Techniques to lower bound extension complexity

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Known lower bounds on extended formulation

	Kaibel, Weltge	Razborov's symmetry arg.	Inform. theory	SA + Fourier
COR/ TSP	yes [KW'13]	yes [FMPTdW'11]	yes [BM12+BP13]	?
approx. COR	?	yes [BFPS'12]	yes [BM12+BP13]	?
matching	?	yes [R'13]	yes [BP'14]	?
approx CSPs	?	?	?	yes [CLRS'13]

Extended formulation

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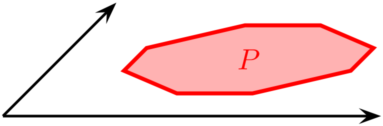

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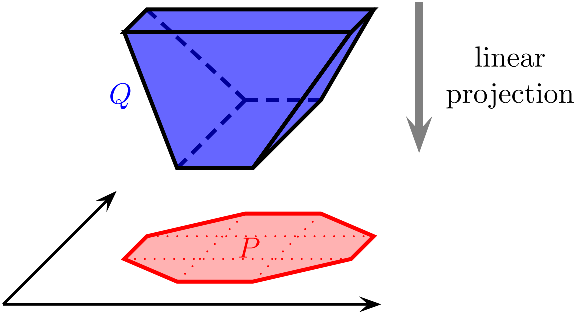
- Write $P = \{x \in \mathbb{R}^n \mid \exists y : Bx + Cy \leq d\}$



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  $\rightarrow$ many inequalities
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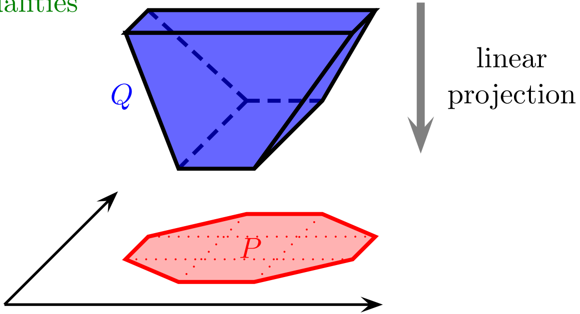

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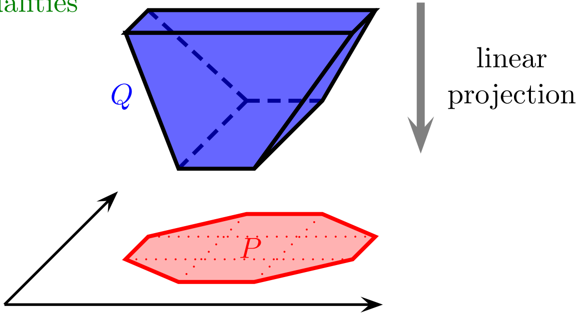
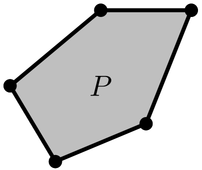


- The **extension complexity** of P is

$$\mathrm{xc}(P) := \min \left\{ \#\text{facets of } Q \;\middle|\; \begin{array}{c} Q \text{ polyhedron} \\ p \text{ linear map} \\ p(Q) = P \end{array} \right\}$$

Slack-matrix

Write: $P = \mathrm{conv}(\{x_1, \ldots, x_v\}) = \{x \in \mathbb{R}^n \mid Ax \le b\}$



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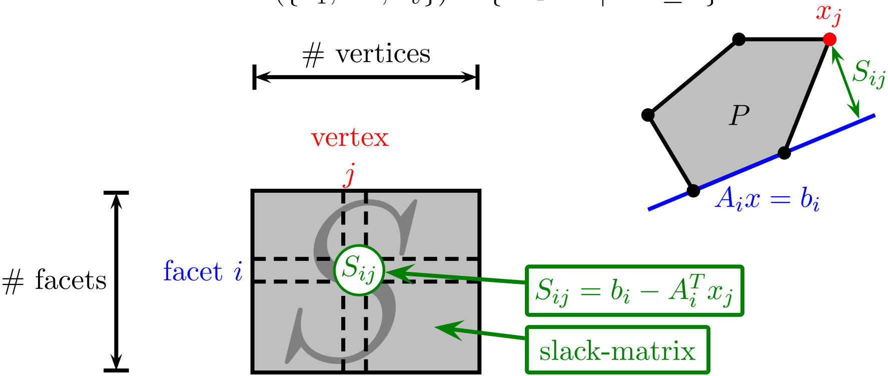

