# The complexity of uniform Nash equilibria and related regular subgraph problems ⋆

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**Abstract**

We investigate the complexity of finding Nash equilibria in which the strategy of each player is uniform on its support set. We show that, even for a restricted class of win-lose bimatrix games, deciding the existence of such uniform equilibria is an **NP**-complete problem. Our proof is graph-theoretical. Motivated by this result, we also give **NP**-completeness results for the problems of finding regular induced subgraphs of large size or regularity, which can be of independent interest.

*Key words:* computational complexity, NP-completeness, uniform Nash equilibrium, regular induced subgraph

## 1 Introduction

The recent interaction between Game Theory and Theoretical Computer Science has led to a deep study of the computational issues underlying basic game theoretic notions. A prominent object of these studies is the hardness of computing Nash equilibria in non-cooperative games [19]. Recent results established evidence of hardness for this problem [6, 10]. Even in the two player case, the best algorithm known has an exponential worst-case running

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time [22]. Furthermore, when one requires equilibria with simple additional properties, the problem immediately becomes **NP**-hard [9, 12].

Motivated by these negative results, recent studies considered the problem of computing other classes of equilibria, such as pure or correlated equilibria [11, 20]. Here we consider *uniform* equilibria, that is, Nash equilibria in which all the strategies played with nonzero probability are played with the same probability. Uniform equilibria can be viewed as falling between pure and mixed Nash equilibria; playing a uniform strategy is arguably the simplest way of mixing pure strategies. Uniform strategies are also easier to implement and thus may be seen as a model for *bounded rationality* [21].

Despite the apparent simplicity of uniform equilibria, we show that even for a very constrained class of games, called *imitation simple bimatrix games* [8], the associated existence problem is **NP**-complete. An imitation simple bimatrix game is a two player game in which the payoffs of both players are in the set $\{0, 1\}$ and the payoff of the row player (the *imitator*) is 1 if and only if he makes the same move as the opponent. We show that it is **NP**-complete to decide if a given imitation simple bimatrix game has a uniform Nash equilibrium [2]. Our proof is essentially graph-theoretical as it relies on a correspondence between equilibrium strategies and some structures in the digraph implicit in the payoff matrix of the column player.

Motivated by this correspondence, we also give **NP**-completeness results for other natural problems concerning regular subgraphs. In particular, we prove that it is **NP**-complete to decide if a graph has an induced regular subgraph of size at least $k$, or if it has an induced regular subgraph of regularity at least $k$, where $k$ is given as input.

The rest of the paper is structured as follows. After discussing related work in Section 1.1, we give the introductory definitions and notation in Section 2. Then, in Section 3, we explain how the game-theoretic hardness result follows from the graph-theoretic result, and we establish the hardness of finding uniform equilibria. The other regular subgraph problems are considered in Section 4.

[2] In a preliminary version of this work we proved a hardness result for the problem of finding a uniform equilibrium with support of size at most (or at least) $k$, where $k$ is given in the input [4]. This result is clearly implied by the result of the present paper.

### 1.1 Related work

As noted above, there has been a considerable amount of work on the complexity of finding mixed Nash equilibria in normal-form games [8–10, 12, 13], culminating in the result by Chen and Deng [6] that finding a mixed equilibrium in a two-player game is a problem complete for the class **PPAD** (defined in Ref. [19]). Also, win-lose games have been shown to be as expressive as general games when one considers mixed equilibria [1].

Pure equilibria in many kind of succinct games have also been studied recently [11, 14]. On the other hand, we are not aware of previous work on uniform equilibria. However, a related result by Lipton et al. [17] is that if we only require an equilibrium that is best response within an accuracy $\epsilon$, then a subexponential algorithm is possible, and the strategies found are uniform on a multiset of size logarithmic in the number of pure strategies. Our setting differs in that we consider strategies uniform on a support *set*, as opposed to a *multiset*, and we do not limit the size of the support.

Our work exploits a connection between uniform equilibria and certain graph structures associated to win-lose games. Similar relations for other classes of equilibria have appeared in other recent works [2, 7, 18].

Problems related to the existence of certain induced subgraphs have been studied in several works, in particular by Lewis [16] and Yannakakis [23]. Notably, these works showed that so-called *hereditary* properties of graphs give rise to **NP**-hard induced subgraph problems (a property is hereditary if it holds for any induced subgraph of $G$ whenever it holds for $G$). However, this result does not apply to the problems we consider, since the property of being a regular subgraph is not hereditary. More recently, the problem of finding large induced subgraphs of *fixed* regularity has been studied by Cardoso et al. [5], who established hardness of the problem, and by Gupta et al. [15], who gave exact exponential-time algorithms that are faster than the naive enumerative approach.

## 2 Definitions and notation

We consider *simple bimatrix games* in normal form. These are specified by two $(0, 1)$ *payoff* matrices $A$ and $B$. The first (resp., second) player is called the *row* (resp., *column*) player. It will be enough to consider $n \times n$ matrices. The rows and columns of both matrices are indexed by the pure strategies of the players. We denote the set of pure strategies of each player by $[n] = \{1, \ldots, n\}$.

