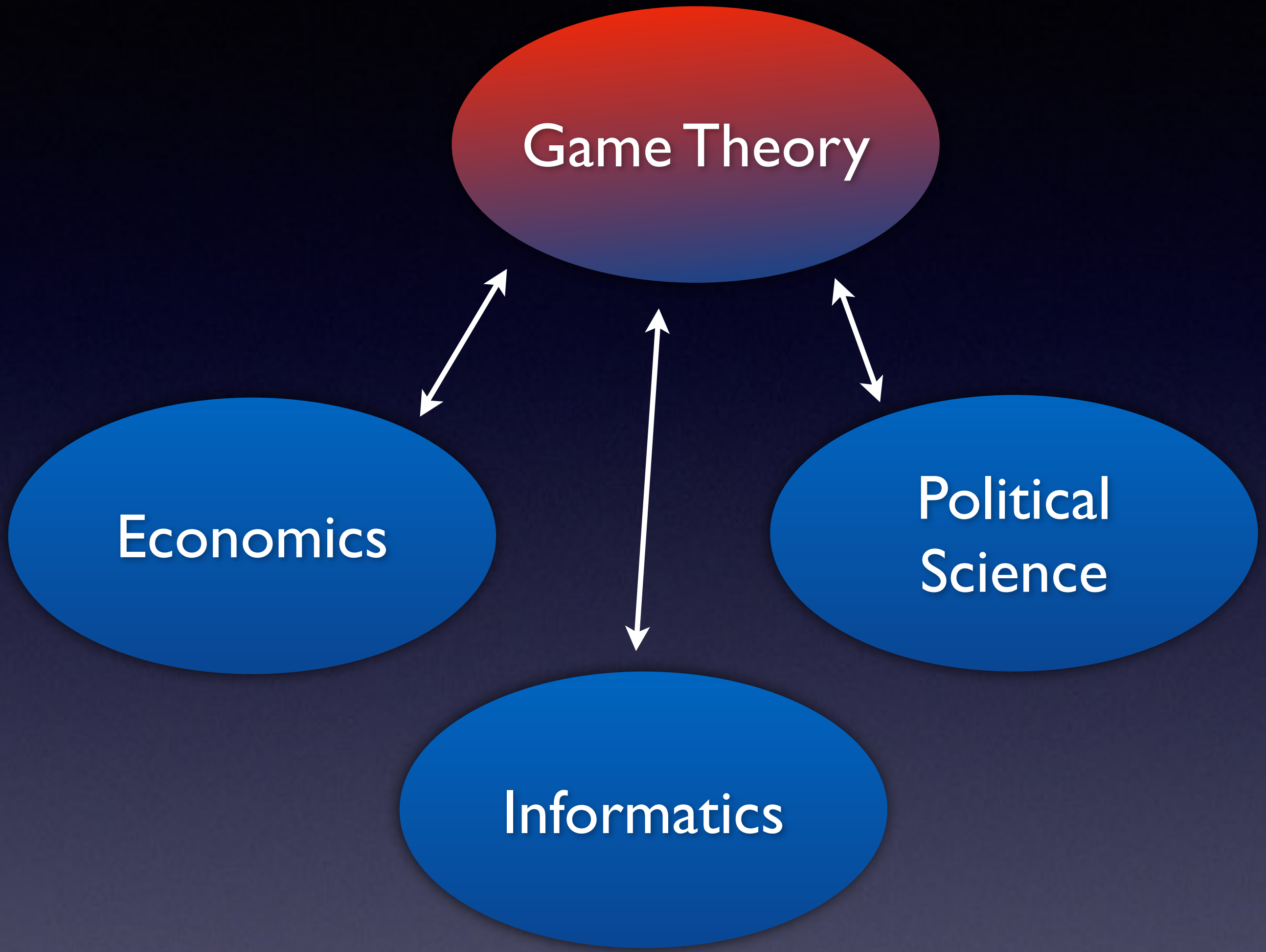
Introduction to Game Theory

What is Game Theory?

- A theory of *interactive decisions*
- Multiple entities - *players* - that need to compete, cooperate, coordinate...
- Every player can choose his *actions*, or *strategies*
- The actions determine the *outcome* of the game

Examples of "Games"

- Games proper:
  - tic-tac-toe, chess...
- Foreign policy
- Financial markets
- Internet and Web interactions
- ...



Example: Rock-Paper-Scissors

- 2 players
- 3 actions per player: Rock, Scissors, Paper
- possible outcomes: victory (+1), tie (0), defeat (-1)
- +1, 0, -1 : *payoff* (or *utility*)

	R	S	P
R	0 0	+1 -1	-1 +1
S	-1 +1	0 0	+1 -1
P	+1 -1	-1 +1	0 0

Normal Form Games

- a set N of *players*; often N = { 1, 2, ..., n }
- for each $i \in N$, a nonempty set S_i
  (*strategies* or *actions* of i)
  - $S = S_1 \times S_2 \times \ldots \times S_n$ is the set of *states*
  - the state determines the outcome of the game
- for each $i \in N$, a function $u_i : S \to \mathbb{R}$
  (*payoff* or *utility function* of i)

What does it mean to analyze a game?

- Identifying *equilibrium points*...
  - choice of "*stable*" actions, that confirm each other
- There are several ways of defining equilibrium ("solution concepts")
- We are not aiming to explain the players' preferences; but to discover what behavior they entail

"*Reason is the slave of the passions*" (D. Hume)

State notation

- Each state s has n components: $s = (s_1, s_2, \ldots, s_i, \ldots, s_n)$
- With the special notation (t_i, r_{-i}) we denote the state $(r_1, r_2, \ldots, t_i, \ldots, r_n)$
  - (t_i, r_{-i}) is the state obtained from r by changing the action of i to t_i
- Example 1: (s_i, s_{-i}) is another way to write state s
- Example 2: (s'_i, s_{-i}) is the state obtained from s by replacing the action of i by s'_i

Prisoner's Dilemma

M. Flood, M. Drescher, A. Tucker (1950)

- 2 suspects, separately interrogated by the police
- Cooperate with police (by naming your accomplice), or stay silent?
  - cooperating implies a discount on time spent in jail
  - if both talk, time in jail will be higher
  - if silent, but indicated as accomplice, time in jail will be maximum

Prisoner's Dilemma: Analysis

- 2 players
- actions:
  stay silent (S),
  confess (C)

