Equivariant semidefinite lifts and sum of squares hierarchies

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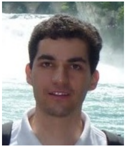


Joint work with **Hamza Fawzi** and **James Saunderson**

Cargese 2014

Question: representability of convex sets

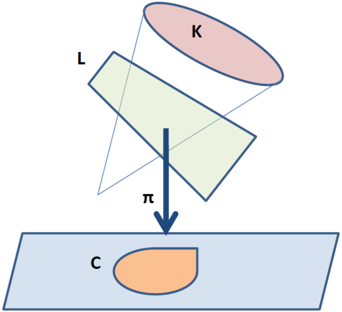
Existence and efficiency:

- When is a convex set representable by conic optimization?
- How to quantify the number of additional variables that are needed?

Given a convex set C, is it possible to represent it as

$$C = \pi(K \cap L)$$

where K is a cone, L is an affine subspace, and π is a linear map?



SDP representations

In full generality, difficult to understand (but we're making progress!)

- Characterized by a Yannakakis-like theorem
- Set C may have many "inequivalent" PSD lifts
- For nonpolyhedral sets, continuity considerations arise
- Constructive techniques (e.g., SOS) have additional properties

Our starting point: "symmetric" (equivariant) lifts.

Lifts and symmetries

A natural requirement: lift should be "symmetric".

- Informally: lift "respects" the symmetries of the convex body C.
- Basic idea: symmetries of C "lift" to symmetries upstairs in $K \cap L$

(Formal definition will follow, examples first!).

Long history: Yannakakis'91, Kaibel-Pashkovich-Theis'10 ("symmetry matters"), Lee-Raghavendra-Steurer-Tan'14 ...

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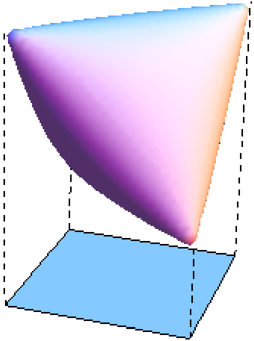
Examples and non-examples (I)

An equivariant psd lift of the square $[-1,1]^2$:

$$[-1,1]^2 = \left\{ (x_1, x_2) \in \mathbb{R}^2 \ : \ \exists u \in \mathbb{R} \ \begin{bmatrix} 1 & x_1 & x_2 \\ x_1 & 1 & u \\ x_2 & u & 1 \end{bmatrix} \succeq 0 \right\}. \tag{1}$$

Square as a projection of the elliptope:



Examples and non-examples (II)

A 3-dimensional hyperboloid:

$$H = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \ : \ x_1, x_2, x_3 \geq 0 \text{ and } x_1 x_2 x_3 \geq 1\}.$$

A non-equivariant psd lift of H of size 6:

$$H = \left\{(x_1, x_2, x_3) \ : \ \exists y, z \geq 0 \quad x_1 x_2 \geq y^2, \ x_3 \geq z^2, \ yz \geq 1\right\}$$
$$= \left\{(x_1, x_2, x_3) \ : \ \exists y, z \quad \begin{bmatrix} x_1 & y \\ y & x_2 \end{bmatrix} \succeq 0, \ \begin{bmatrix} x_3 & z \\ z & 1 \end{bmatrix} \succeq 0, \ \begin{bmatrix} y & 1 \\ 1 & z \end{bmatrix} \succeq 0 \right\}.$$

H is invariant under permutation of coordinates, but the lift does not respect this symmetry (role of variables is different).

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Equivariant lifts

Let $P \subset \mathbb{R}^n$ be a polytope invariant under the action of a group $G \subset GL(\mathbb{R}^n)$, with a lift $P = \pi(\mathcal{S}_+^d \cap L)$.

Definition: The lift is G-equivariant if there is a group homomorphism $\rho : G \to GL(\mathbb{R}^d)$ such that:

1. Subspace L is invariant under conjugation by ρ:

$$Y \in L \qquad \Longrightarrow \qquad \rho(T) Y \rho(T)^T \in L \quad \forall T \in G.$$

2. ρ "intertwines" the lift map

$$\pi(\rho(T) Y \rho(T)^T) = T\pi(Y), \qquad \forall T \in G, \forall Y \in \mathcal{S}_+^d \cap L.$$

Comments

Unlike in the LP case, several slightly different definitions are possible (mainly, affine-equivariance vs. projective-equivariance).

