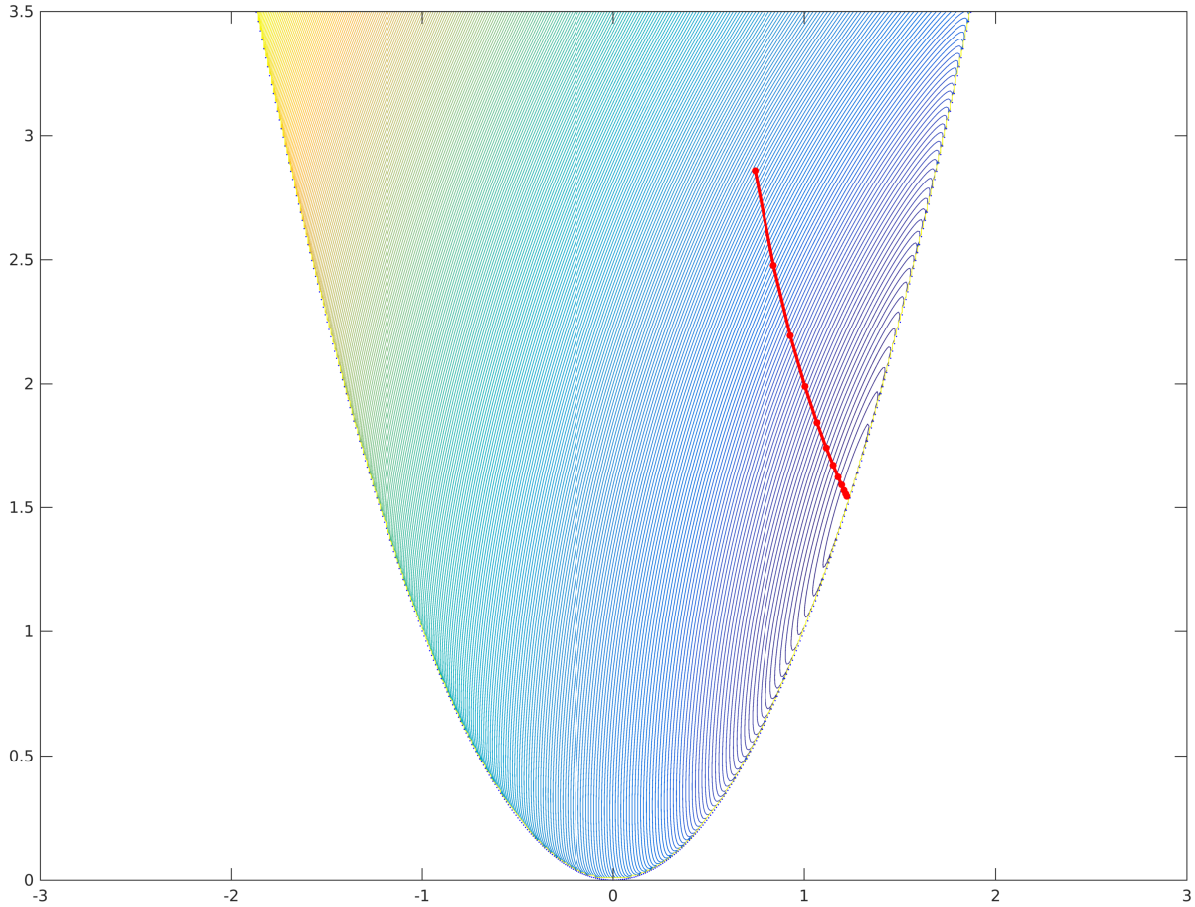
μ =2.284e-1

Example 3



μ =1.522e-1

Example 3



μ =1.015e-1