A *mixed* strategy is a probability distribution over pure strategies, that is, a vector $x \in \mathbb{R}^n$ such that $\sum_i x_i = 1$ and for every $i \in [n]$, $x_i \geq 0$. The *support* $\mathrm{supp}(x)$ of a mixed strategy $x$ is the set of pure strategies $i$ such that $x_i > 0$. When the row player plays mixed strategy $x$ and the column player plays mixed strategy $y$, their expected payoffs will be, respectively, $x^t Ay$ and $x^t By$. A mixed strategy $x$ is *uniform* if for every $i \in \mathrm{supp}(x)$, $x_i = 1/|\mathrm{supp}(x)|$.

A *Nash equilibrium* of the game $(A, B)$ is a pair of mixed strategies $(x, y)$ from which neither player has an incentive to deviate: for all mixed strategies $\overline{x}$ and $\overline{y}$, $x^t Ay \geq \overline{x}^t Ay$ and $x^t By \geq x^t B\overline{y}$. A *uniform equilibrium* is a Nash equilibrium in which both players play uniform strategies. A *uniform equilibrium strategy* is a uniform strategy played in some uniform equilibrium.

We will consider only *imitation* simple bimatrix games. A bimatrix game is an imitation game if the row player, called the *imitator*, has payoff 1 if he plays the same pure strategy as the opponent, and 0 otherwise. Thus, in an imitation simple bimatrix game matrix $A$ is the identity matrix $I_n$. We will only consider games $(I_n, B)$ where the matrix $B$ is zero along the main diagonal – otherwise a pure equilibrium clearly exists.

We now describe our graph-theoretical notation. Given a digraph $G = (V, E)$, we will use $G(S)$ to denote the subgraph induced by the nodes in the subset $S \subseteq V$. When the digraph $G$ is clear from the context, with slight abuse of notation we will also use $S$ to refer to the induced subgraph $G(S)$. If $v \in V$, as a shorthand for $S \cup \{v\}$ we will write $S + v$. We will use $d^-(v, S)$ to denote the in-degree of $v$ in $G(S)$. In Section 4 we use a similar notation $d(v, S)$ in case of an undirected graph.

## 3 Games and graphs

### *3.1 Uniform equilibria and induced subgraphs*

In this section we formulate our result on uniform equilibria and explain its connection with regular subgraph problems.

Let UNIFORM NASH be the problem of deciding the existence of a uniform Nash equilibrium in an explicitly given imitation simple bimatrix game. Our main result is the following.

**Theorem 1** UNIFORM NASH *is* **NP**-*complete.*

In order to prove Theorem 1, we define certain subgraph structures (Definition 2) and we show that they are tightly related to uniform Nash equilibria in the

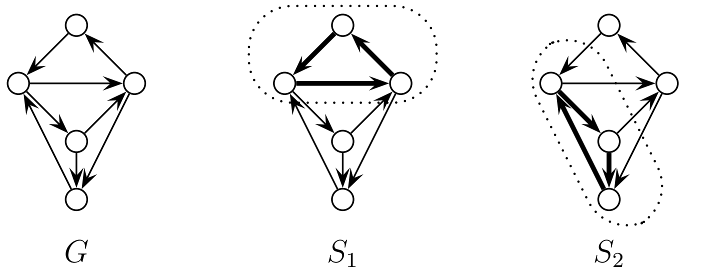

Figure 1. Induced subgraph $S_1$ is a DRIS in $G$, while $S_2$ is not

game associated to a given graph (Lemma 5). We then show that finding such structures in a given graph is an **NP**-complete problem (Lemma 8), from which the **NP**-completeness of UNIFORM NASH will follow.

The graph-theoretic definition we will need in order to prove Theorem 1 is the following.

**Definition 2** *Let $G = (V, E)$ be a digraph. We call a set $S \subseteq V$ a* dominant-regular induced subgraph *(DRIS) of $G$ if there is an integer $r$ such that*

*(i) for all $v \in S$, $d^-(v, S) = r$;*
*(ii) for all $v \in V$, $d^-(v, S + v) \leq r$.*

Figure 1 shows a digraph $G$ and two induced subgraphs, only one of which is a DRIS in $G$.

**Proposition 3** *Given a digraph $G = (V, E)$, $S \subseteq V$ is a DRIS in $G$ if and only if $S$ is a DRIS in $G(T)$ for all $T$ such that $S \subseteq T \subseteq V$.*

**PROOF.** Immediate from Definition 2.

□

**Proposition 4** *Given a DRIS $S$ in a digraph $G$, if $T \subseteq S$ and there is no arc from $S - T$ to $T$ in $G$, then $T$ is a DRIS in $G$.*

**PROOF.** For all $v \in T$, $d^-(v, T) = d^-(v, S)$ by the assumption that no arc can cross the cut $(S - T, T)$. Thus there is an $r$ such that part (i) of Definition 2 holds. Part (ii) holds because $T \subseteq S$, thus $d^-(v, T + v) \leq d^-(v, S + v) \leq r$ for any node $v$.

□

Our proof of Theorem 1 is based on the following Lemma.