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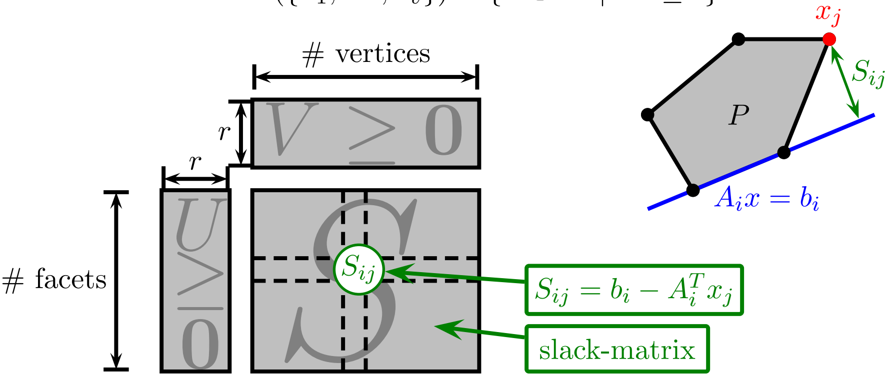

Non-negative rank:

$$\mathrm{rk}_+(S) = \min\{r \mid \exists U \in \mathbb{R}^{f \times r}_{\geq 0}, V \in \mathbb{R}^{r \times v}_{\geq 0} : S = UV\}$$

Yannakakis' Theorem

Theorem (Yannakakis '88)

If S is the **slack-matrix** for $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$, then $\mathrm{xc}(P) = \mathrm{rk}_+(S)$.

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- $A_i x > b_i \Longrightarrow A_i x + \underbrace{U_i y}_{\geq 0} > b_i$.

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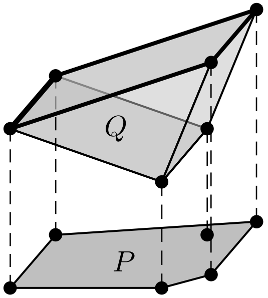

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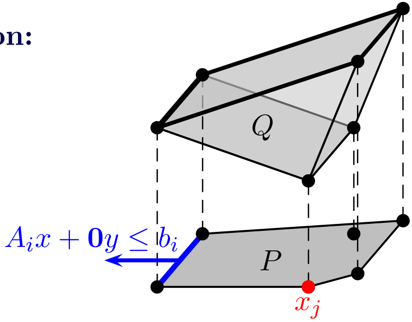

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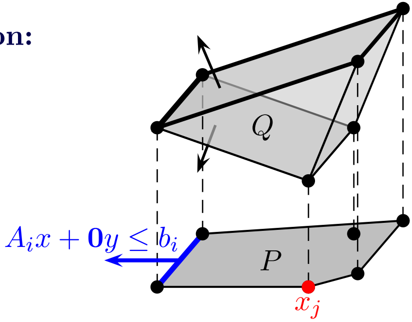

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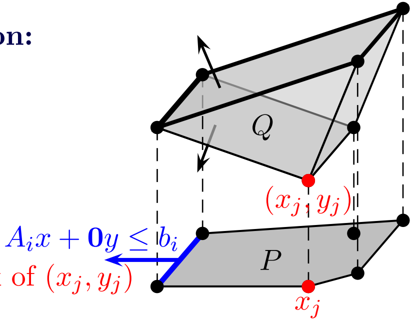

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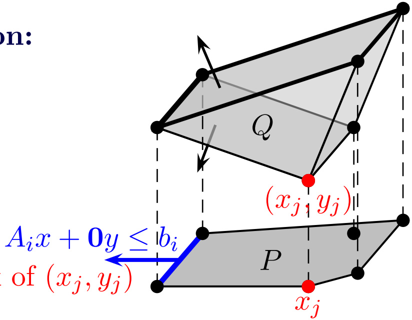

$$\langle u(i), v(j) \rangle = \underbrace{u(i)^T d}_{=b_i} - \underbrace{u(i)B}_{=A_i} x_j - \underbrace{u(i)C}_{=\mathbf{0}} y_j = S_{ij}$$

Rectangle covering lower bound

Observation

$\mathrm{rk}_+(S) \geq$ rectangle-covering-number(S).