We prefer this one, for a few reasons:

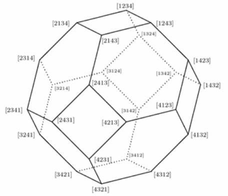
- More natural in affine setting
- Sum of squares hierarchies are intrinsically affine-equivariant
- Consistent with symmetry-reduction techniques for SDP/SOS (e.g., Gatermann-P.'04)

Orbitopes

Special class of convex bodies: orbitopes

$$C = \{\mathrm{conv}(g \cdot x_0) \; : \; g \in G\},$$

where G is a compact group.



Many important examples: hypercubes, hyperspheres, Grassmannians, Birkhoff polytope, permutahedra, parity polytope, cut polytope, . . .

SDP aspects analyzed in Sanyal-Sottile-Sturmfels'11, earlier appearances in Barvinok-Vershik'88, Barvinok-Blekherman'05, etc.

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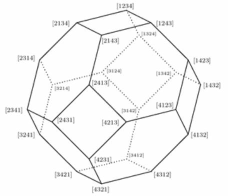

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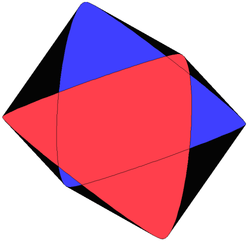
Example: SO(n)-orbitope

Consider $SO(n)$, the group of $n \times n$ matrices with determinant one. This is the orbit of I under $O(n)$ action.

Convex hull is of interest in optimization problems involving rotation matrices.



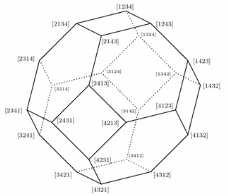
$SO(n)$ orbitope has an SDP representation!

Explicit construction based on the double cover of $SO(n)$ with spin group. (Saunderson-P.-Willsky, arXiv:1403:4914)

Regular orbitopes

Convex hull of a group orbit

$$C = \{\mathrm{conv}(g \cdot x_0) \,:\, g \in G\},$$



An orbitope is regular if the stabilizer of a point is the trivial subgroup. Equivalently, a bijection between group elements and extreme points.

E.g., for the symmetric group S_n (permutahedron), if all entries of x_0 are distinct, then C is regular.

A structure theorem for equivariant lifts

Equivariant lifts of orbitopes are particularly nice.

Why?: Every *equivariant* SDP lift is of *sum of squares* type.

More formally:

Theorem [FSP 13]: Let P be a G-regular orbitope, with a G-equivariant lift of size d. Then for any linear form ℓ, there exist functions $f_j \in V$ such that

$$\ell_{\max} - \ell(x) = \sum_j f_j(x)^2 \qquad \forall x \in X$$

where $X = \mathrm{ext}(P)$, and V is a G-invariant subspace of $\mathcal{F}(X)$, where $\mathcal{F}(X)$ is the space of real-valued functions on X.

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Factorization theorem

Let P be a polytope, with extreme points $X = \text{ext}(P)$ and a PSD lift $P = \pi(\mathcal{S}_+^d \cap L)$.

Recall the generalization of Yannakakis' theorem, characterizing the existence of SDP lifts:

Theeorem [GPT11]: There exists a map $A : X \to \mathcal{S}_+^d$, such that for any facet-defining inequality $\ell(x) \leq \ell_{max}$, there is $B(\ell) \in \mathcal{S}_+^d$ with

$$\ell_{max} - \ell(x) = \langle A(x), B(\ell) \rangle \quad \forall x \in X.$$

Proof sketch

Since orbitope is regular, we can associate a group element $\imath(x) \in G$ to every extreme point. We have then:

$$\begin{aligned} \ell_{\max} - \ell(x) &= \langle A(x) B(\ell) \rangle \\ &= \langle A(\imath(x) \cdot x_0) B(\ell) \rangle \\ &= \langle \rho(\imath(x)) A(x_0) \rho(\imath(x))^T B(\ell) \rangle \\ &= \mathrm{vec}(\rho(\imath(x)))^T \underbrace{(A(x_0) \otimes B(\ell))}_{\text{psd}} \mathrm{vec}(\rho(\imath(x))) \end{aligned}$$

and $\rho(\imath(x)))$ defines a G-invariant subspace of functions on X.

Equivariant lifts are of SOS-type

Theorem [FSP 13]: Let P be a G-regular orbitope, with a G-equivariant lift of size d. Then for any linear form ℓ, there exist functions $f_j \in V$ such that

$$\ell_{\max} - \ell(x) = \sum_j f_j(x)^2 \qquad \forall x \in X$$

where $X = \operatorname{ext}(P)$, and V is a G-invariant subspace of $\mathcal{F}(X)$.

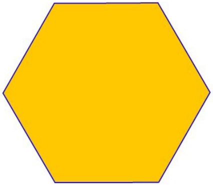
Why is this useful?