	S	C
S	-1 -1	-10 0
C	0 -10	-5 -5

Dominant Strategies

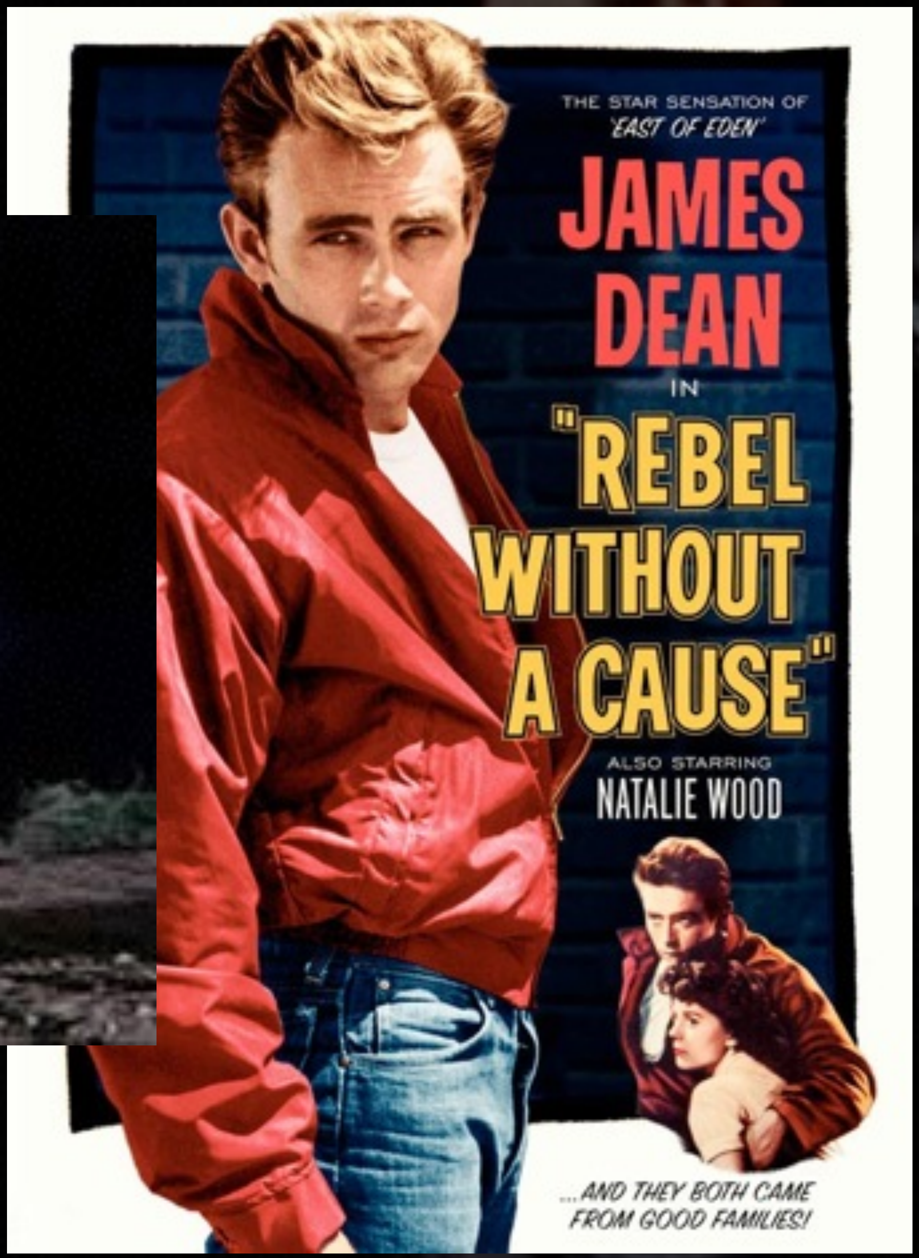
- In Prisoner's Dilemma, Confessing is a *dominant strategy*
- In general, action s_i is a dominant strategy for player i if
  - $u_i(s_i, s'_{-i}) \geq u_i(s'_i, s'_{-i})$ for all $s' \in S$
- In a *dominant strategy equilibrium* (DSE), each player chooses a dominant strategy
- In words:

  The current action (s_i) of each player (i) is at least as good as any alternative (s'_i), irrespective of what the choices of the other players (s'_{-i}) are

DSE Computation

- Suppose a game is given explicitly as input
  - size of input is proportional to |S|
- What is the complexity of finding a DSE ?
- We can enumerate each action of each player
  - For each action, check whether it is dominant
    - Time $O(|S|)$ suffices to check the definition
- Total complexity is $O(n|S|^2)$
- If payoffs are given implicitly, much more difficult!

The Game of "Chicken"



Movie: *Rebel Without a Cause* (1955)

The Game of "Chicken"

- 2 players
- actions: jump (J) or drive on (D)

	J	D
J	0 0	-1 1
D	1 -1	-100 -100

Movie: *Rebel Without a Cause* (1955)

Pure Nash equilibria

- A state $s \in S$ is a *pure Nash equilibrium* (PNE) if:
  - $u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$ for all $i \in N$ and $s'_i \in S_i$
- In words:

  The current action (s_i) of each player (i) is at least as good as any other alternative she has (s'_i), <u>given</u> the actions of the other players (s_{-i})
- In "Chicken", there are two PNE:
  (J, D) and (D, J)
- A DSE is a special type of PNE

PNE Computation

- Suppose a game is given explicitly as input
  - size of input is proportional to |S|
- What is the complexity of finding a PNE ?
- We can enumerate each state of the game
  - For each state, check whether it is a PNE
    - Time O(|S|) suffices to check the definition
- Total complexity is $O(|S|^2)$
- If payoffs are given implicitly, much more difficult!

The Bandwidth Sharing Game

- n users share a common Internet connection
- each decides how much bandwidth he tries to use
- payoff depends on required bandwidth and on free bandwidth (to model latency)
- $N = \{ 1, 2, ..., n \}$
- $S_i = [0, 1]$ (note: infinite set of actions!)
- $u_i(s) = s_i \cdot (1 - \sum_j s_j)$

The Bandwidth Sharing Game

- Let $t = \sum_{j \neq i} s_j \Rightarrow u_i(s) = s_i \cdot (1 - t - s_i)$
- From perspective of player i, t is a constant
- s_i should optimize $u_i(s)$, given t:

$$(\partial/\partial s_i)\, u_i(s) = 0 \quad \Rightarrow \quad 1 - t - 2\, s_i = 0 \quad \Rightarrow \quad s_i = (1-t)/2$$

- So, all s_i are equal $\Rightarrow s_i = (1-(n-1)s_i)/2$
- Solving for s_i, we find $s_i = 1/(n+1)$ (for all i)

Tragedy of the Commons

- We found the equilibrium $s_i = 1/(n+1)$ (for all i)
  - Together, the users almost consume the entire bandwidth!
  - The payoff of each user is $\Theta(1/n^2)$
- If $s_1 = \ldots = s_n = 1/2n$, the payoff of each user would be much higher, $\Theta(1/n)$
- An example of the **Tragedy of the Commons:** users act contrary to the common good

Equilibria

- *Is there always at least one equilibrium point?*
- Rock-Paper-Scissors: no PNE!
- Idea: allow *mixed (random)* actions
- Example:
  - Rock 33,3...%
  - Scissors 33,3...%
  - Paper 33,3...%

	R	S	P
R	0 0	+1 -1	-1 +1
S	-1 +1	0 0	+1 -1
P	+1 -1	-1 +1	0 0

Mixed Strategies and States

- A *mixed strategy* for player i is a probability distribution on the set S_i
  - Function $p_i : S_i \rightarrow [0,1]$ such that $\sum_{s_i \in S_i} p_i(s_i) = 1$
- A *mixed state* is a collection $(p_i)_{i \in N}$ of mixed strategies, one for every player
  - Induces a probability $p(s) = p_1(s_1) \cdot p_2(s_2) \cdot \ldots \cdot p_n(s_n)$ for every pure state of the game
  - Induces an *expected payoff* for player i:

$$u_i(p) = \sum_{s \in S} p(s) \cdot u_i(s)$$

Mixed Nash Equilibria

- A mixed Nash equilibrium (MNE) is a mixed state in which no player can unilaterally improve his expected payoff by switching to a different mixed strategy

- A PNE is a special type of MNE
- For any game G,
  - DSE(G) $\subseteq$ PNE(G) $\subseteq$ MNE(G)
- However, DSE(G) and PNE(G) can be empty sets!

