

Approximation Algorithms for Submodular Optimization

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Symmetric Submodular Functions

$$f : 2^V \rightarrow \mathbf{R}$$

$$\text{Symmetric } f(X) = f(V \setminus X), \quad \forall X \subseteq V.$$

Symmetric Submodular Function Minimization

$$\min\{f(X) \mid \emptyset \neq X \neq V\}?$$

$$O(n^3 \gamma) \quad \text{Queyranne (1998)}$$

Minimum Cut Algorithm by MA-ordering

Nagamochi & Ibaraki (1992)

Minimum Degree Ordering

Nagamochi (2007)

Submodular Partition

Minimize
$$\sum_{i=1}^k f(V_i)$$

subject to
$$V = V_1 \cup \dots \cup V_k$$

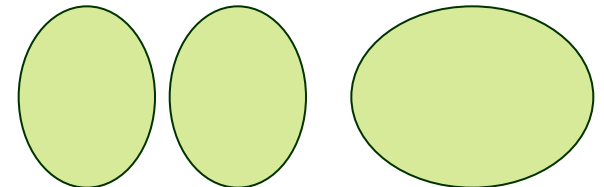
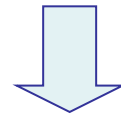
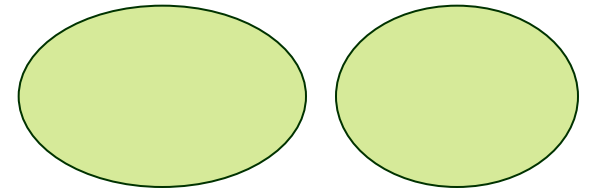
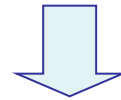
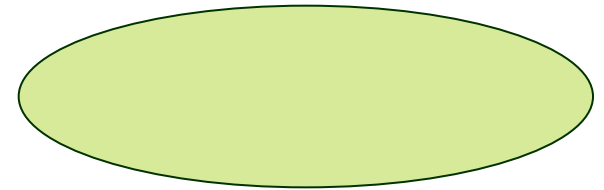
$$V_i \cap V_j = \phi \quad (i \neq j)$$

f : Monotone or Symmetric

$2(1 - 1/k)$ -Approximation

Zhao, Nagamochi, Ibaraki (2005)

Greedy Split



Questions

What Kind of Approximation Algorithms
Can Be Extended to Optimization Problems
with Submodular Cost or Constraints ?

Cf. Submodular Flow (Edmonds & Giles, 1977)

How Can We Exploit Discrete Convexity
in Design of Approximation Algorithms?

Cf. Ellipsoid Method

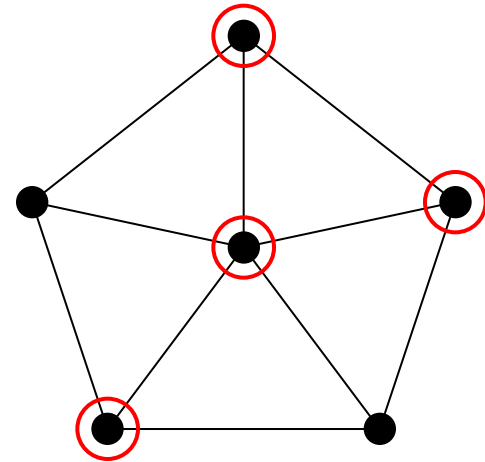
(Grötschel, Lovász & Schrijver, 1981)

Submodular Vertex Cover

Graph $G = (V, E)$

Submodular Function

$$f : 2^V \rightarrow \mathbf{R}_+$$



Find a Vertex Cover $S \subseteq V$ Minimizing $f(S)$

2-Approximation Algorithm

Goel, Karande, Tripathi, Wang (FOCS 2009)

Iwata & Nagano (FOCS 2009)

Relaxation Problem

Convex Programming Relaxation (CPR)

Minimize $\hat{f}(x)$

subject to $x(u) + x(v) \geq 1 \quad (\forall e = (u, v) \in E)$

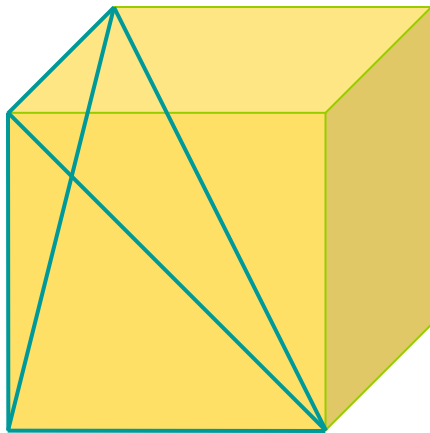
$x(v) \geq 0 \quad (\forall v \in V)$

CPR has a half-integral optimal solution.