Rectangle covering lower bound

$$V = \begin{bmatrix} 0 & 0 & 2 & 1 & 0 \\ 0 & 2 & 2 & 0 & 3 \end{bmatrix}$$

$$U = \begin{bmatrix} 3 & 2 \\ 1 & 1 \\ 0 & 2 \\ 0 & 0 \\ 2 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 4 & 10 & 3 & 5 \\ 0 & 2 & 4 & 1 & 3 \\ 0 & 4 & 4 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 \end{bmatrix} = S$$

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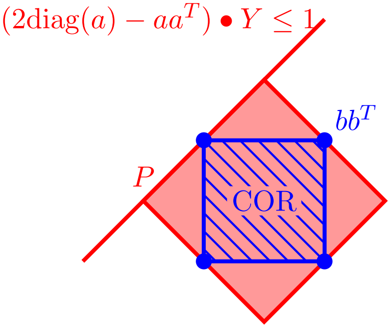
Theorem (Fiorini, Massar, Pokutta, Tiwary, de Wolf '12)

$\mathrm{xc}(\mathrm{COR}) \geq 2^{\Omega(n)}$.

Correlation polytope (2)

Lemma

For all $a \in \{0,1\}^n$, $(2\mathrm{diag}(a) - aa^T) \bullet Y \leq 1$ is a feasible inequality for $Y \in COR$.



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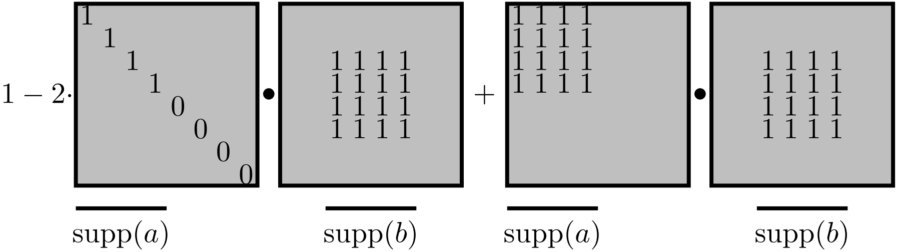

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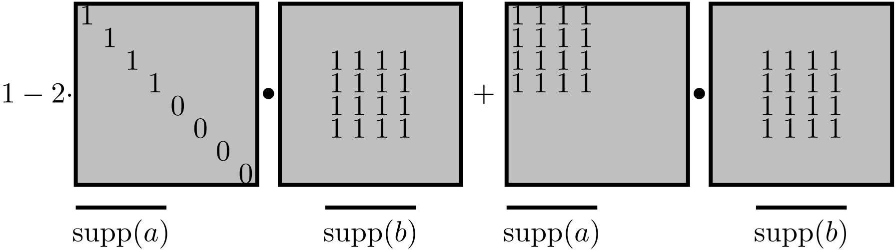
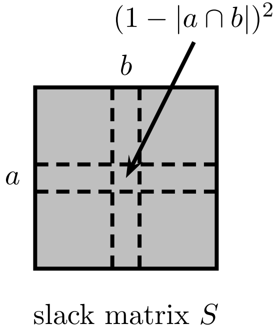
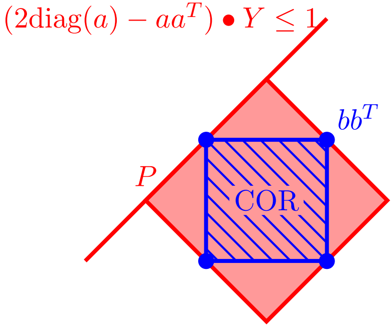


$$= 1 - 2|a \cap b| + |a \cap b|^2 = (1 - |a \cap b|)^2 \geq 0$$

Correlation polytope (3)