Purely Competitive Games

- Games with *zero sum*:
  the sum of the players' payoffs
  is the same in all outcomes of the game

- Examples:
  - Rock-Paper-Scissors
  - Division of a cake between two persons
    - One person divides it, the other chooses a piece

A Fundamental Contribution

- John von Neumann (1903–1957)
- A father of computing...
- ...and of Game Theory
    - He proved that *every zero-sum game admits an equilibrium* (in 1928 for 2 players; in 1944 for many players)
- Proponent of *mutually assured destruction* doctrine; president of USA committee for intercontinental ballistic missiles

Dangerous Games: Cold War

- Theory of nuclear deterrence
- The "*Doomsday Machine*"

Movie: *Dr. Strangelove* (1964)

1962: Cuban Missile Crisis

- USA President - John F. Kennedy
  URSS Premier - Nikita Krusciov
- USA nuclear missiles in Turkey & Italy
- URSS nuclear missiles in Cuba
- Embargo and diplomatic crisis

Hawks vs Doves

- 2 players
  USA, URSS
- strategies:
  aggressive hawk (H)
  or peaceful dove (D)
- same as the
  "Chicken" game!

	D	H
D	0 0	-1 1
H	1 -1	

Zero-Sum Games when n = 2

- Payoff matrices $A = (a_{ij})$, $-A = (-a_{ij})$ $\in R^{m1 \times m2}$
- In this case we can compute a MNE in polynomial time!
  - How? Linear programming!
- Say P.2 knew that P.1 was playing mixed strategy x
  - P.2 looks at expected payoff vector $x \cdot A$
  - P.2 chooses any column achieving <u>minimum</u> value
- So P.1 can secure himself payoff v if <u>all</u> the entries of $x \cdot A$ are <u>at least</u> v

Zero-Sum Games when n = 2

- In other words, P.1 wants to optimize this linear program:

$$
\begin{aligned}
v^* &= \max v \\
&\sum_{i \in S_1} x_i a_{ij} \geq v \qquad \forall j \in S_2 \\
&\sum_{i \in S_1} x_i = 1 \\
&x_i \geq 0 \qquad \forall i \in S_1.
\end{aligned}
$$

Zero-Sum Games when n = 2

- Similarly, P.2 wants to optimize the linear program:

$$u^* = \min u$$
$$\sum_{j \in S_2} y_j a_{ij} \leq u \qquad \forall i \in S_1$$
$$\sum_{j \in S_2} y_j = 1$$
$$y_j \geq 0 \qquad \forall j \in S_2.$$

- It can be shown that $v^* = u^*$ and the LPs are each the dual of the other! Their solutions yield the equilibrium

Example

$$\max v$$

$$x_1 \cdot 2 + x_2 \cdot 1 \geq v$$

$$x_1 \cdot (-1) + x_2 \cdot 3 \geq v$$

$$x_1 + x_2 = 1$$

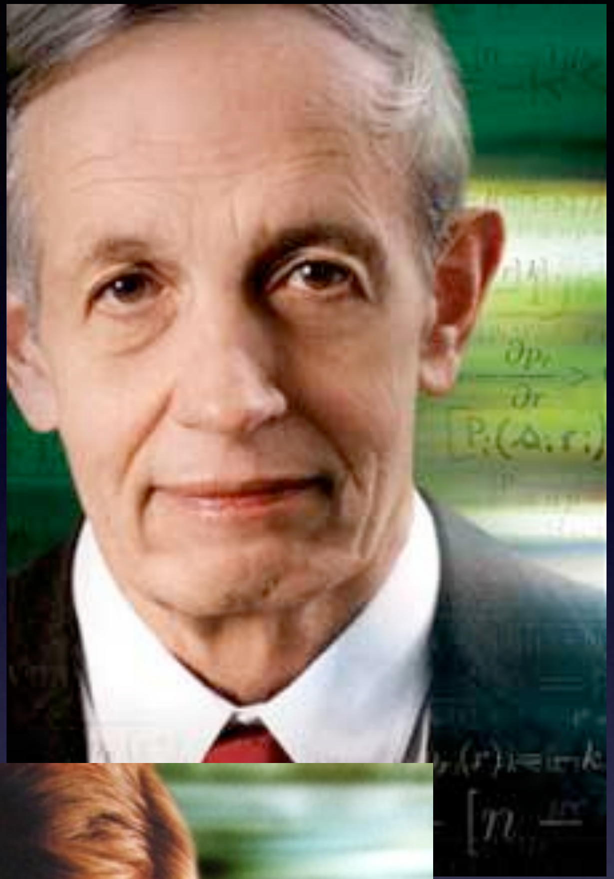
$$x_1, x_2 \geq 0.$$

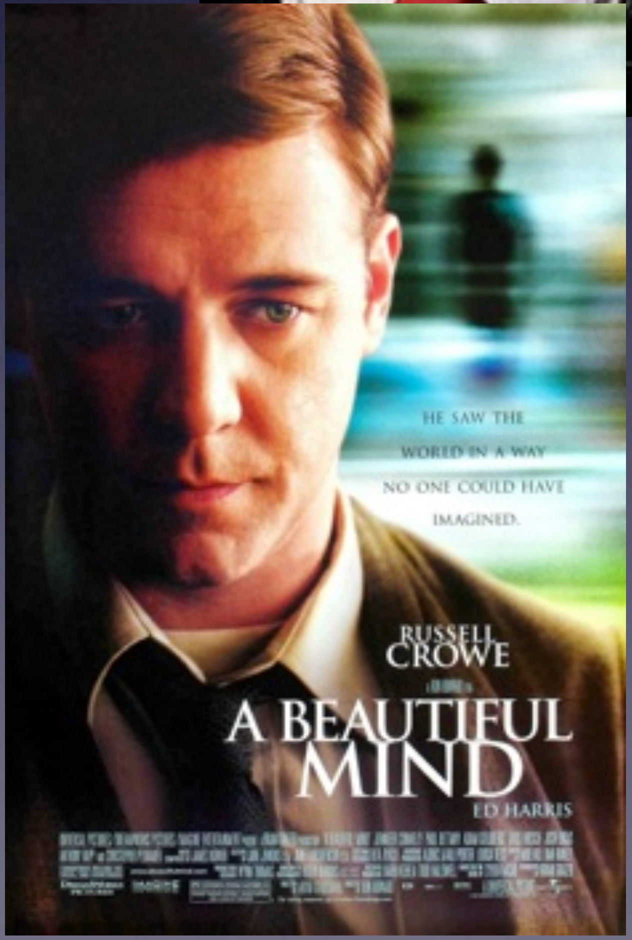
- Solving graphically, we get
  $x_1 = 2/5$,
  $x_2 = 3/5$,
  $v^* = 7/5$