**Lemma 5** *Let $\Gamma = (I_n, M)$ be an imitation simple bimatrix game and let $G$ be the digraph whose adjacency matrix is $M$. Then $\Gamma$ has a uniform equilibrium if and only if $G$ has a DRIS.*

**PROOF.** Let $S$ be a dominant-regularity induced subgraph with in-degree $r$. Consider the unique uniform strategy $x$ having support $S$ (that is, $x_i = 1/|S|$ if $i \in S$ and $x_i = 0$ otherwise). We show that $(x, x)$ is a uniform equilibrium. By definition of $x$, $|S|x^t M$ is a row vector whose $i$-th coordinate gives the in-degree of node $i$ in $G(S)$. But then, by definition of a DRIS, $x^t M$ is maximal on coordinates $i \in S$; thus, if the row player plays $x$, the column player has no incentive to deviate from $x$. But if the second player plays $x$, the vector of incentives for the first player is $I_n x = x$ and hence $(x, x)$ is a uniform equilibrium for $\Gamma$.

In the other direction, let $(x, y)$ be a uniform equilibrium. We show that $\mathrm{supp}(x)$ is a DRIS. Since the game is an imitation game, it can be easily verified that the support of $x$ has to be included in the support of $y$. Let $S = \mathrm{supp}(x)$. Since the column player has no incentive to deviate, for every $l \in [n]$ and for every $i \in \mathrm{supp}(y)$, and in particular for every $i \in S$, $(x^t M)_i \geq (x^t M)_l$. Now $|S|(x^t M)_i = \sum_{j \in S} M_{ji}$ so we have

$$d^-(i, S) = \sum_{j \in S} M_{ji} \geq \sum_{j \in S} M_{jl} = d^-(l, S + l)$$

for every $i \in S$ and $l \in N$. Thus $S$ is a dominant-regularity induced subgraph in $G$.

□

Let DOMINANT-REGULAR INDUCED SUBGRAPH be the problem of deciding whether a given digraph admits a DRIS. Then Lemma 5 immediately implies the following.

**Corollary 6** *There is a polynomial-time reduction from* DOMINANT-REGULAR INDUCED SUBGRAPH *to* UNIFORM NASH*.*

Incidentally, we observe that the problem of deciding the existence of a uniform equilibrium has always a positive answer if we consider imitation simple bimatrix games $(I_n, M)$ with $M$ symmetric: any maximal clique in the corresponding undirected graph is a DRIS and so it corresponds to a uniform equilibrium by Lemma 5.

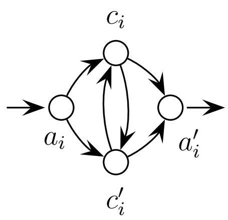


Figure 2. The gadget used in Lemma 7

### *3.2 Hardness of finding dominant-regular subgraphs*

In this section we conclude the proof of Theorem 1 by showing that DOMINANT-REGULAR INDUCED SUBGRAPH is **NP**-complete. The Theorem will then follow by Corollary 6.

We start by observing that in a general digraph the existence of a DRIS is not guaranteed.

**Lemma 7** *For every $k > 0$, there is a graph $G = (V, E)$ with $|V| = 4k$ that has no DRIS and that contains an independent set of size $k$.*

**PROOF.** $G$ is constructed by starting from a cycle on $k$ nodes and then replacing the $i$-th node of the cycle with the 4-node gadget in Figure 3.2. Figure 3.2 shows the resulting graph when $k = 6$.

Clearly the nodes $c_i$ form an independent set of size $k$. To show that no $S \subseteq V$ can be a DRIS in $G$, notice first that the only possible values $r$ could take are 0, 1 and 2. We thus consider three cases.

> Case $r = 0$. It is clearly impossible that $S = V$. But then there exists a node $v \notin S$ with $d^-(v, S + v) > 0$, contradiction.
> Case $r = 1$. In this case it is easy to verify that $S$ should include some $c_i$ or $c'_i$. Assume that $c_i \in S$. Then $c'_i \notin S$, otherwise $d^-(a'_i, S + a'_i) > 1$, violating condition (ii) in Definition 2. So for $c_i$ to have in-degree 1 in $S$, $a_i$ has to be in $S$. But then $d^-(c'_i, S + c'_i) > 1$.
> Case $r = 2$. Since $d^-(a_i, V) = 1$, $S$ does not include any $a_i$. If $S$ includes any node $a'_i$, it should also include its two in-neighbors $c_i, c'_i$, and by the same argument also $a_i$, contradiction. Similarly we get a contradiction if $S$ contains any node $c_i$ or $c'_i$.

□

**Lemma 8** DOMINANT-REGULAR INDUCED SUBGRAPH *is* **NP**-*complete.*

We prove the lemma by reduction from 3SAT. We show that given any 3SAT

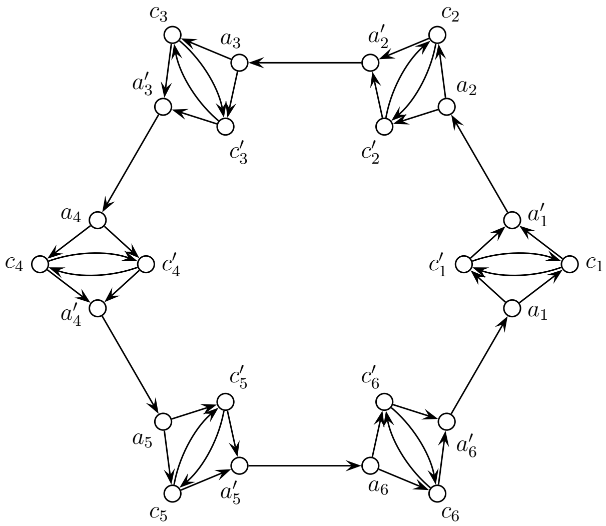


Figure 3. Example of a graph with no DRIS

instance, we can construct a digraph that has a DRIS if and only if the 3SAT instance is satisfiable.