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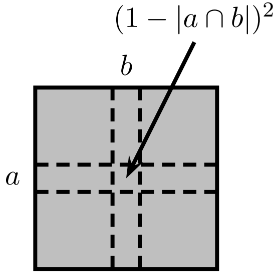
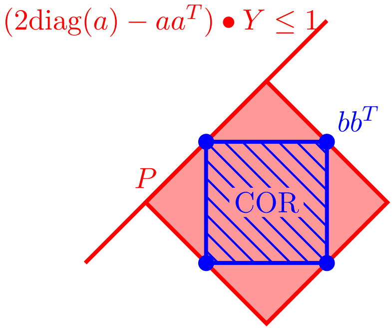


slack matrix S



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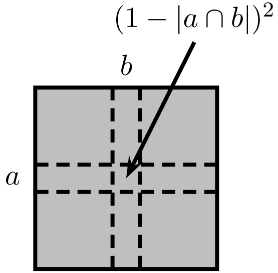

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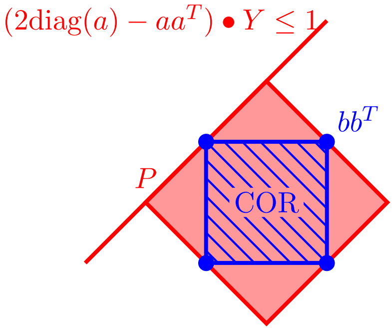

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$$S_{ab} = \begin{cases} 1 & |a \cap b| = 0 \\ 0 & |a \cap b| = 1 \end{cases}$$

Incomplete slack matrices

Lemma

For a polytope $P = \{x \mid Ax \leq b\}$ and $X = \{x_1, \ldots, x_v\} \subseteq P$ define a matrix S with $S_{i,j} := b_i - A_i x_j$. Then

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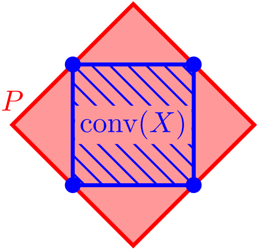

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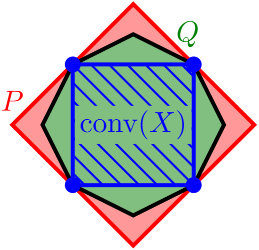

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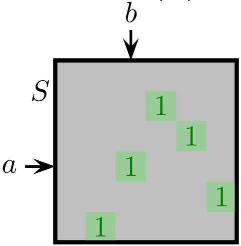

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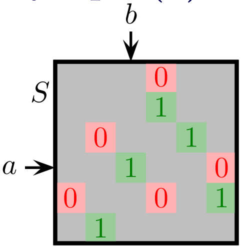

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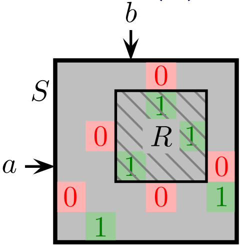

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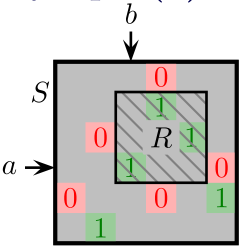

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Any rectangle R has $\mu_0(R) \le (1 + \varepsilon)\mu_1(R) + 2^{-\Theta(n)}$.

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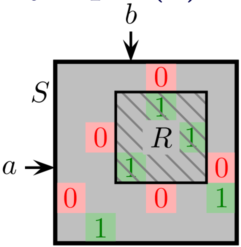

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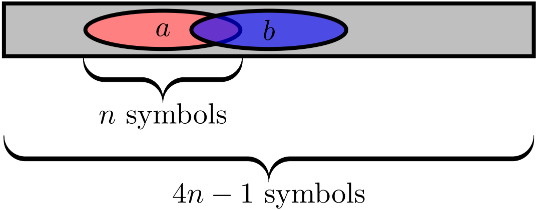
- Applying Razborov

$$\mu_0(R) \leq (1+\varepsilon)\underbrace{\mu_1(R)}_{=0} + 2^{-\Theta(n)} \leq 2^{-\Theta(n)}$$