A=(a	action 1	action 2
action 1	2	-1
action 2	1	3

A Beautiful Mind

- *Does an equilibrium always exist in a game?* (whether zero-sum or not)
- John Nash (1928–2015)
- 1949: proved his famous result: *every (finite) game has an equilibrium*
- 1961-1970: admitted for paranoid schizophrenia
- 1994: Nobel prize for economics





Movie: *A Beautiful Mind* (2002)

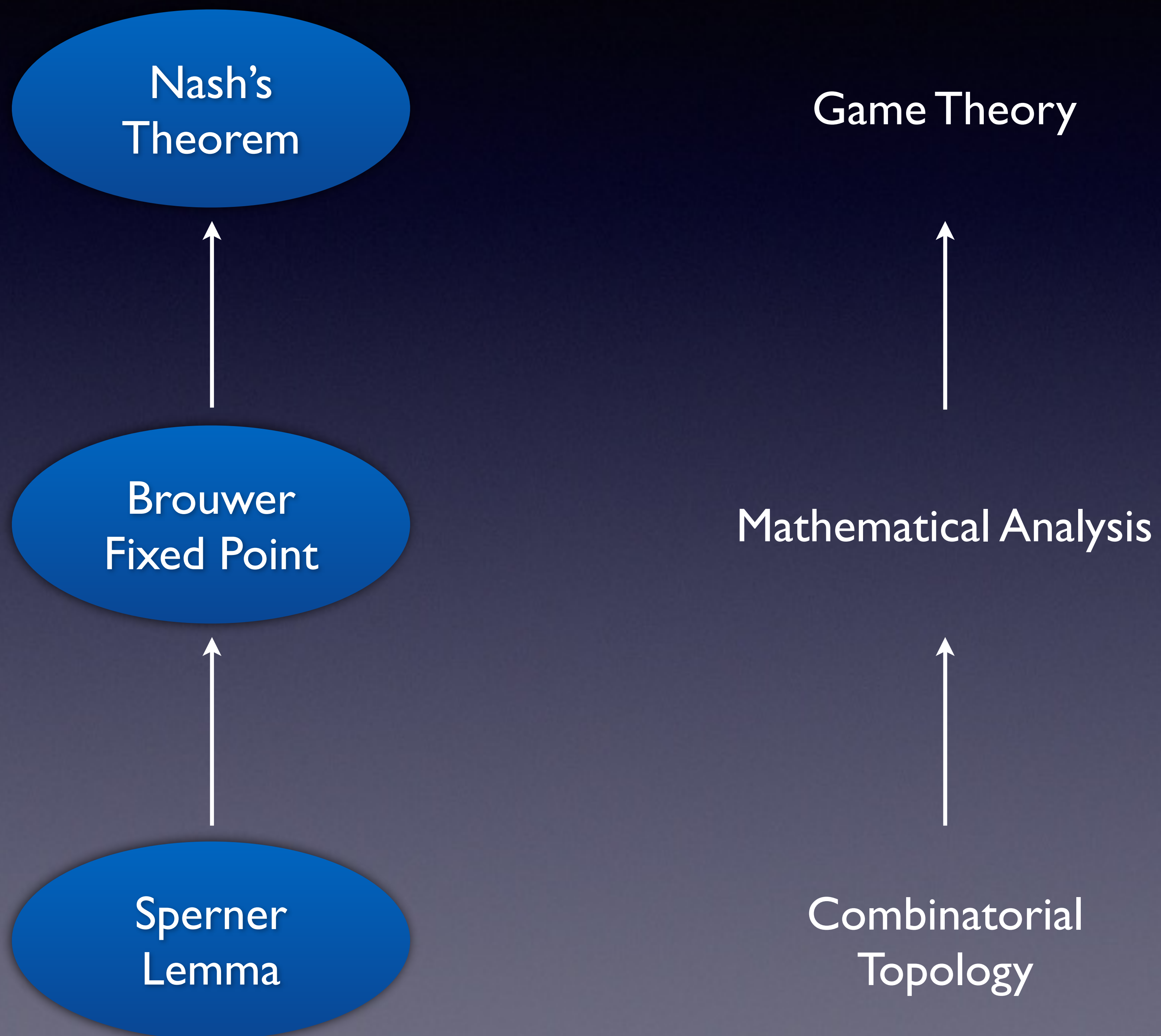
Brouwer's Fixed Point Theorem

- Nash's Theorem is based on Brouwer's Fixed Point Theorem (1910):

  Every continuous function F from a closed disk to itself has a fixed point

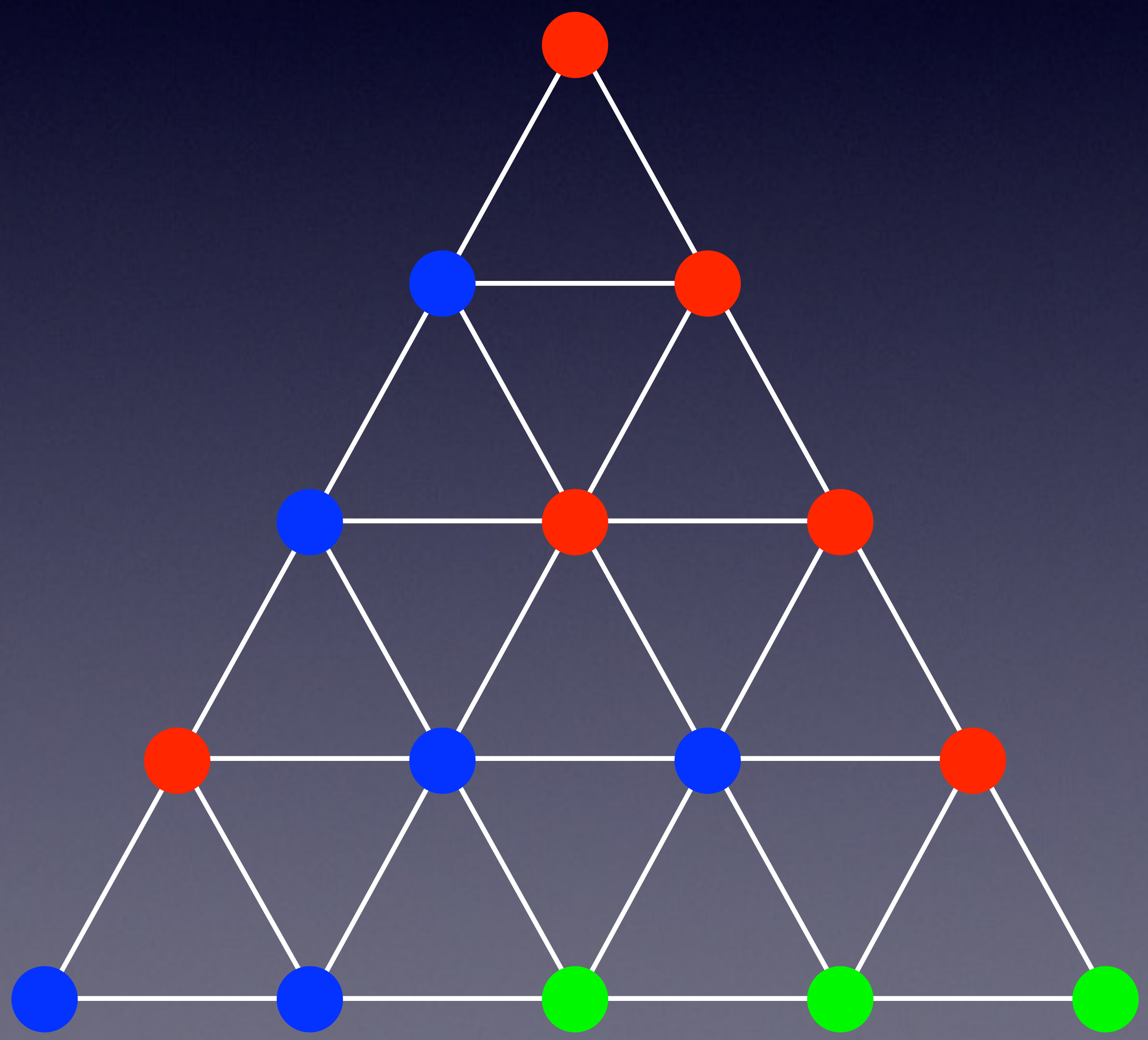
- A fixed point is a vector x such that F(x) = x
- Example: Mixing your coffee
  - x = location of a molecule of coffee before mixing
  - F(x) = location of same molecule after mixing