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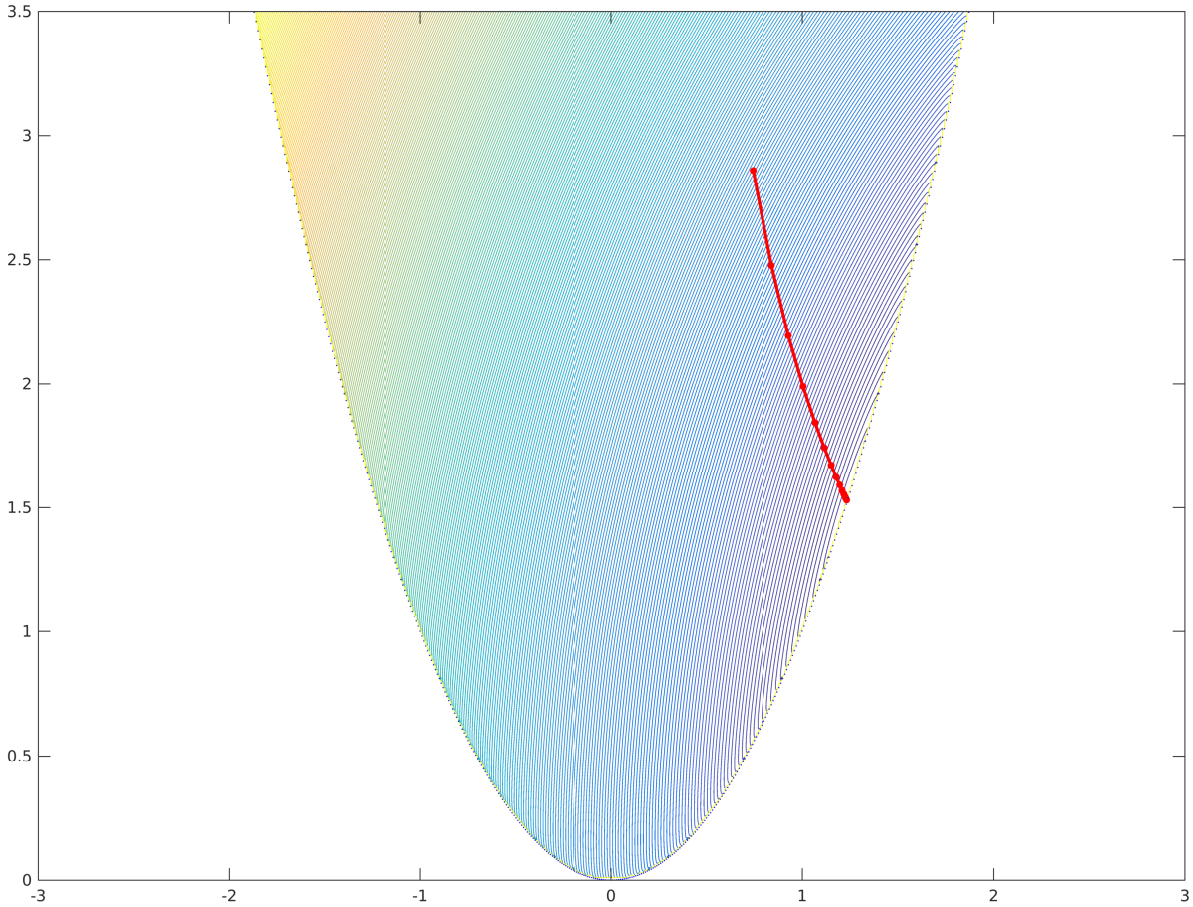