→ 2-Approximation Algorithm

Proof of Half-Integrality

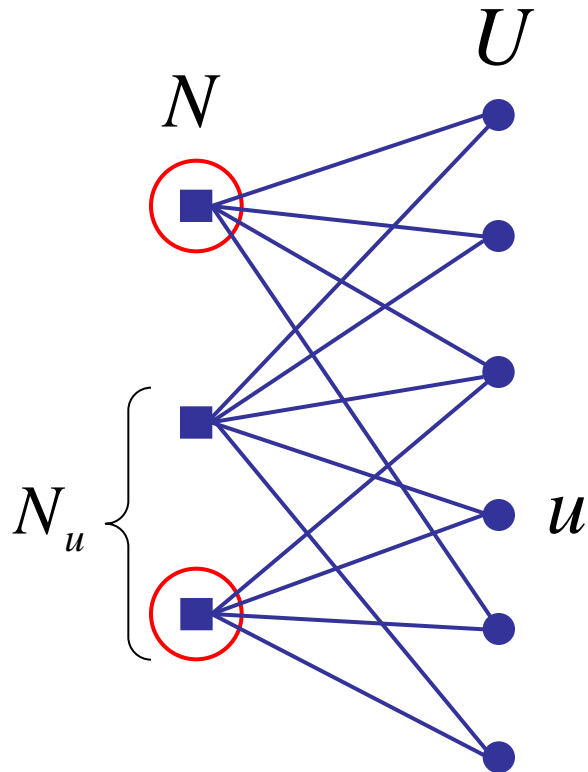
$$\sigma : V \rightarrow \{1, \dots, n\}$$



$$\left(\begin{array}{l} x_u - x_v \geq 0 \quad (\sigma(u) < \sigma(v)) \\ x_u + x_v \geq 1 \quad (\forall e = (u, v) \in E) \\ x_v \geq 0 \quad (\forall v \in V) \end{array} \right)$$

Every nonsingular submatrix
in the coefficient matrix has
a half-integral inverse.

Submodular Cost Set Cover



Find $X \subseteq N$ Covering U with Minimum $f(X)$.

$$\eta := \max_{u \in U} |N_u|$$

η -Approximation

Rounding Algorithm

Primal-Dual Algorithm

Koufogiannakis & Young (ICALP 2009)
Greedy η -Approximation Algorithm for Monotone Submodular Functions

Submodular Multiway Partition

Chekuri & Ene (FOCS 2011)