Thus, consider a 3CNF formula $f$ in which, wlog, no clause contains both a variable and its negation. Let the sets of variables and clauses of $f$ be $\{x_1, \ldots, x_n\}$ and $\{c_1, \ldots, c_m\}$ respectively. There is one node in our digraph $G = (V, E)$ for each literal of $f$, and one node for each clause. $G$ also contains an additional node $x_0$. We denote by $X$ the set of nodes corresponding to literals and to $x_0$, and by $C$ the set of nodes corresponding to clauses, so that $V = X \cup C$. Arcs are as follows:

- an arc from each literal node $x_i$ to each other node in $X - \{\overline{x}_i\}$;
- an arc from $x_0$ to each other node in $X$;
- an arc to each node $C_j$ from all the nodes in $X$ except the three corresponding to the literals that form $C_j$.

Figure 4 shows the graph corresponding to a generic 3SAT instance.

We begin by proving the following lemma.

**Lemma 9** *The graph $G$ has the following properties:*

*(i) if $f$ is satisfiable, then $G$ has a DRIS $S \subseteq X$ such that $|S| = n + 1$;*
*(ii) if $S \subseteq X$ is a DRIS in $G$, then $f$ is satisfiable.*

**PROOF.**

(i) Consider a satisfying assignment for $f$. Let $S$ the subset of $X$ corresponding to the literals having value `true` in this assignment, plus the node $x_0$. Note

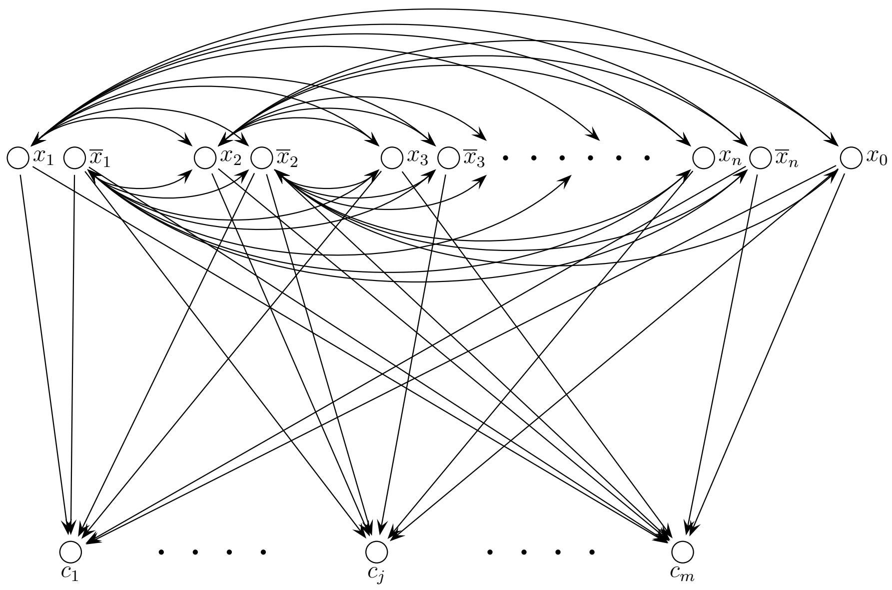


Figure 4. The graph constructed in the proof of Lemma 8

that $|S| = n + 1$, since for each variable $x_i$ only one between $x_i$ and $\overline{x}_i$ can be `true`. Now $S$ is a DRIS in $G$: $d^-(x, S + x) = n$ for each $x \in X$, and $d^-(c, S + c) \leq n$ for all $c \in C$ because at least one of the literals appearing in $c$ is `true`.

(ii) Let $S \subseteq X$ be a DRIS in $G$. $S$ must contain $x_0$, otherwise the second condition in Definition 2 is violated. Thus $d^-(v, S) = d^-(x_0, S)$ for each $v \in S$. It follows that for each $i \in [n]$ *at most* one of the two nodes $x_i, \overline{x}_i$ can be included in $S$: otherwise $x_i$ and $x_0$ would have different in-degree in $S$, violating the first condition in Definition 2. On the other hand, $S$ must contain *at least* one of $x_i, \overline{x}_i$, otherwise

$$d^-(x_i, S + x_i) = |S| > d^-(x_0, S).$$

So indeed $|S| = n + 1$ and the literals corresponding to $S$ define a truth assignment. This assignment satisfies $f$ because for each clause $c$,

$$d^-(c, S + c) \leq d^-(x_0, S) = n,$$

thus there is a node $x \in S$ such that $(x, c)$ is not an arc of $G$, meaning that $c$ must contain at least one `true` literal.

□

Notice that Lemma 9 implies that DOMINANT-REGULAR INDUCED SUBGRAPH is **NP**-hard if we additionally require the DRIS to be contained in some specified subset $X$ of the nodes. To relax this assumption, we enrich our graph $G$ constructed from $f$ as follows: consider the graph $G_0 = (V_0, E_0)$ obtained by applying the construction in Lemma 7 with $k = n$. Let $G'$ be the union of $G$ and $G_0$ (notice that $G$ and $G_0$ share the nodes $c_1, \ldots, c_m$).