The setting

- We consider tuples $a, b \subseteq [4n-1]$ with $|a| = |b| = n$



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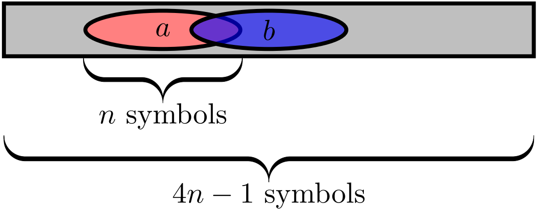

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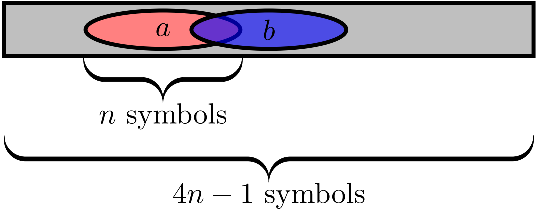

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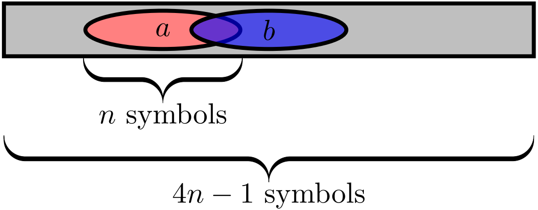

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- A **rectangle** is of the form $R = A \times B$
- **Measure** of rectangle: $\mu_0(R) = \Pr_{(a,b)\in Q_0}[(a,b) \in R]$

Example rectangles

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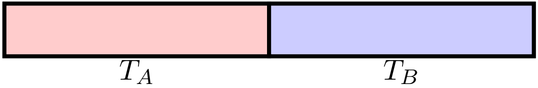
- Partition $[4n-1] = T_A \dot{\cup} T_B$ with $|T_A| \approx |T_B|$



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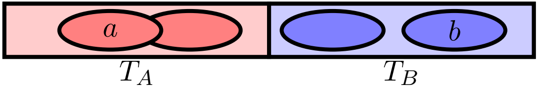

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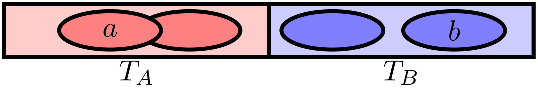

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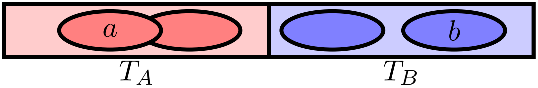

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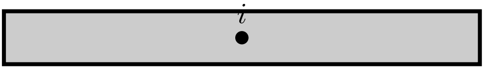

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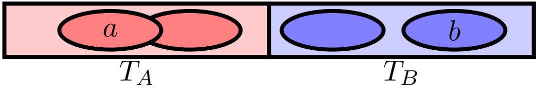

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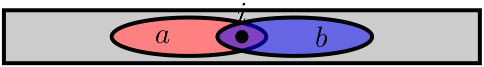

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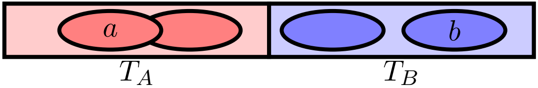

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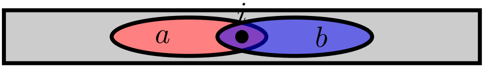
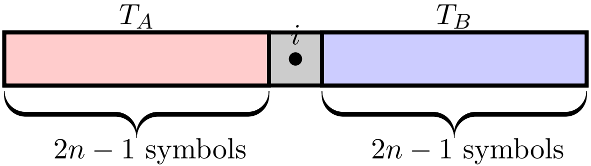


- Let $A := \{a : i \in a\}$ and $B := \{b : i \in b\} \to R := A \times B$
- The measures are

$$\mu_1(R) = \Theta\left(\frac{1}{n}\right) \text{ and } \mu_0(R) = 0$$

Partitions

A **partition** is a tuple $T = (T_A, T_B, i)$



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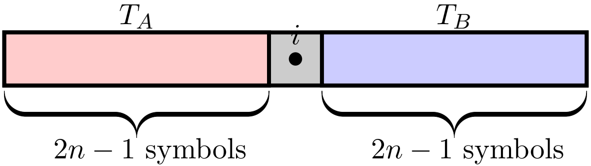