μ =3.007e-2

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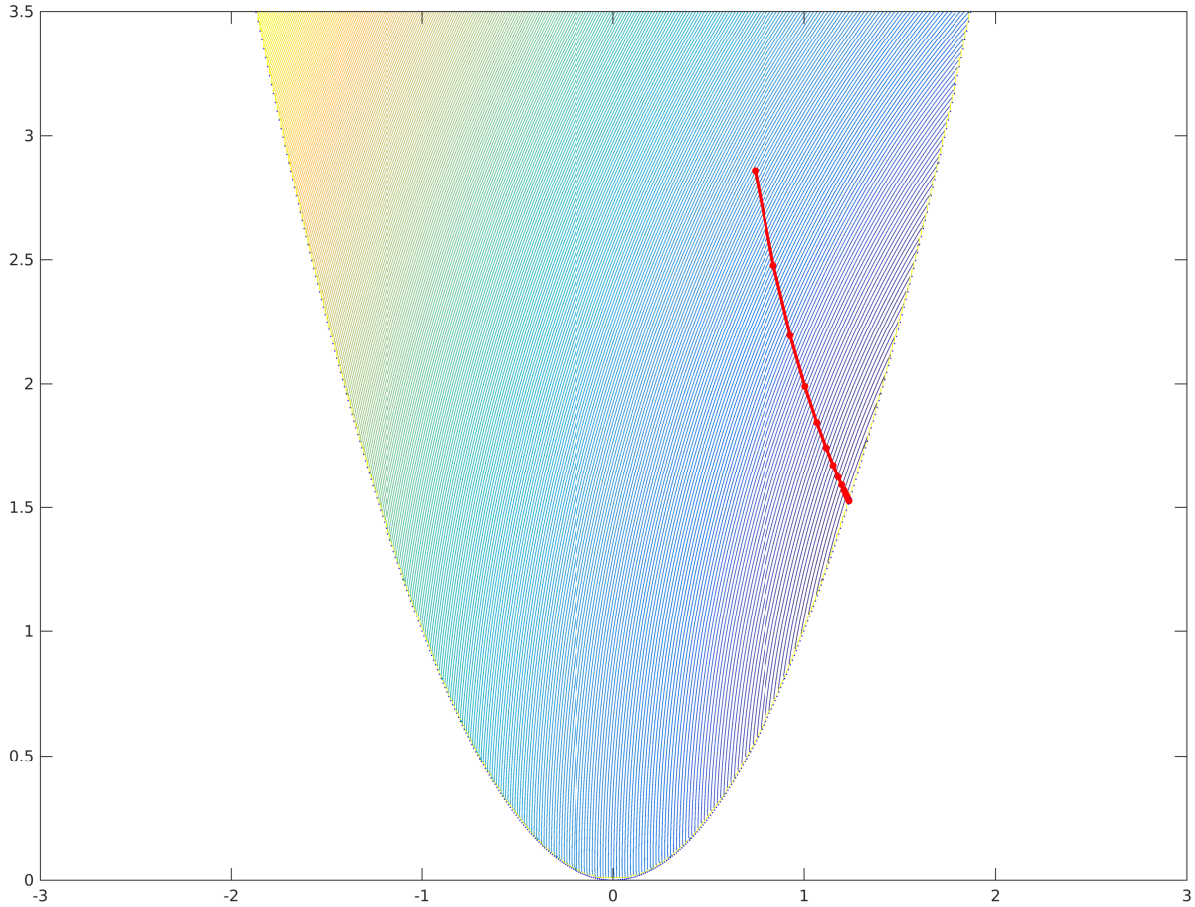
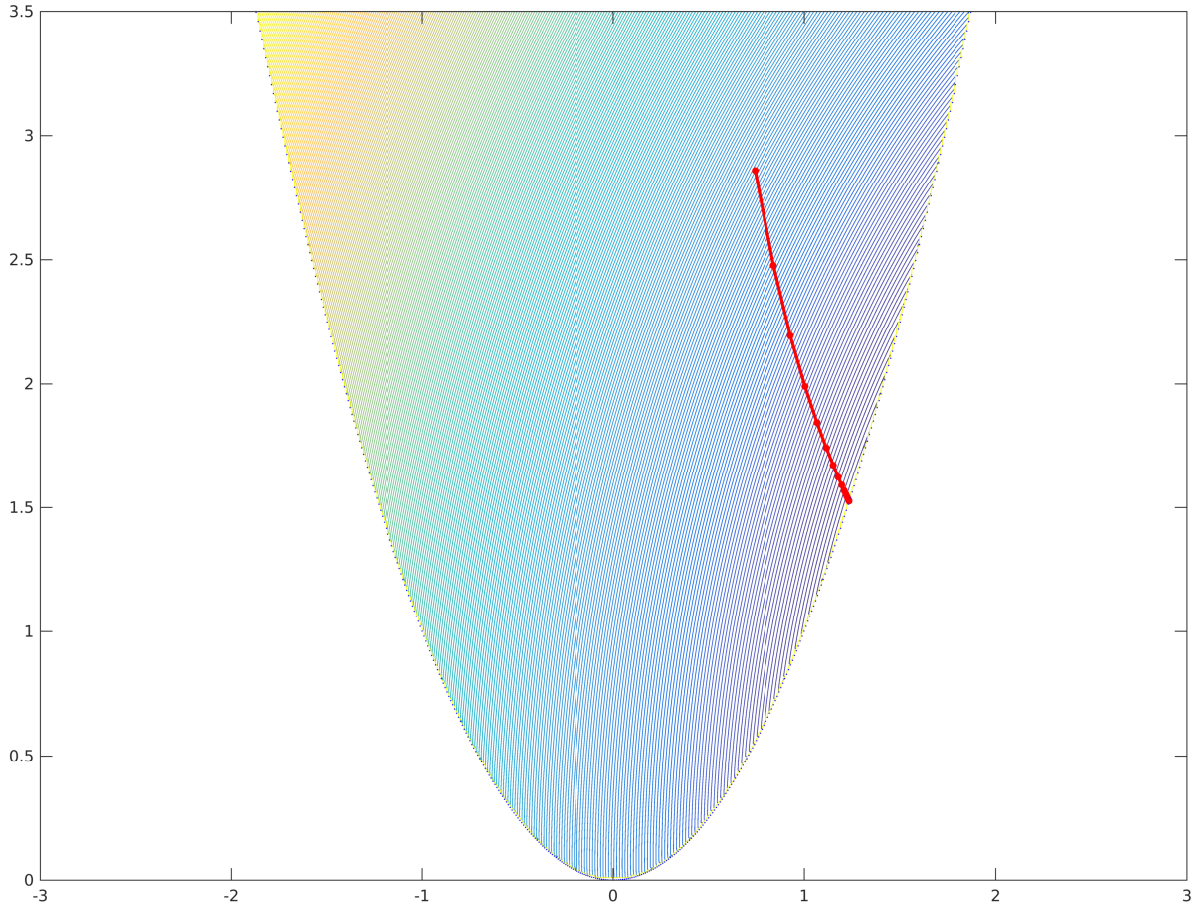