Can use representation theory to understand invariant subspaces of $\mathcal{F}(X)$ (isotypic decomposition).

For polytopes, these are finite-dimensional subspaces of polynomials.

Regular polygons as regular orbitopes

A regular polygon in the plane. Invariant under dihedral group (rotations/flips).



Functions on vertices X can be represented by

$$V = V_0 \oplus V_1 \oplus \cdots \oplus V_d$$

where V_k are subspaces of polynomials.

For the case of the square, these invariant subspaces are

$$\{1\}, \{x, y\}, \{xy\}$$

i.e.,

$$\langle(1,1,1,1)\rangle, \quad \langle(1,1,-1,-1),(1,-1,1,-1)\rangle, \quad \langle(1,-1,-1,1)\rangle,$$

Invariant subspaces

Every invariant subspace of $\mathcal{F}(X)$ is a sum of (possibly many) of the V_i.

Thus, the size of every equivariant lift of an orbitope corresponds to the sum of dimensions of the subspaces V_i that appear in V.

Understanding which V_i can (or cannot) appear in an SOS representation, will allow us to produce (or bound) equivariant representations.

Example: For the square, this argument easily yields that no equivariant lift of size 2 can exist.

Regular polygons

For instance, for the regular hexagon we have the invariant subspaces

$$\{\cos t, \sin t\}, \quad \{\cos 2t, \sin 2t\}, \quad \{\cos 3t, \sin 3t\}.$$

Picking the first two, we obtain the SDP lift:

$$\begin{bmatrix} 1 & x & y & t \\ x & (1+r)/2 & s/2 & r \\ y & s/2 & (1-r)/2 & -s \\ t & r & -s & 1 \end{bmatrix} \succeq 0$$

In general, carefully choosing the subspaces V_i can yield equivariant lifts that are exponentially better than "naive" SOS.
(Fawzi-Saunderson-P.'14, Hamza's talk later in the week?)

Lower bounding size of representations

Can use *representation theory* to understand invariant subspaces of $\mathcal{F}(X)$.

For polytopes, these are finite-dimensional subspaces of polynomials. Computing their dimensions, we obtain lower bounds on symmetric representations.

Next, two examples of important polytopes in combinatorial optimization, and a nonpolyhedral orbitope.

Parity polytope

The *parity polytope* PAR_n is the convex hull of all points $x \in \{-1, 1\}^n$ that have an even number of -1.

Theorem [FSP13]: Let PAR_n be the parity polytope. Then, any Γ_{parity}-equivariant psd lift of PAR_n must have size $\geq \binom{n}{\lceil n/4 \rceil}$.

Remark: Weakening the symmetry requirements (e.g., only permutations, or only even sign-flips), PAR_n has polynomial-size LP/SDP lifts.

Cut polytope

The *cut polytope* is defined as

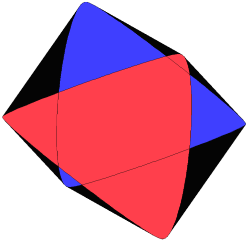
$$CUT_n = \text{conv}(xx^T \; : \; x \in \{-1, 1\}^n).$$

Theorem [FSP13]: Any psd lift of CUT_n that is equivariant with respect to the cube (hyperoctahedral) group must have size $\geq \binom{n}{\lceil n/4 \rceil}$.

Related work in:
J. Lee, P. Raghavendra, D. Steurer, and N. Tan, *On the power of symmetric LP and SDP relaxations*, CCC 2014.

Example: SO(n)-orbitope

Recall the SO(n) orbitope (convex hull of rotation matrices).



Diagonal slice is the parity polytope PAR_n, and can show that we inherit its lower bounds.

As a consequence, the spin-based construction (which is equivariant, and exponential-sized) is optimal!

(Saunderson-P.-Willsky, arXiv:1403:4914)

Summary

- Equivariant lifts of regular orbitopes can be understood
- Structure theorem: all equivariant lifts are of SOS type
- Lower bounds from representation theory

If you want to know more:

- H. Fawzi, J. Saunderson, P.A. Parrilo, Equivariant semidefinite lifts and sum-of-squares hierarchies `arXiv:1312.6662`.
- J. Saunderson, P.A. Parrilo and A. Willsky, Semidefinite descriptions of the convex hull of rotation matrices, `arXiv:1403.4914`.
- H. Fawzi, J. Gouveia, P.A. Parrilo, R. Robinson, R. Thomas, Positive semidefinite rank, `arXiv:1407.4095`.
- J. Gouveia, P.A. Parrilo, R. Thomas, Lifts of convex sets and cone factorizations, *Mathematics of Operations Research*, 38:2, 2013. `arXiv:1111.3164`.

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