# Submodular Maximization

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**Lecture 2**

**4th Cargese Workshop on Combinatorial Optimization**

### Constrained Submodular Maximization

Family of allowed subsets $\mathcal{M} \subseteq 2^{\mathcal{N}}$.

$$\begin{aligned} \max \quad & f(S) \\ s.t. \quad & S \in \mathcal{M} \end{aligned}$$

# Constrained Maximization - Problem I

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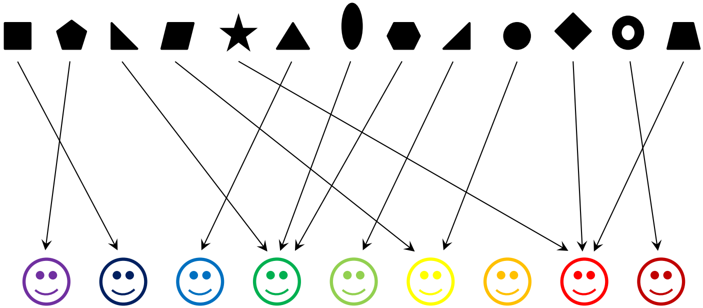

# Constrained Maximization - Problem I (Cont.)

**Problem I - Submodular Welfare**

**Input:**

1. Collection $\mathcal{Q}$ of unsplittable items.
2. $f_i : 2^{\mathcal{Q}} \to \mathcal{R}_+$ monotone submodular utility, $1 \leqslant i \leqslant k$.

**Goal:** Assign all items to maximize social welfare: $\sum_{i=1}^{k} f_i(Q_i)$.

Arises in the context of **combinatorial auctions**. [Lehman-Lehman-Nisan-01]

**Problem II - Submodular Maximization Over a Matroid**

**Input:** Matroid $\mathcal{M} = (\mathcal{N}, \mathcal{I})$ and submodular $f : 2^{\mathcal{N}} \rightarrow \mathcal{R}_+$.
**Goal:** Find $S \in \mathcal{I}$ maximizing $f(S)$.

Case of monotone $f$ captures: Submodular Welfare, Max-$k$-Coverage, Generalized-Assignment ...

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**Combinatorial Approach:**

- Greedy and local search techniques.
- For some cases provides best-known/tight approximations:

| | |
|---|---|
| Knapsack constraint | [Sviridenko-04] |
| intersection of $k$ matroids | [Lee-Sviridenko-Vondrák-09],[Ward-12] |
| $k$-exchange systems | [Feldman-Naor-S-Ward-11] |

**[Nemhauer-Wolsey-Fisher-78]**

Greedy is a $(1/2)$-approximation for maximizing a monotone submodular $f$ over a matroid.

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**Uniform Matroid:**

- Greedy is a $\left(1 - \frac{1}{e}\right)$-approximation [Nemhauser-Wolsey-Fisher-78].
- Captures Max-$k$-Coverage.
- Tight for coverage functions [Feige-98].

# The Greedy Approach

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### Non-monotone $f$ over a matroid:

- $\approx 0.309$-approximation (fractional local search). [Vondrák-09]
- $\approx 0.325$-approximation (simulated annealing). [Gharan-Vondrák-11]
- $\approx 0.478$-hard absolute! [Gharan-Vondrák-11]

**Notation:** $f_S(u) = f(S \cup u) - f(S)$

## Greedy Algorithm

1. $S_0 \leftarrow \emptyset$.
2. **for** $i = 1$ **to** $k$ **do**:
   $u_i \leftarrow argmax_{u \notin S_{i-1}} \{f_{S_{i-1}}(u)\}$.
3. **Return** $S_k$.

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## Theorem [Nemhauer-Wolsey-Fisher-78]

For monotone submodular $f$,

$$f(S_k) \geqslant \left(1 - \left(1 - \frac{1}{k}\right)^k\right) \cdot f(OPT) \geqslant \left(1 - \frac{1}{e}\right) \cdot f(OPT)$$

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## Non-Monotone Submodular Functions

- $1/e$ is best factor (continuous approach via multilinear extension)

## Randomized Greedy Algorithm

1. $S_0 \leftarrow \emptyset$.
2. **for** $i = 1$ **to** $k$ **do**:
   $u_i \leftarrow$ uniformly choose in random an element from $M_i$.
   $S_i \leftarrow S_{i-1} \cup u_i$.
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## How is $M_i$ defined?

$M_i \subseteq \mathcal{N} \setminus S_{i-1}$:

$$\max \sum_{u \in M_i} f_{S_{i-1}}(u) \ \textbf{s.t.} \ |M_i| = k.$$

Assumptions: (w.l.o.g. by adding dummy elements)

- $|\mathcal{N} \setminus S_{i-1}| \geqslant k$
- $\forall u \in \mathcal{N} \setminus S_{i-1}, f_{S_{i-1}}(u) \geqslant 0$

comment: “empty” iteration if a dummy element is chosen.