μ =8.910e-3 KKT error = 8.910e-3

Example 3



μ =8.910e-3 KKT error = 8.910e-3

KKT conditions – stationary points of $P(x;\mu)$

KKT point

Let (x, λ) be a KKT pair

$$\begin{aligned} \nabla f(x) - \lambda^\top \nabla g(x) &= 0 \\ g(x) &\geq 0 \\ \lambda &\geq 0 \\ \lambda^\top g(x) &= 0 \end{aligned}$$

Stationary point of $P(x;\mu)$

Let $x(\mu)$ be such that

$$\nabla f(x(\mu)) - \sum_{i=1}^{m} \lambda_i(\mu) \nabla g_i(x(\mu)) = 0$$

$$\lambda_i(\mu) = \frac{\mu}{g_i(x(\mu))}$$

The pair $(x(\mu), \lambda(\mu))$

- defines the so-called **central-path** or **interior-path**
- it is "almost" a KKT pair; complementary slackness $\lambda^\top g(x) = 0$ is violated
- but we have:

$$\lambda_i(\mu) g_i(x(\mu)) = \mu$$

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An inexact solution method

Algorithm `LOG-BARRIER`

Data: $\mu_\circ > \mu_{tol} > 0$, `maxit`, x_0 s.t. $g(x_0) > 0$

for $k = 0, 1, \ldots,$`maxit`

Compute $x_k = x(\mu_k)$ t.c. $\|\nabla P(x_k; \mu_k)\| \leq \mu_k$

if $\mu_k < \mu_{tol}$ **then**

$x^* \leftarrow x_k$, $\lambda^* \leftarrow \mu_k / g(x_k)$, STOP

endif

endfor

Return: (x^*, λ^*)

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Equality constraints

Let us consider the problem

$$\begin{array}{ll} \min\limits_{x} & f(x) \\ c.v. & g(x) \geq 0 \\ & h(x) = 0 \end{array}$$

The log-barrier function is

$$\begin{aligned} P(x;\mu) &= f(x) - \mu \sum_{i=1}^{m} \log g_i(x) + \frac{1}{2\mu} \|h(x)\|^2 \\ \nabla_x P(x;\mu) &= \nabla f(x) - \sum_{i=1}^{m} \frac{\mu}{g_i(x)} \nabla g_i(x) + \frac{1}{\mu} \sum_{i=1}^{p} h_i(x) \nabla h_i(x) \end{aligned}$$

Initial point

since,

$$P(x;\mu) < +\infty \text{ iff } g(x) > 0,$$

then ...

in order to use $P(x;\mu)$ within a practical solution method a strictly feasible point x_0, i.e. $g(x_0) > 0$, must be known (see `LOG-BARRIER`).