**Lemma 10** *The graph $G'$ has the following properties:*

*(i) if $f$ is satisfiable, then $G'$ has a DRIS;*
*(ii) if $G'$ has a DRIS, then $f$ is satisfiable.*

**PROOF.** Part (i) can be proved exactly as Lemma 9(i). To prove (ii), first notice that $G'(V) = G$ and $G'(V_0) = G_0$, as in $G'$ there is no arc of the form $(c_i, c_j)$.

Now let $S$ be a DRIS in $G'$. Then $S \nsubseteq V_0$, otherwise $S$ would also be a DRIS in $G_0$, contradicting Lemma 7. Thus $S \cap X$ is nonempty. Since by construction $G'$ contains no arc from $V \cup V_0 - X$ to $X$, and in particular no arc from $S - (S \cap X)$ to $S \cap X$, by Proposition 4 $S \cap X$ is also a DRIS in $G'$. Then, by Proposition 3, $S \cap X$ is a DRIS in $G'(V) = G$. We can conclude by Lemma 9(ii) that $f$ is satisfiable.

□

Lemma 8 now immediately follows from Lemma 10 and the **NP**-hardness of 3SAT.

## 4 Hardness of other regular subgraph problems

In this section we give **NP**-completeness results for other natural variations of the problem of finding regular induced subgraphs. In particular, we show that both of the following problems are **NP**-complete.

MAXIMUM REGULAR INDUCED SUBGRAPH (MAX-RIS)
*Instance*: a graph $G(V, E)$ and an integer $k$.
*Question*: is there a set $S \subseteq V$ such that $G(S)$ is regular and $|S| \geq k$?

MAXIMUM-REGULARITY INDUCED SUBGRAPH (MAX-RRIS)
*Instance*: a graph $G(V, E)$ and an integer $k$.
*Question*: is there a set $S \subseteq V$ such that $G(S)$ is $r$-regular, for some $r \geq k$?

**Theorem 11** MAX-RIS *is* **NP***-complete.*

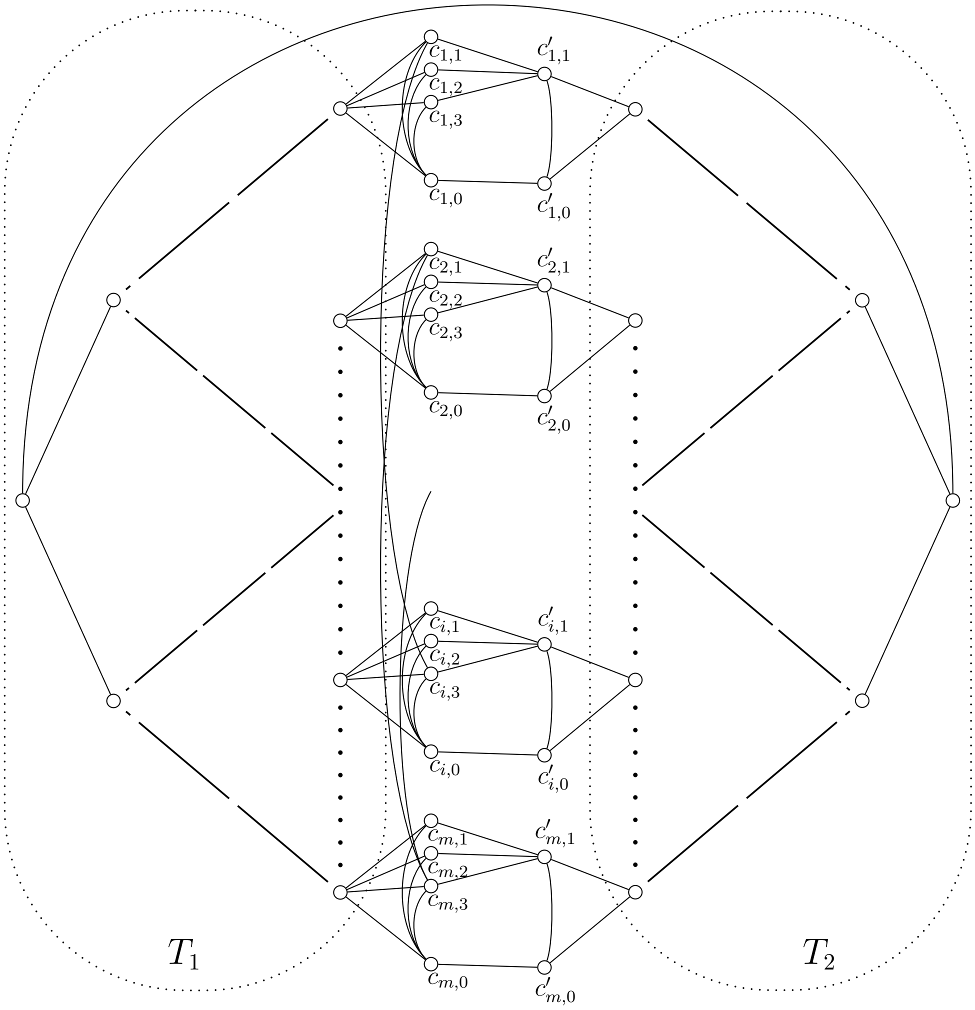


Figure 5. The graph constructed in the proof of Theorem 11

**PROOF.** Consider a generic instance of 3SAT consisting of a formula $f$ with $m$ clauses over $n$ variables; we assume, wlog, that $m = 2^q$ for some integer $q > 1$: note that it is always possible to build such a formula $f'$, satisfiable if and only if $f$ is, by adding at most $m$ copies of a clause of $f$. We create the corresponding instance of MAX-RIS as follows (see Figure 5):