# Performance of Randomized Greedy

**Theorem** [Buchbinder-Feldman-N-Schwartz-14]

For monotone submodular $f$,

$$\mathbb{E}[f(S_k)] \geqslant \left(1 - \left(1 - \frac{1}{k}\right)^k\right) \cdot f(OPT) \geqslant \left(1 - \frac{1}{e}\right) \cdot f(OPT)$$

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**Theorem** [Buchbinder-Feldman-N-Schwartz-14]

For non-monotone submodular $f$,

$$\mathbb{E}[f(S_k)] \geqslant \left(1 - \frac{1}{k}\right)^k \cdot f(OPT) \geqslant \left(\frac{1}{e}\right) \cdot f(OPT)$$

# Monotone Submodular Functions

**condition on first $i-1$ steps:**

**expected gain at $i$th step:**

$$\mathbb{E}[f_{S_{i-1}}(u_i)] = \frac{1}{k} \cdot \sum_{u \in M_i} f_{S_{i-1}}(u) \geqslant \frac{1}{k} \cdot \sum_{u \in OPT \setminus S_{i-1}} f_{S_{i-1}}(u)$$

$$\geqslant \frac{f(OPT \cup S_{i-1}) - f(S_{i-1})}{k} \geqslant \frac{f(OPT) - f(S_{i-1})}{k}$$

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**taking expectations over all outcomes:**

$$\mathbb{E}[f_{S_{i-1}}(u_i)] \geqslant \frac{f(OPT) - \mathbb{E}[f(S_{i-1})]}{k}$$

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**rearranging:** $(\mathbb{E}[f(S_i)] = \mathbb{E}[f(S_{i-1})] + \mathbb{E}[f_{S_{i-1}}(u_i)])$

$$f(OPT) - \mathbb{E}[f(S_i)] \leqslant \left(1 - \frac{1}{k}\right) \cdot [f(OPT) - \mathbb{E}[f(S_{i-1})]]$$

**implying:**

$$f(OPT) - \mathbb{E}[f(S_i)] \leqslant \left(1 - \frac{1}{k}\right)^i \cdot [f(OPT) - \mathbb{E}[f(S_0)]]$$
$$\leqslant \left(1 - \frac{1}{k}\right)^i \cdot f(OPT)$$

**implying:**

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$$\leqslant \left(1 - \frac{1}{k}\right)^i \cdot f(OPT)$$

**thus:**

$$\mathbb{E}[f(S_k)] \geqslant \left(1 - \left(1 - \frac{1}{k}\right)^k\right) \cdot f(OPT) \geqslant \left(1 - \frac{1}{e}\right) \cdot f(OPT)$$

**completing the proof.**

**condition on first $i-1$ steps:**

**expected gain at $i$th step:**

$$\mathbb{E}[f_{S_{i-1}}(u_i)] = \frac{1}{k} \cdot \sum_{u \in M_i} f_{S_{i-1}}(u) \geqslant \frac{1}{k} \cdot \sum_{u \in OPT \setminus S_{i-1}} f_{S_{i-1}}(u)$$

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**but what is $f(OPT \cup S_{i-1})$ for non-monotone $f$?**

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**but what is $f(OPT \cup S_{i-1})$ for non-monotone $f$?**

**Lemma**

For all $0 \leqslant i \leqslant k$,

$$\mathbb{E}[f(OPT \cup S_i)] \geqslant \left(1 - \frac{1}{k}\right)^i \cdot f(OPT)$$

**proof deferred for now ...**

**taking expectations over all outcomes:**

$$\mathbb{E}[f_{S_{i-1}}(u_i)] \geqslant \mathbb{E}\left[\frac{f(OPT \cup S_{i-1}) - f(S_{i-1})}{k}\right]$$
$$\geqslant \frac{\left(1-\frac{1}{k}\right)^{i-1} \cdot f(OPT) - \mathbb{E}[f(S_{i-1})]}{k}$$

# Non-Monotone Submodular Functions

## taking expectations over all outcomes:

$$\mathbb{E}[f_{S_{i-1}}(u_i)] \geqslant \mathbb{E}\left[\frac{f(OPT \cup S_{i-1}) - f(S_{i-1})}{k}\right]$$
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## it can be proved by induction that:

$$\mathbb{E}[f(S_i)] \geqslant \frac{i}{k} \cdot \left(1 - \frac{1}{k}\right)^{i-1} \cdot f(OPT)$$