Minimize
$$\sum_{i=1}^k f(V_i)$$

subject to
$$V = V_1 \cup \dots \cup V_k$$

$$V_i \cap V_j = \phi \quad (i \neq j)$$

$$s_i \in V_i \quad (i = 1, \dots, k)$$

f : Nonnegative Submodular

Relaxation Problem

Convex Programming Relaxation

$$\text{Minimize } \sum_{i=1}^k \hat{f}(x_i)$$

$$\text{subject to } \sum_{i=1}^k x_i(v) = 1 \quad (\forall v \in V)$$

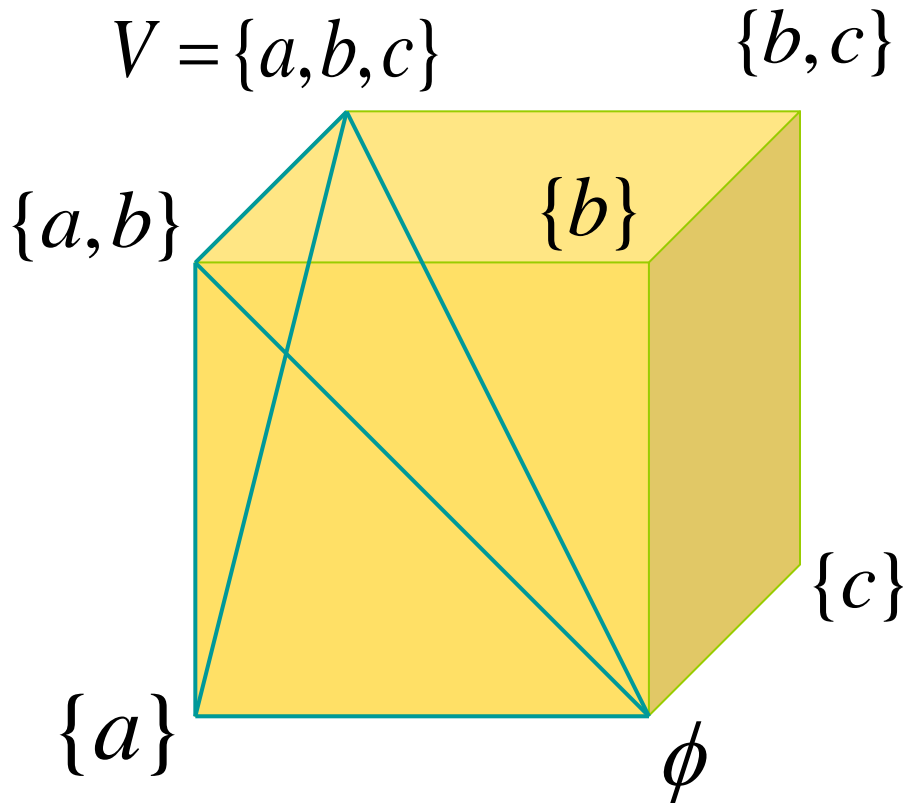
$$x_i(s_i) = 1 \quad (i = 1, \dots, k)$$

$$x_i(v) \geq 0 \quad (i = 1, \dots, k, \forall v \in V)$$

Ellipsoid Method \longrightarrow x_i^* : Optimal Solution

Discrete Convexity

Lovász (1983)



\hat{f} : Linear Interpolation

\hat{f} : Convex



f : Submodular

$\theta \in [0, 1]$: Chosen Uniformly at Random

$X := \{v \mid x(v) \geq \theta\}, \quad \hat{f}(x) = \mathbb{E}[f(X)]$

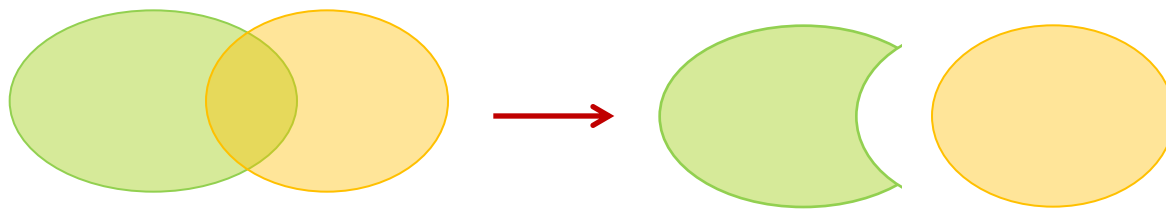
Rounding Scheme

f : Symmetric Submodular

$\theta \in [0,1]$: Chosen Uniformly at Random

$$A_i := \{v \mid x_i(v) \geq \theta\} \quad (i = 1, \dots, k), \quad U := V - \bigcup_{i=1}^k A_i$$

Uncross (A_1, \dots, A_k)