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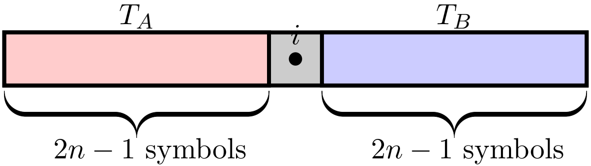

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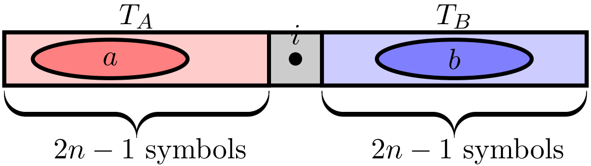

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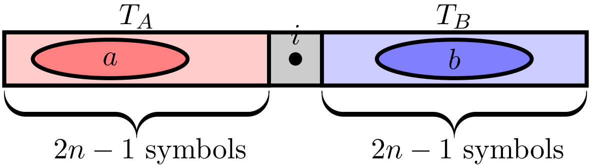

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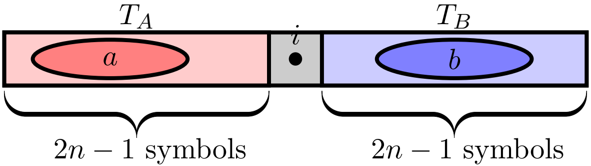

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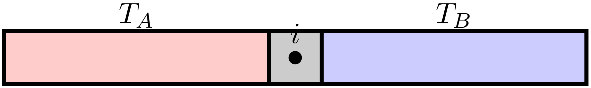

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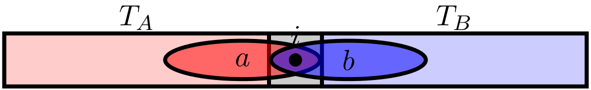

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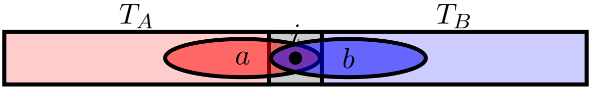

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Goal: $\mu_0(R) \leq (1+\varepsilon)\mu_1(R) + 2^{-\Theta(n)}$

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Easy case I - Small partitions

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Assumption: Suppose that all partitions T have

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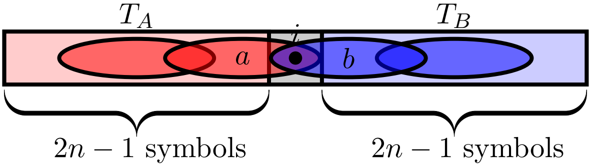

Then

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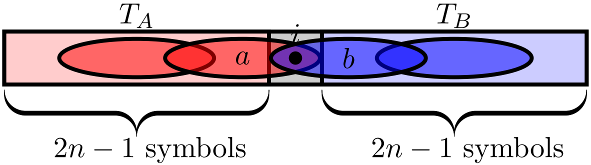

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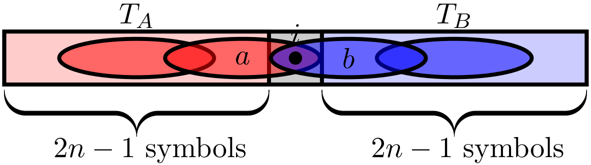

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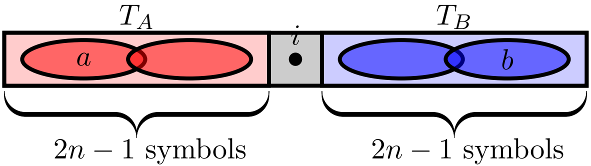
\end{aligned}$$

An example for a bad partition

Example: Consider a partition T and rectangle

$$R = A \times B \quad \text{with} \quad A := \{a \subseteq T_A\} \quad \text{and} \quad B := \{b \subseteq T_B\}$$