- for each clause $c_i$ we add three nodes, denoted by $c_{i,1}$, $c_{i,2}$ and $c_{i,3}$, one for each literal in $c_i$; we denote by $L$ the set of all these nodes;
- for each clause $c_i$ we also add three auxiliary nodes, denoted by $c_{i,0}$, $c'_{i,0}$ and $c'_{i,1}$;
- for $1 \leq i \leq m$ we connect
  - $c_{i,0}$ with $c_{i,1}$, $c_{i,2}$ and $c_{i,3}$;
  - $c'_{i,1}$ with $c_{i,1}$, $c_{i,2}$ and $c_{i,3}$;
  - $c'_{i,0}$ with $c_{i,0}$ and $c'_{i,1}$;

- we add two binary trees $T_1$ and $T_2$, where $|T_1| = |T_2| = 2m - 1$. Note that both $T_1$ and $T_2$ have $m$ leaves; for $1 \leq i \leq m$ we connect the $i$-th leaf of $T_1$ with $c_{i,0}$, $c_{i,1}$, $c_{i,2}$ and $c_{i,3}$; the $i$-th leaf of $T_2$ is connected with $c'_{i,0}$ and $c'_{i,1}$; finally we connect together the two roots;
- for $1 \leq i, i' \leq m$ and $1 \leq t, t' \leq 3$ we connect $c_{i,t}$ with $c_{i',t'}$ if and only if they correspond to opposed literals;
- we set $k := 8m - 2$.

Note that in the graph defined above, the following nodes have degree 3: all the internal nodes in $T_1$, all the nodes in $T_2$ and the nodes $c'_{i,0}$. We denote by $Q$ the set of all the other nodes, i.e. the nodes $x$ such that $d(x, V) \neq 3$.

We first show that if $f$ is satisfiable the instance of MAX-RIS has a solution. Given a truth assignment function that satisfies $f$, pick, for each clause $c_i$, exactly one `true` literal $c_{i,t}$. Let $S$ include all the nodes in $V - L$ together with the $m$ nodes, in $L$, corresponding to the literals selected in this way. Then $|S| = 8m - 2 = k$, and since the truth assignment satisfies $f$ there can't be an edge between any two nodes in $S \cap L$. Therefore for all $v \in S$ we have that $d(v, S) = 3$, i.e. $G(S)$ is regular.

Now assume that MAX-RIS has a solution $S$, where $G(S)$ is $r$-regular and $|S| \geq 8m - 2$. We show that this implies that $f$ is satisfiable by proving the following points:

(1) $r \geq 3$;
(2) $S \not\subseteq Q$;
(3) $S$ determines a truth assignment that satisfies $f$.

(1) Assume by contradiction that $r \leq 2$; note that at most three nodes between $c_{i,0}$, $c_{i,1}$, $c_{i,2}$, $c_{i,3}$ and the $i$-th leaf of $T_1$ belong to $S$. Furthermore, not all the internal nodes of $T_1$ and the nodes of $T_2$ (remember that the size of both $T_1$ and $T_2$ is $2m-1$) can be included in $S$ because there will be at least two nodes (the roots) with degree 3. It follows that $|S| < 3m + (3m - 2) + 2m = 8m - 2$ (here $2m$ is the total number of nodes $c'_{i,0}$ and $c'_{i,1}$).

(2) If $S = Q$, for every $1 \leq i \leq m$, $d(c_{i,0}, S) = 4$ and $d(c'_{i,1}, S) = 3$, it follows that $G(S)$ is not regular. If $S \subset Q$ then $|S| < 6m \leq 8m - 2$.

(3) From the previous point $|S - Q| > 0$, so there must be $x \in S$ such that $d(x, V) = 3$. Using (1), all the nodes connected to $x$ must belong to $S$, and, in particular, at least one node either in $T_1$ or in $T_2$; due to the recursive structure of the trees all the nodes of $T_1$ and $T_2$ must belong to $S$. Consider now the leaves of $T_2$: their degree must be 3, and this implies that $c'_{i,0} \in S$ and $c'_{i,1} \in S$ for all $1 \leq i \leq m$; similarly, $c_{i,0} \in S$ for all $1 \leq i \leq m$. Hence, of each triple $c_{i,1}$, $c_{i,2}$ and $c_{i,3}$, exactly one node must be in $S$ so that, for any $i$ and any leaf $v$ of $T_1$, $d(v, S) = 3$.

We denote by $L_S$ the set of nodes in $L$ that belong to $S$, i.e. $L_S = S \cap L$. Note that (i) $|L_S| = m$, and there is exactly one node in $L_S$ for each clause, and (ii) there can't be an edge between two nodes in $L_S$, otherwise their degree would be greater than 3. Therefore from $L_S$ we can derive a truth assignment by setting `true` the literals of $f$ corresponding to nodes in $L_S$. The truth assignment is valid because of (ii), while $f$ is satisfied because of (i).