How Nash’s Theorem is proved



Sperner's Lemma

Every valid coloring of a triangulated triangle has at least one trichromatic cell



Sperner's Lemma

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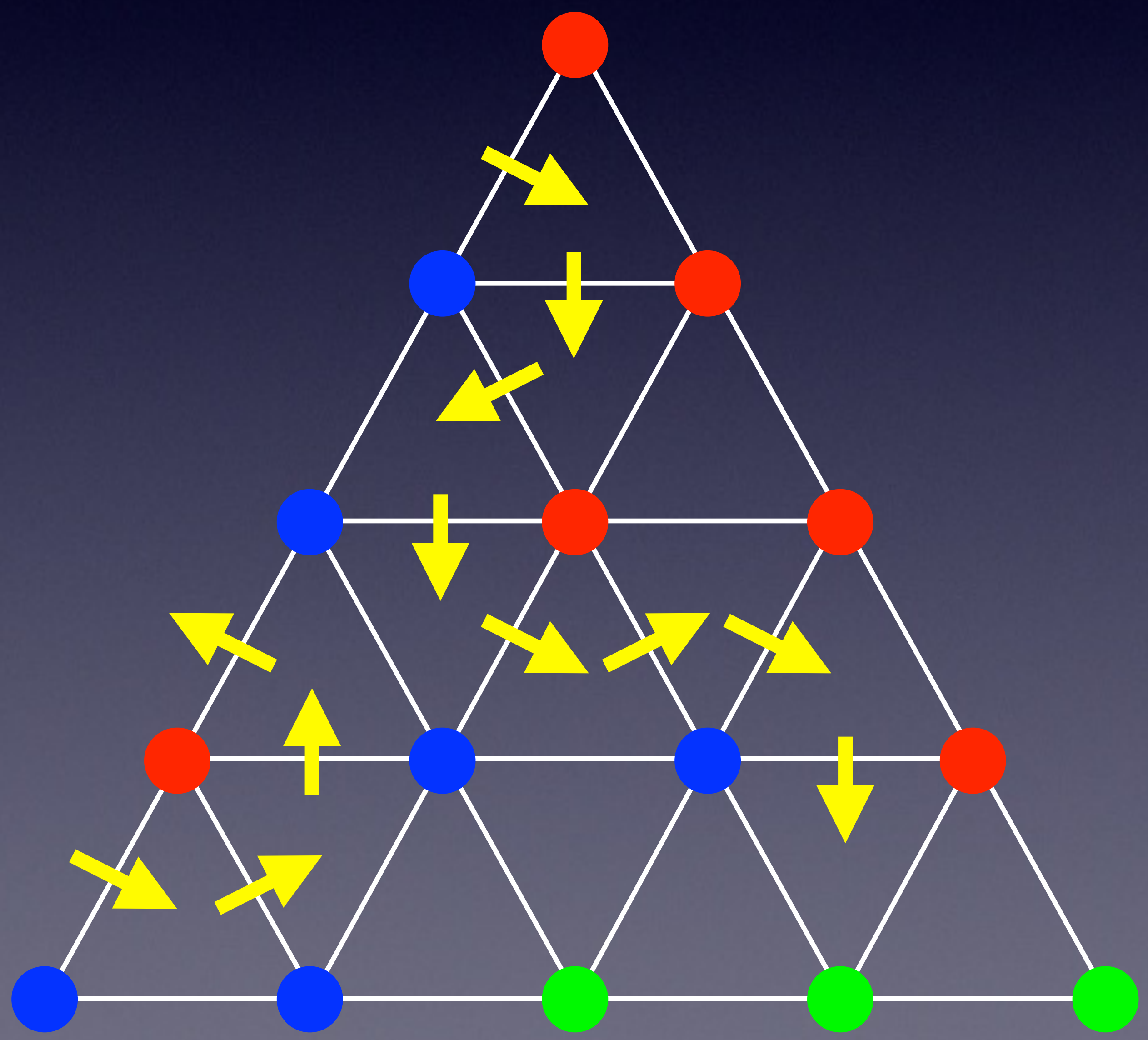
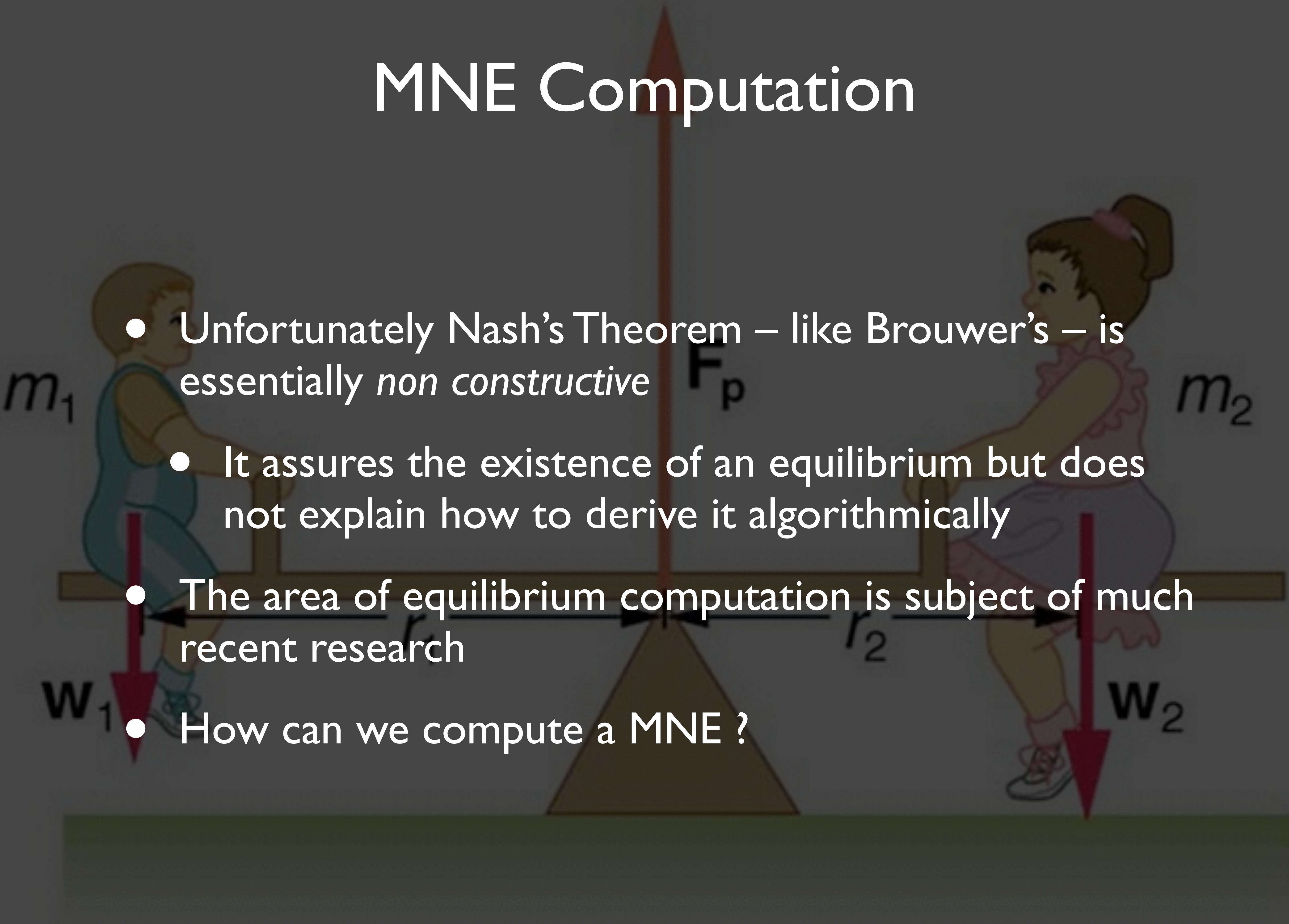


