Erratum to "Physarum Can Computer Shortest Paths: Convergence Proofs and Complexity Bounds" by Luca Becchetti, Vincenzo Bonifaci, Michael Dirnberger, Andreas Karrenbauer, and Kurt Mehlhorn, ICALP 2013, LNCS 7966, 472-483

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In chapter 4 of the paper mentioned in the title, the non-uniform directed Physarum dynamics

$$\dot{x}_e(t) = a_e(q_e(t) - x_e(t)) \tag{1}$$

is studied and a convergence result is claimed. A directed graph G with node set N, edge set E, positive edge lengths $\ell_e > 0$ and positive edge reactivities $a_e > 0$ for all $e \in E$, and two distinguished nodes s_0 and s_1 is given. It is assumed that there is a directed path from s_0 to s_1.

The dynamics evolves a state vector $x \in \mathbb{R}^E_{>0}$ according to (1). The vector $q(t) \in \mathbb{R}^E$ is the electrical flow in the undirected network G, where the conductivity of edge e is equal to $x_e(t)/\ell_e$ and one unit of current is sent from s_0 to s_1; $q_e(t)$ is positive if the electrical flow is in the direction of the edge e and negative otherwise. For each edge e, its reactivity determines how fast the edge reacts to the difference between $q_e(t)$ and $x_e(t)$.

If $a_e = 1$ for all e, it was shown in [IJNT11] that the dynamics (1) converges to the shortest directed s_0-s_1 path in the following sense. For the edges e on the shortest path, $x_e(t)$ converges to 1 as $t \to \infty$ and for the edges not on the shortest path $x_e(t)$ converges to zero. This assumes that the shortest path P^* from s_0 to s_1 is unique.

In [BBD$^+$13, Theorem 2], the same claim is made for general positive reactivities a_e. We quote.

Theorem 2 ([BBD$^+$13]) *Assume (A1) - (A4) and let $\varepsilon \in (0,1)$ be arbitrary. If $t \geq \ldots$, then $x_e(t) \geq 1 - 2\varepsilon$ for $e \in P^*$ and $x_e \leq \varepsilon$ for $e \notin P^*$.*

Only a proof sketch is given. It is claimed that a key part of the proof in [IJNT11] generalizes. We quote:

Lemma 10 ([BBD$^+$13]) *Assume (A1) to (A2): For $t \geq t_0 \stackrel{\text{def}}{=} (1/a_{\min})\ln(3mX_0)$, there is a nonnegative-non-circulatory flow $f(t)$ with*

$$|f_e(t) - x_e(t)| \leq 5mX_0 e^{-a_{\min} t}.$$

Proof: We follow the analysis in [IJNT11], taking reactivities into account. ∎

In our notes we have the following argument.

$$\frac{d}{ds} x_e e^{a_e s} = \dot{x}_e e^{a_e s} + a_e x_e e^{a_e s} = a_e(q_e - x_e) e^{a_e s} + a_e x_e e^{a_e s} = a_e q_e e^{a_e s}$$

we have

$$x_e(t) e^{a_e t} - x_e(0) = \int_0^t a_e q_e(s) e^{a_e s} ds$$

and hence

$$x_e(t) = x_e(0) e^{-a_e t} + \int_0^t a_e q_e(s) e^{-a_e(t-s)} ds = x_e(0) e^{-a_e t} + (1 - e^{-a_e t}) \int_0^t a_e q_e(s) \frac{e^{-a_e(t-s)}}{1 - e^{-a_e t}} ds.$$

Let

$$\bar{q}_e(t) = \int_0^t a_e q_e(s) \frac{e^{-a_e(t-s)}}{1 - e^{-a_e t}} ds.$$

Since $\int_0^t e^{-a_e(t-s)} ds = (1 - e^{-a_e t})/a_e$, $\bar{q}(t)$ is a convex combination of flows and hence a flow.

This argument is incorrect as was pointed out by Damian Straszak and Nisheeth Vishnoi (personal communication to Kurt Mehlhorn) after inspection of our notes. It is true that for each edge e, $\bar{q}_e$ is a convex combination of the values $q_e(s)$, $s < t$. However, these convex combinations are not uniform over edges as the weight $a_e \frac{e^{-a_e(t-s)}}{1-e^{-a_e t}}$ with which $q_e(s)$ contributes to $q_e(t)$ depends on a_e. Therefore $\bar{q}$ is NOT a convex combination of flows. This invalidates the proof of the Lemma and hence the proof of [BBD+13, Theorem 2].

A correct proof of [BBD+13, Theorem 2] was recently given in [FKKM19]. The proof is not along the lines of the proof for the uniform case in [IJNT11], but introduces a Lyapunov function for (1).

References

[BBD+13] Luca Becchetti, Vincenzo Bonifaci, Michael Dirnberger, Andreas Karrenbauer, and Kurt Mehlhorn. Physarum Can Compute Shortest Paths: Convergence Proofs and Complexity Bounds. In *ICALP*, volume 7966 of *LNCS*, pages 472–483, 2013.

[FKKM19] Enrico Facca, Andreas Karrenbauer, Pavel Kolev, and Kurt Mehlhorn. Convergence of the Non-Uniform Directed Physarum Dynamics. *CoRR*, abs/1906.077811, 2019.

[IJNT11] Kentaro Ito, Anders Johansson, Toshiyuki Nakagaki, and Atsushi Tero. Convergence properties for the Physarum solver. arXiv:1101.5249v1, January 2011.