□

**Theorem 12** MAX-RRIS *is* **NP**-*complete.*

**PROOF.** We transform 3SAT to MAX-RRIS. The instance of 3SAT is a formula $f$ with $m$ clauses and $n$ variables and, without loss of generality, we assume that $m \geq 2$ and that there is no clause $c$ that includes two opposite literals. Note that for each formula $f$ there is a formula $f'$, with $m' = 3(m-1)$ clauses and $n' = n+1$ variables, such that $f'$ is satisfiable if and only if $f$ is satisfiable: we build $f'$ by adding $2m - 3$ identical clauses that include only one literal, corresponding to a new variable, repeated three times. We now transform the generic instance of $f'$ into an instance of MAX-RRIS in the following way (see Figure 6):

- for each clause $c_i$ we add
  - three nodes $c_{i,1}$, $c_{i,2}$ and $c_{i,3}$; they represent the literals of the clause, and we refer to all of them, for $1 \leq i \leq m'$, as the set $L$;
  - three nodes $a_i$, $b_i$ and $c_i$; let $A$, $B$ e $C$ denotes, respectively, the set of all the nodes $a_i$, $b_i$ and $c_i$ for $1 \leq i \leq m'$;
  - the following sets of nodes: $A_{i,1}$, $A_{i,2}$, $A_{i,3}$, $B_i$; each of these sets has cardinality equal to $m'$;
- we add two root nodes, denoted by $r_1$ and $r_2$;
- we connect each node in a set $A_{i,1}$, $A_{i,2}$, $A_{i,3}$, $B_i$ with all the nodes in the same set, i.e. all these sets are cliques of size $m'$;
- we connect
  - each node $a_i$ with all the nodes in $A_{i,1} \cup A_{i,2} \cup A_{i,3}$;
  - each node $b_i$ with all the nodes in $B_i$;
  - each node $c_i$ with all the nodes in $B_i$;
  - each node $c_{i,1}$ with all the nodes in $A_{i,1}$;
  - each node $c_{i,2}$ with all the nodes in $A_{i,2}$;
  - each node $c_{i,3}$ with all the nodes in $A_{i,3}$;
- we connect each node $c_i$ with the corresponding nodes $c_{i,1}$, $c_{i,2}$ and $c_{i,3}$;
- we connect $r_1$ with each node $a_i$ and the node $r_2$ with each node $b_i$;
- we connect $r_1$ to $r_2$;
- we connect each occurrence of a literal $c_{i,j}$ with all occurrences $c_{i',j'}$ that represent the opposite literal;
- finally, we let $k := m' + 1$.