MNE Computation

- Unfortunately Nash's Theorem – like Brouwer's – is essentially *non constructive*
  - It assures the existence of an equilibrium but does not explain how to derive it algorithmically
- The area of equilibrium computation is subject of much recent research
- How can we compute a MNE ?



Best Response

- A mixed strategy p_i is a *best response* to strategies $p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_n$ if for all mixed strategies p'_i of player i,

$$\sum_{s \in S} p_1(s_1) \ldots p_i(s_i) \ldots p_n(s_n) \cdot u_i(s) \geq \sum_{s \in S} p_1(s_1) \ldots p'_i(s_i) \ldots p_n(s_n) u_i(s)$$

- That is, p_i is a maximizer of i's expected payoff
- In a MNE, every player is playing a best response strategy

Support of a Mixed Strategy

- The *support* of mixed strategy p_i is the set of all actions played with nonzero probability:
    - $supp(p_i) = \{ j \in S_i : p_i(j) > 0 \}$
- Example:
    - $p_i = (1/3, 0, 0, 1/2, 1/6)$
    - $supp(p_i) = \{ 1, 4, 5 \}$

Characterization of Best Response

- Theorem:
  A mixed strategy p_i is a best response
  $\Leftrightarrow$ all pure strategies in $supp(p_i)$ are best responses

Computing MNE with given supports

- If n = 2 and we knew the support of a MNE, we could compute the MNE by solving a linear program!
- Say payoff matrices A = (a_{ij}), B = (b_{ij}) $\in R^{m1 \times m2}$
- If we knew support sets I $\subseteq S_1$, J $\subseteq S_2$, we could solve:

$$\sum_{j\in J} y_j a_{kj} \leq \sum_{j\in J} y_j a_{ij} \qquad \forall k \in S_1, \forall i \in I$$

$$\sum_{i\in I} x_i b_{ik} \leq \sum_{i\in I} x_i b_{ij} \qquad \forall k \in S_2, \forall j \in J$$

$$\sum_{i\in I} x_i = 1, \qquad \sum_{i\in J} y_j = 1$$

$$x_i \geq 0 \quad \forall i \in I, \qquad y_j \geq 0 \quad \forall j \in J.$$

An Algorithm for MNE when n = 2

- Of course, we don't know the supports in a MNE
- But we can enumerate them (although inefficiently)
- Try all supports $I \subseteq S_1$, $J \subseteq S_2$ and check if the linear constraints are feasible
- At least one pair of supports will yield a MNE
- Complexity: $2^{m1+m2} \cdot \text{poly}(\text{bits}(A) + \text{bits}(B))$
- Can we do better? (Open problem)

Example: Chicken

- Let's try I = { jump, drive }, J = { jump, drive }
- Variables x_j, x_d, y_j, y_d

$$y_j \cdot 0 + y_d \cdot (-1) = y_j \cdot 1 + y_d \cdot (-100)$$
$$x_j \cdot 0 + x_d \cdot (-1) = x_j \cdot 1 + x_d \cdot (-100)$$
$$y_j + y_d = 1$$
$$x_j + x_d = 1$$
$$x_j, x_d, y_j, y_d \geq 0.$$

- We find a new (non-pure) mixed Nash equilibrium: $x_j = 0.99$, $x_d = 0.01$, $y_j = 0.99$, $y_d = 0.01$

Mechanism Design

- Analysis of a game:
  - Game → analysis → forecast outcomes
- Mechanism design:
  - Desired outcomes → synthesis → game
- Goals:
  - Simplicity of equilibrium strategies
  - Efficiency of resource allocation
  - Fairness
  - Payoff (for the "legislating" authority of the game)

Auctions

- How to sell an object via an auction by mail?
- "First price" auction:
  - sealed-envelope offers
  - highest bid wins and pays corresponding amount
- "Second price" auction (Vickrey auction):
  - sealed-envelope offers
  - highest bid wins, but pays the amount given by *the second highest bid*
- In use since 500 A.C. and 1893 D.C., respectively