Return $(A_1, \dots, A_{k-1}, A_k \cup U)$

\longrightarrow 3/2-Approximate Solution

Rounding Scheme

f : Nonnegative Submodular

$\delta \in (\frac{1}{2}, 1]$: Chosen Uniformly at Random

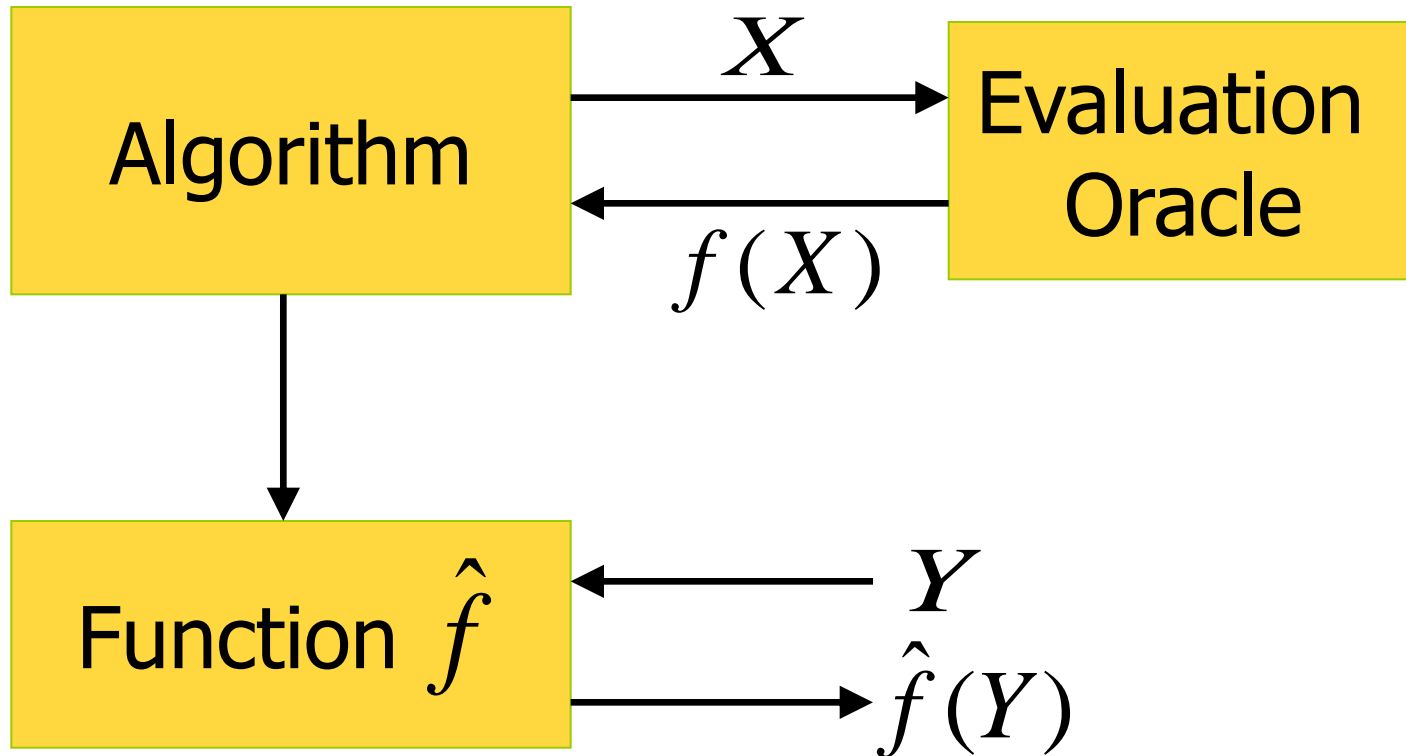
$A_i := \{v \mid x_i(v) \geq \theta\}$ ($i = 1, \dots, k$), $U := V - \bigcup_{i=1}^k A_i$

Return $(A_1, \dots, A_{k-1}, A_k \cup U)$

→ 2-Approximate Solution

Improvement over the $(k - 1)$ -Approximation
by Zhao, Nagamochi, & Ibaraki (2005)

Approximating Submodular Functions



Approximating Submodular Functions

Goemans, Harvey, Iwata & Mirrokni (SODA 2009)

$$f(\phi) = 0, \quad f(X) \geq 0, \quad \forall X \subseteq V.$$

Construct a set function \hat{f} such that

$$\hat{f}(X) \leq f(X) \leq \alpha(n)\hat{f}(X), \quad \forall X \subseteq V.$$

For what function α is this possible?

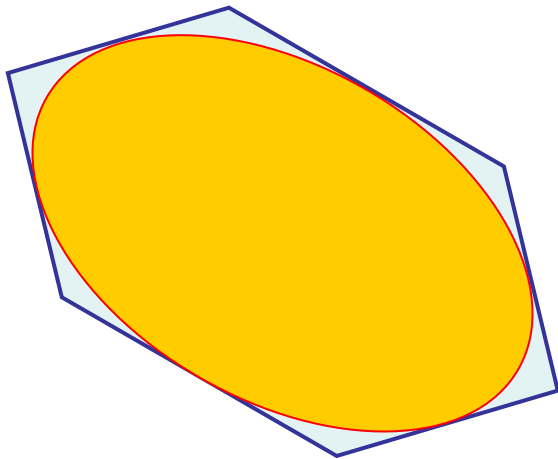
Algorithm with $\alpha(n) = O(\sqrt{n} \log n)$
for monotone submodular functions