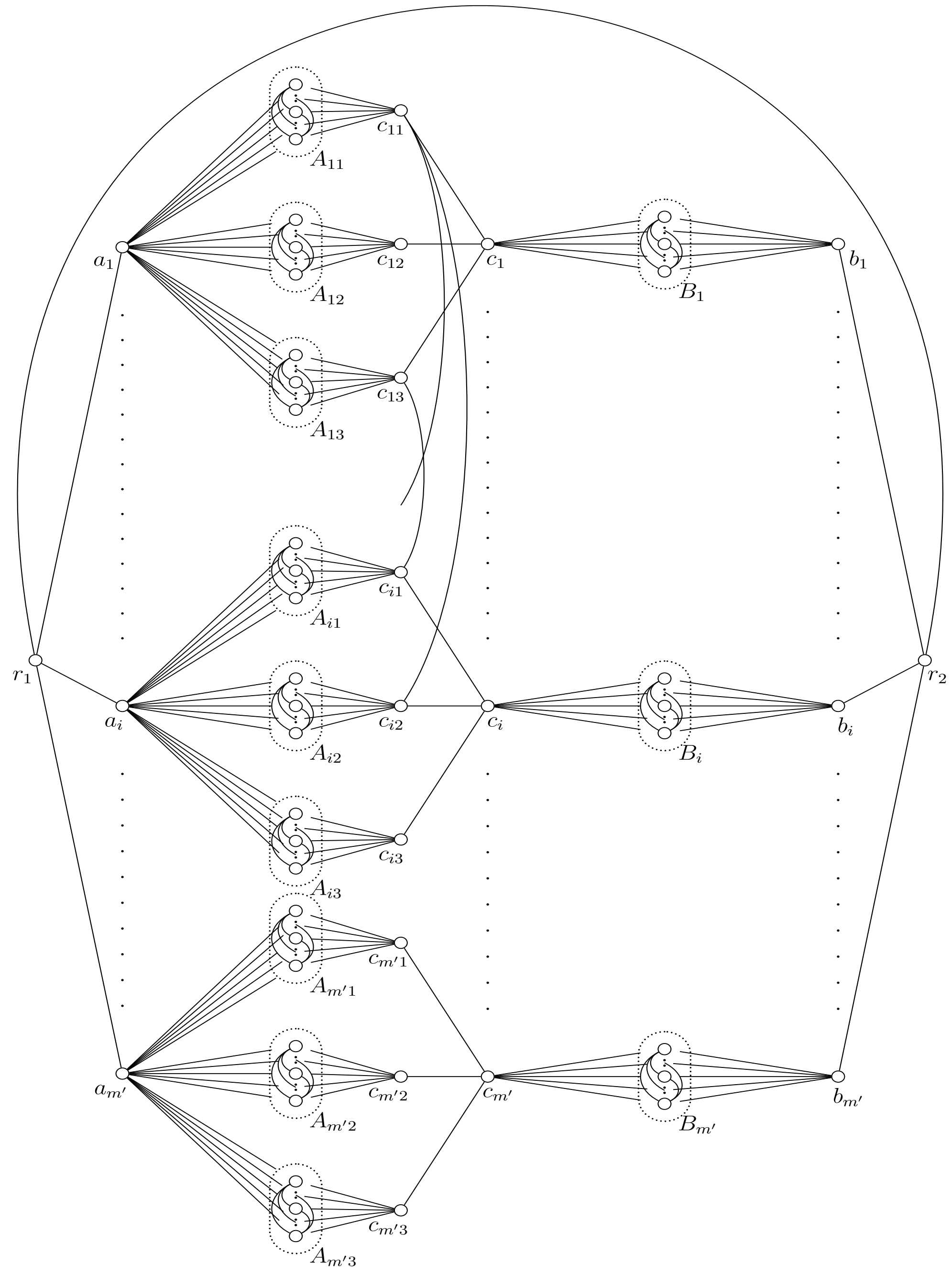


Figure 6. The graph constructed in the proof of Theorem 12

We now show that if $f'$ is satisfiable then there is a solution $S$ to the corresponding instance of MAX-RRIS. Consider a truth assignment satisfying $f'$. We build $S$ in the following way: for each clause we arbitrarily pick one node $c_{i,t}$ that corresponds to a `true` literal, and we also include in $S$ the corresponding set $A_{i,t}$. Additionally, we include for $1 \leq i' \leq m'$ all nodes $a_{i'}$, $b_{i'}$, $c_{i'}$, all the sets $B_{i'}$ and the two roots $r_1$ and $r_2$. It is easy to verify that such a set $S$ induces a $(m'+1)$-regular graph on $G$ (since the nodes $c_{i,t}$ are chosen according to a truth assignment, no pair of nodes in $S \cap L$ is connected by an edge).

Now assume that $S \subseteq V$ is a solution of MAX-RRIS, i.e. it is a set inducing an $r$-regular subgraph with $r \geq m' + 1$. To show that this implies that $f'$ is satisfiable we prove the following points:

(1) $S \nsubseteq L$;
(2) for each set $X \in \{A_{i,1}, A_{i,2}, A_{i,3}, B_i\}$, if $S \cap X \neq \emptyset$, then $X \subset S$;
(3) if $c_i \in S$ then $B_i \subset S$;
(4) if $S \cap A_{i,t} \neq \emptyset$ then $a_i \in S$;
(5) if $a_i \in S \cap A$ then there is exactly one $t \in \{1, 2, 3\}$ such that $A_{i,t} \subset S$;
(6) $S \cap L$ determines a truth assignment that satisfies $f'$.

(1) Each node $c_{i,t}$ is connected to at most $3(m-1)$ nodes in $L$, where $3(m-1) < 3m - 2 = m' + 1$: we built $f'$ from $f$, and only the nodes that correspond to the literals in $f$ ($3m$ literals) can be connected together. If we consider only nodes in $L$ it is not possible to reach the minimum degree $k = m' + 1$.

(2) Note that $X$ is a clique of size $m'$, and every node in $X$ is connected to only two nodes not in $X$. Therefore, if a node $x$ belongs to $S \cap X$, to reach the minimum degree $m' + 1$ all the nodes connected to it must belong to $S$, so that $X \subset S$.

(3) If $c_i \in S$, since $m' + 1 > 3$ there is $\beta \in S \cap B_i$, so by (2) $B_i \subset S$.

(4) Let $\alpha \in S \cap A_{i,t}$. Since $d(\alpha, V) = m' + 1$, to achieve degree $m' + 1$ all the neighbors of $\alpha$ must be in $S$, and in particular $a_i \in S$.

(5) Consider a node $a_i \in S \cap A$; since $m' + 1 > 1$ there must be a node $\alpha \in S \cap A_{i,t}$. From (2) it follows that $A_{i,t} \subset S$. If there were also a node $\alpha' \in S \cap A_{i,t'}$, with $t' \neq t$, then we would similarly have $A_{i,t'} \subset S$. But then the degree of $a_i$ in $S$ would be $d(a_i, S) \geq 2m' > m' + 1 = d(\alpha, S)$ and so the induced subgraph would not be regular; this implies the uniqueness of $A_{i,t}$.

(6) From (1) we know that $S - L \neq \emptyset$. By (2–5) and the fact that $d(r_1, V) = d(r_2, V) = d(b_i, V) = m' + 1$, one can deduce that for each triple $c_{i,1}$, $c_{i,2}$, $c_{i,3}$, there is only one $t$ such that $c_{i,t} \in S$. Moreover this implies $A_{i,t} \subset S$ and $c_i \in S$, so that $G(S)$ must be $m' + 1$ regular and $c_{i,t}$ is only connected to $c_i$ and the set $A_{i,t}$. This means that in $S \cap L$ there cannot be two adjacent nodes, i.e. among the literals in $S \cap L$ there is no pair of opposite literals. Therefore it is possible to assign the value `true` to all the literals whose corresponding nodes are in $S \cap L$; this gives a truth assignment which satisfies $f'$ and, hence, also satisfies $f$.

$\square$

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