Second Price Auction

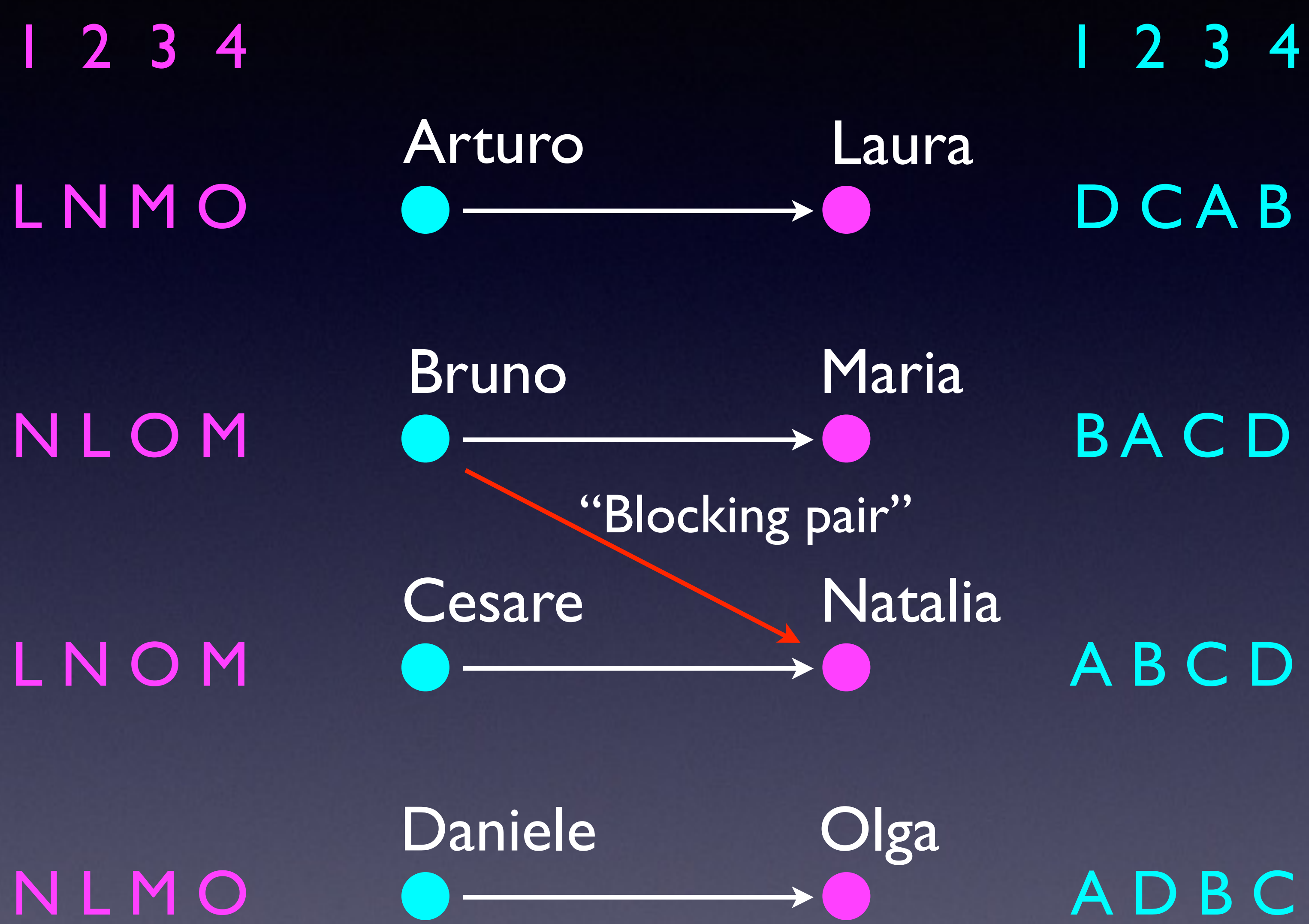
- G1's valuation = 10 €
  G2's valuation = 20 €
- If tied, assign to G1
- Payoff =
  valuation - payment
- Second price auction induces
  efficient and *truthful* outcomes

	€ 10	€ 20
€ 10	0 0	0 10
€ 20	0 0	-10 0

Applications of Auctions

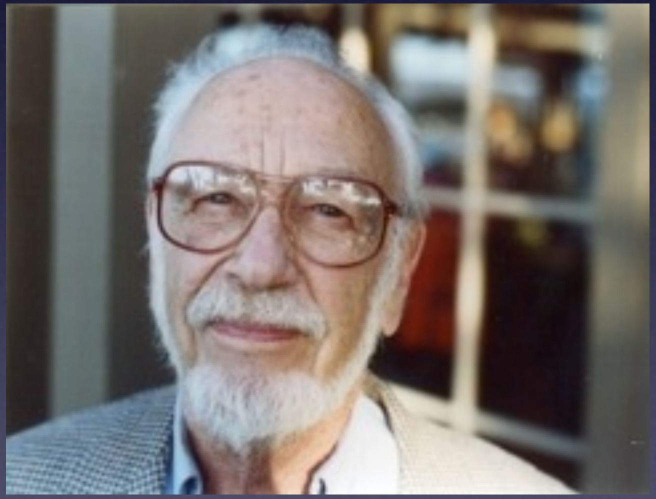
- eBay: online auctions for goods
  - Bidding by proxy (automatic increase)
  - revenue: 16 billion $ (in 2013)
- Google: online auctions for ad slots
  - revenue: 42 billion $ (in 2012)
- National auctions for assigning transmission frequencies