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**it can be proved by induction that:**

$$\mathbb{E}[f(S_i)] \geqslant \frac{i}{k} \cdot \left(1 - \frac{1}{k}\right)^{i-1} \cdot f(OPT)$$

**setting $i = k$:**

$$\mathbb{E}[f(S_k)] \geqslant \frac{k}{k} \cdot \left(1 - \frac{1}{k}\right)^{k-1} \cdot f(OPT) \geqslant \frac{1}{e} \cdot f(OPT)$$

**completing the proof**

**we first prove:**

**Lemma** [closely related to Feige-Mirrokni-Vondrak-11]

Let $\mathcal{N}(p)$ be a random subset where each element is chosen with probability at most $p$ (not necessarily independently). Then,

$$\mathbb{E}[f(\mathcal{N}(p))] \geqslant (1-p)f(\varnothing)$$

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Let $\mathcal{N}(p)$ be a random subset where each element is chosen with probability at most $p$ (not necessarily independently). Then,

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**Proof:**

$\mathcal{N}$ is sorted with respect to probability of inclusion in $\mathcal{N}(p)$:

$$\forall i \leqslant j: \quad \Pr[u_i \in \mathcal{N}(p)] \geqslant \Pr[u_j \in \mathcal{N}(p)]$$

Terminology:

- $\mathcal{N}_i = \{u_1, \ldots, u_i\}$
- $p_i$ - probability that $u_i$ is chosen
- $X_i$ - indicator for the event that $u_i$ is chosen

**Thus:**

$$
\begin{aligned}
\mathbb{E}[f(\mathcal{N}(p))] &= \mathbb{E}\left[f(\varnothing) + \sum_{i=1}^{n} X_i \cdot f_{\mathcal{N}_{i-1}\cap\mathcal{N}(p)}(u_i)\right] \\
&\geqslant \mathbb{E}\left[f(\varnothing) + \sum_{i=1}^{n} X_i \cdot f_{\mathcal{N}_{i-1}}(u_i)\right] \quad (\textbf{submodularity}) \\
&= f(\varnothing) + \sum_{i=1}^{n} \mathbb{E}[X_i] \cdot f_{\mathcal{N}_{i-1}}(u_i) \\
&= f(\varnothing) + \sum_{i=1}^{n} p_i \cdot f_{\mathcal{N}_{i-1}}(u_i) \\
&= (1-p_1)\cdot f(\varnothing) + \left[\sum_{i=1}^{n-1} (p_{i-1} - p_i)\cdot f(\mathcal{N}_i)\right] + p_n \cdot f(\mathcal{N}_n) \\
&\geqslant (1-p)\cdot f(\varnothing) \quad (\textbf{since } p \geqslant p_1 \geqslant p_2 \geqslant \ldots \geqslant p_n) \qquad \square
\end{aligned}
$$

**Lemma**

For all $0 \leqslant i \leqslant k$,

$$\mathbb{E}[f(OPT \cup S_i)] \geqslant \left(1 - \frac{1}{k}\right)^i \cdot f(OPT)$$

**observations:**

- $g(S) = f(S \cup OPT)$ is a submodular function
- in iteration $i$, each element of $\mathcal{N} \setminus S_{i-1}$ is **not** chosen to $S_i$ with probability at least $1 - 1/k$
- an element belongs to $S_i$ with probability at most $1 - (1 - 1/k)^i$
- reminder: $\mathbb{E}[g(\mathcal{N}(p))] \geqslant (1 - p)g(\varnothing)$

## Lemma

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- reminder: $\mathbb{E}[g(\mathcal{N}(p))] \geqslant (1 - p)g(\varnothing)$

## completing the proof:

$$\mathbb{E}[f(OPT \cup S_i)] = \mathbb{E}[g(S_i \setminus OPT)] \geqslant \left(1 - \frac{1}{k}\right)^i \cdot g(\varnothing) = \left(1 - \frac{1}{k}\right)^i \cdot f(OPT)$$

## Main Ideas

Random greedy: $|M_i|$ has variable size

- If the marginal values of the additional elements is significant, then the performance improves.
- Otherwise, OPT is "mostly" contained in $M_i$ and then a continuous version of the double greedy algorithm can be used, since $|M_i|$ is $O(k)$.

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Random greedy: $|M_i|$ has variable size

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- Otherwise, OPT is "mostly" contained in $M_i$ and then a continuous version of the double greedy algorithm can be used, since $|M_i|$ is $O(k)$.

## Theorem [Buchbinder-Feldman-N-Schwartz-14]

There is an efficient algorithm that achieves an approximation factor of $\frac{1}{e} + 0.004$ for non-monotone submodular function maximization over a uniform matroid.