Ellipsoidal Approximation

K : Centrally Symmetric Convex Body

$$(x \in K \Rightarrow -x \in K)$$

$E(A)$: Maximum Volume Inscribed Ellipsoid
(The John Ellipsoid)



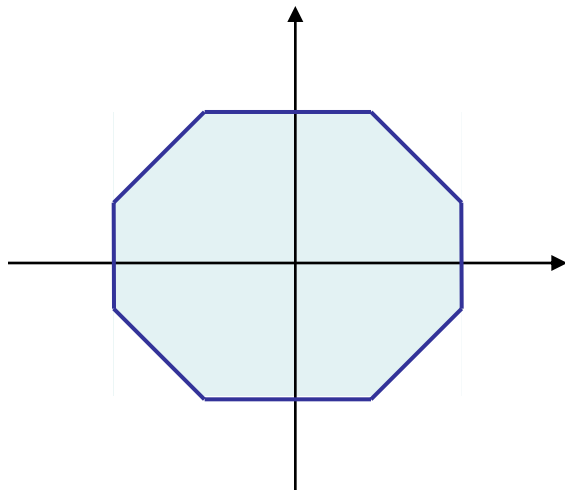
$$E(A) \subseteq K \subseteq \sqrt{n}E(A)$$

Symmetrized Polymatroids

f : Monotone Submodular Function $f(\emptyset) = 0$

$$P(f) = \{x \mid x \in \mathbf{R}_+^V, \sum_{v \in S} x(v) \leq f(S), \forall S \subseteq V\}$$

$$Q(f) = \{x \mid x \in \mathbf{R}^V, \sum_{v \in S} |x(v)| \leq f(S), \forall S \subseteq V\}$$



John Ellipsoid of $Q(f)$

Axis-Aligned $E(D)$

D : Diagonal

Symmetrized Polymatroids

$$E(D) : \text{John Ellipsoid of } Q(f) \quad p(u) = \frac{1}{D_{uu}} \quad (u \in V)$$

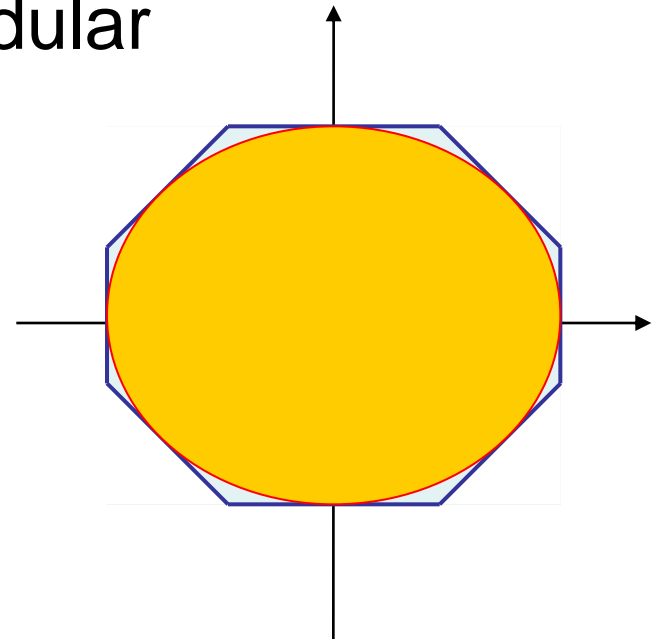
$$\hat{f}(S) := \sqrt{\sum_{u \in S} p(u)} = \max\{ \chi_S y \mid y \in E(D) \}$$

\hat{f} : Monotone Submodular

$$f(S) = \max\{ \chi_S y \mid y \in Q(f) \}$$

$$E(D) \subseteq Q(f) \subseteq \sqrt{n}E(D)$$

$$\hat{f}(S) \leq f(S) \leq \sqrt{n} \hat{f}(S)$$



Submodular Load Balancing

Svitkina & Fleischer (FOCS 2008)

f_1, \dots, f_m : Monotone Submodular Functions

$$\min_{\{V_1, \dots, V_m\}} \max_j f_j(V_j) ?$$

$$f_j(X) := \sum_{v \in X} p_j(v) \longrightarrow \text{Scheduling}$$

2-Approximation Algorithm

Lenstra, Shmoys, Tardos (1990)

$O(\sqrt{n} \log n)$ -Approximation Algorithm