The Stable Marriage Problem



Stable Marriage: history & applications

- Proposed by D. Gale and L. Shapley in 1962

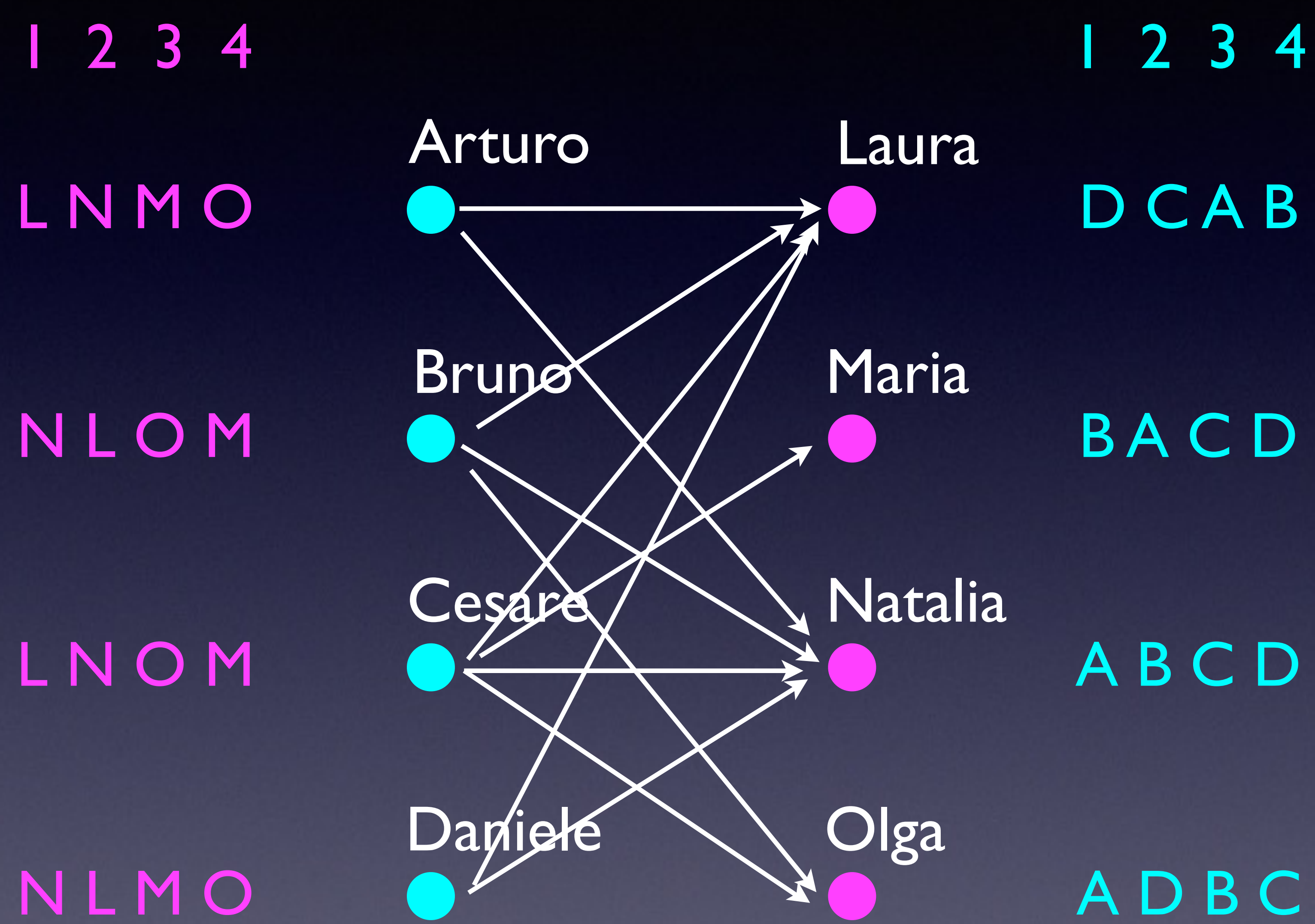


- Matching hospitals and doctors
- Matching universities and students
- Kidney transplants (A. Roth)
- Shapley & Roth (2012): Nobel prize for Economics, in part for this work

How to Arrange Stable Marriages

- Repeat the following iteration until necessary
  - Consider the next single man, i
  - i proposes to the next woman j on his list from whom he has not yet been rejected
  - If j is single or prefers i to her current partner, she accepts (rejecting her partner);
    otherwise she rejects i
  - If the former partner of j is now single, restart

Stable Marriage – Example

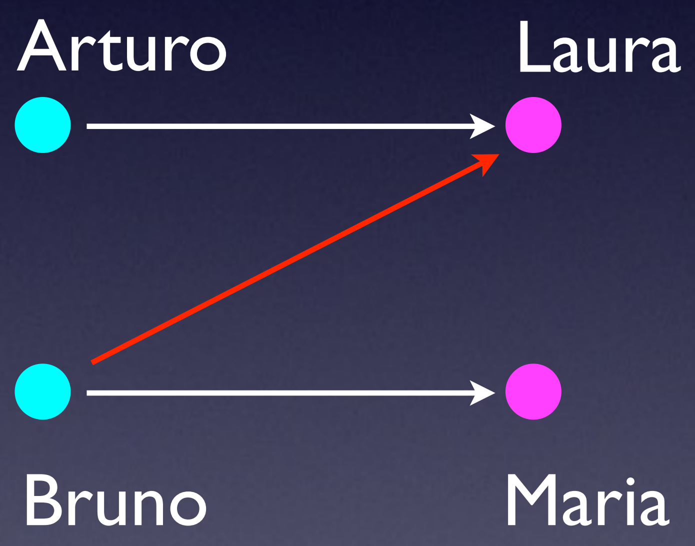


Termination and number of steps

- Once a woman is married, she stays married (her partner can change)
- When the partner of a woman changes, this is to a more preferable partner for her: at most n-1 times
- Every step, either a single woman becomes married, or a married woman changes partner: at most n^2 steps

Stability of the final solution

- Suppose final matching is not stable:



- So, L prefers B to A.
  Two cases:

- When B proposes to L, L has a husband C preferable to B; C is also preferable to A, but in the algorithm women, only get more preferable partners, contradiction.

- When B proposes to L, L is free but B is later replaced by someone preferable to B. Again, L can never end up with A

Graphical Games

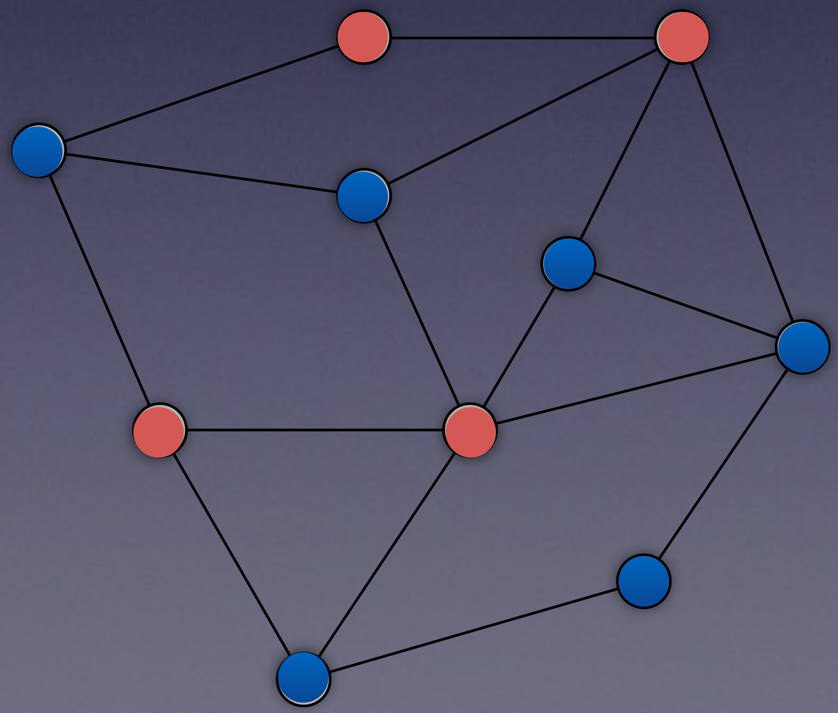
- A representation of multiplayer games that exploits locality of interactions
- Described by an undirected graph G
    - The players of the game are the nodes of G
    - The payoff of a node only depends on its action and the actions of its neighbors
- This representation can be much more compact than the normal form representation

Graphical Games

- A graphical game is a pair (G, M) where G is a graph over {1,2,…,n} and M = (M_1,…,M_n) is a sequence of local game matrices
- Let s^i be the projection of state s onto the players in the neighborhood of i, N(i)
- Each local game matrix specifies the payoff $M_i(s^i)$ for player i, which depends only on the actions of players in N(i)

Party Affiliation Game

- Support either <u>D</u>emocratic or <u>R</u>epublican
- If you (i) and your friend (j) both support same party, you both get +1
- If you (i) and your friend (j) support opposite parties, you both get -1



For Further Reading

- Laszlo Mero
  Calcoli morali. Teoria dei giochi, logica e fragilità umana
  Edizioni Dedalo, 2000

- Laszlo Mero
  Moral Calculations. Game Theory, Logic and Human Frailty
  Springer, 1998