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Slack variables

$$\begin{array}{ll} \min\limits_{x} & f(x) \\ c.v. & g(x) \geq 0 \end{array} \quad \rightsquigarrow \quad \begin{array}{ll} \min\limits_{x,s} & f(x) \\ c.v. & g_i(x) - s_i = 0 \\ & \boxed{s_i \geq 0} \end{array}$$

Slack variables

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$$P(x, s; \mu) = f(x) - \mu \sum_{i=1}^{m} \log s_i + \frac{1}{2\mu} \sum_{i=1}^{m} (g_i(x) - s_i)^2$$

Infeasible Interior Point Methods

N.B. $P(x; \mu)$ defined even when $g(x) \not> 0$

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Infeasible Interior Point Methods

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Relation with KKT

$$\begin{aligned} \min_{x,s} \quad & f(x) \\ c.v. \quad & g_i(x) - s_i = 0 \\ & s_i \geq 0 \end{aligned}$$

Lagrangian function: $L(x, s, \rho, \lambda) = f(x) + \rho^\top (g - s) - \lambda^\top s$

KKT conditions:

$$\begin{aligned} \nabla_x L &= \nabla f(x) + \sum_{i=1}^{m} \rho_i \nabla g_i(x) = 0 \\ \nabla_s L &= -\rho - \lambda = 0 \Rightarrow \rho = -\lambda \\ g - s &= 0 \\ \lambda &\geq 0,\ s \geq 0,\ \lambda^\top s = 0 \end{aligned}$$

Relation with KKT

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Relation with KKT

$$\min_{x,s} \quad P(x, s; \mu)$$

$$\begin{aligned}
P(x, s; \mu) &= f(x) - \mu \sum_{i=1}^{m} \log s_i + \frac{1}{2\mu} \sum_{i=1}^{m} (g_i(x) - s_i)^2 \\
\nabla_x P &= \nabla f(x) + \sum_{i=1}^{m} \frac{g_i(x) - s_i}{\mu} \nabla g_i(x) = 0 \\
\nabla_{s_i} P &= -\frac{\mu}{s_i} - \frac{g_i(x) - s_i}{\mu} = 0 \Rightarrow \frac{g_i(x) - s_i}{\mu} = -\frac{\mu}{s_i} = -\lambda_i \\
\lambda_i s_i &= \mu, \quad g_i(x) - s_i = -\frac{\mu^2}{s_i} = -\lambda_i \mu
\end{aligned}$$

Relation with KKT

$$\min_{x,s} \quad P(x,s;\mu)$$

$$\begin{aligned}
P(x,s;\mu) &= f(x) - \mu \sum_{i=1}^{m} \log s_i + \frac{1}{2\mu} \sum_{i=1}^{m} (g_i(x) - s_i)^2 \\
\nabla_x P &= \nabla f(x) + \sum_{i=1}^{m} \frac{g_i(x) - s_i}{\mu} \nabla g_i(x) = 0 \\
\nabla_{s_i} P &= -\frac{\mu}{s_i} - \frac{g_i(x) - s_i}{\mu} = 0 \;\Rightarrow\; \frac{g_i(x) - s_i}{\mu} = -\frac{\mu}{s_i} = -\lambda_i \\
\lambda_i s_i &= \mu, \quad g_i(x) - s_i = -\frac{\mu^2}{s_i} = -\lambda_i \mu
\end{aligned}$$

Relation with KKT

$$\min_{x,s} \quad P(x,s;\mu)$$

$$\begin{aligned}
P(x,s;\mu) &= f(x) - \mu \sum_{i=1}^{m} \log s_i + \frac{1}{2\mu} \sum_{i=1}^{m} (g_i(x) - s_i)^2 \\
\nabla_x P &= \nabla f(x) - \sum_{i=1}^{m} \lambda_i \nabla g_i(x) = 0 \\
\nabla_{s_i} P &= -\frac{\mu}{s_i} - \frac{g_i(x) - s_i}{\mu} = 0 \Rightarrow \frac{g_i(x) - s_i}{\mu} = -\frac{\mu}{s_i} = -\lambda_i \\
\lambda_i s_i &= \mu, \quad g_i(x) - s_i = -\frac{\mu^2}{s_i} = -\lambda_i \mu
\end{aligned}$$