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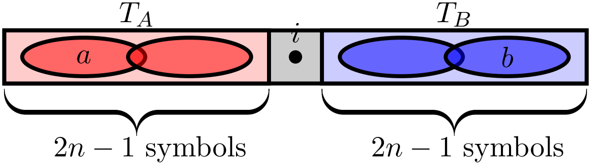

Then

$$\underbrace{\Pr_{a \subseteq T_A \cup \{i\}}[a \in A : i \notin a] = 1}_{\text{contribution to } \mu_0(R)} \qquad \underbrace{\Pr_{a \in T_A \cup \{i\}}[a \in A \mid i \in a] = 0}_{\text{contribution to } \mu_1(R)}$$

Fraction of bad partitions

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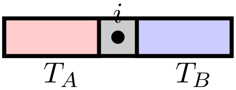
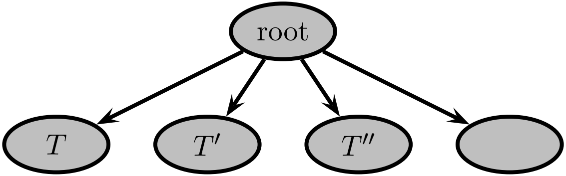
Phase I: Pick T





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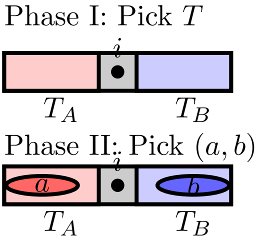

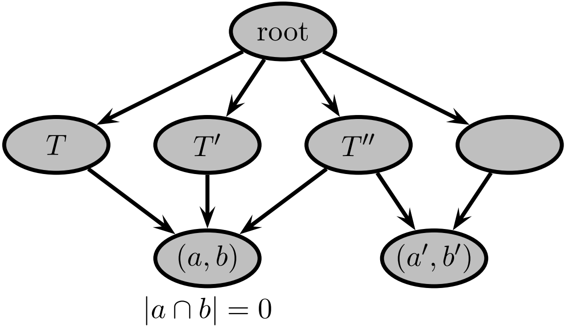

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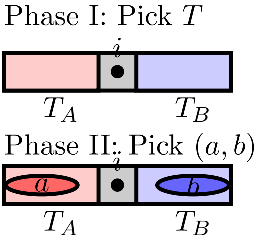

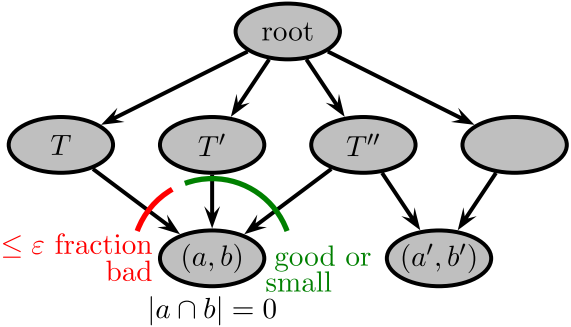

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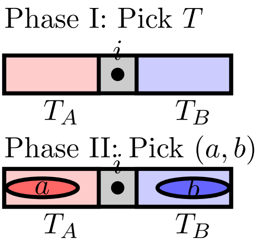

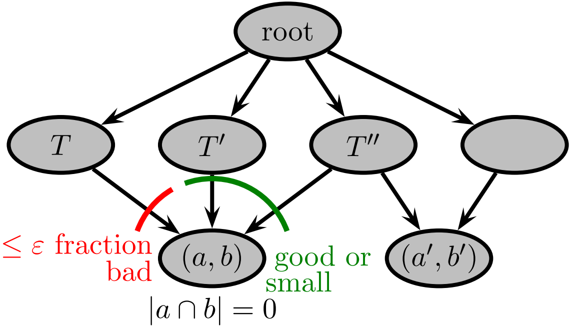

- Suffices to show:

Lemma

For any disjoint pair (a, b), take a random partition T with $a \subseteq T_A, b \subseteq T_B$. Then

$$\Pr[T \text{ is bad}] \leq \varepsilon.$$

Pseudo-random behaviour of large set systems

Imagine the following setting:

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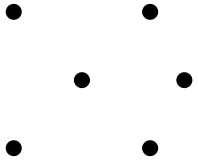

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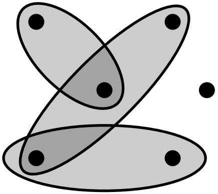
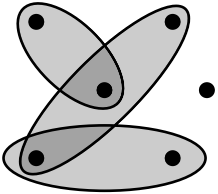


- n elements
- set system $\mathcal{S}$ with $2^{(1-o(1))n}$ sets

Pseudo-random behaviour of large set systems



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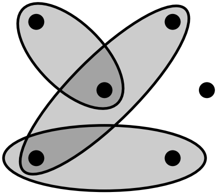

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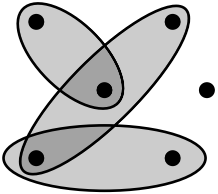

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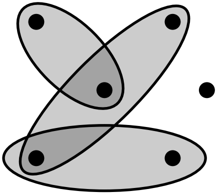

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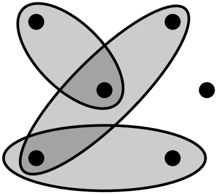

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Proof:

- Take a random set from $\mathcal{S}$

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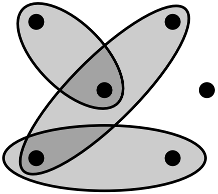

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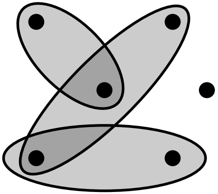
Proof:

- Take a random set from $\mathcal{S}$
- Denote char. vector as $x \in \{0,1\}^n$

Pseudo-random behaviour of large set systems

Imagine the following setting:



- n elements
- set system $\mathcal{S}$ with $2^{(1-o(1))n}$ sets

Questions:

- Is it possible that $\geq$ **1%** of elements are in **no** set at all?
  NO! The $0.99n$ active elements form at most $2^{0.99n}$ sets
- Is it possible that $\geq$ **1%** elements are in $\leq$ **49%** of sets?
  NO!

Proof:

- Take a random set from $\mathcal{S}$
- Denote char. vector as $x \in \{0,1\}^n$

$$\log |\mathcal{S}| = H(x)$$

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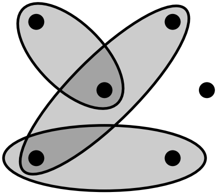

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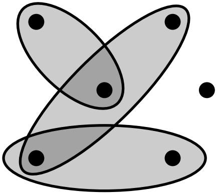

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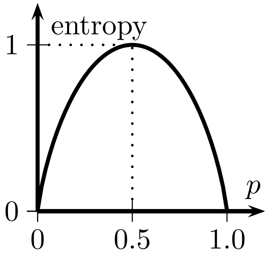

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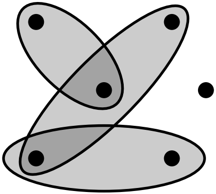

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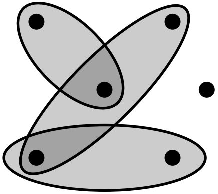

Lemma

If $|\mathcal{S}| \geq 2^{(1-\Theta(\varepsilon^3))n}$, then a $(1-\varepsilon)$-fraction of elements i lies in a $(\frac{1}{2} \pm \varepsilon)$-fraction of sets.

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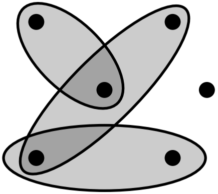

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For such an i:

$$\begin{aligned} \Pr_{S\subseteq[n]}[S \in \mathcal{S} \mid i \in S] &= \underbrace{\Pr_{S\subseteq[n]}[i \in S \mid S \in \mathcal{S}]}_{\in \frac{1}{2}\pm\varepsilon} \cdot \frac{\Pr_{S\subseteq[n]}[S \in \mathcal{S}]}{\underbrace{\Pr_{S\subseteq[n]}[i \in S]}_{=1/2}} \\ &= (1 \pm O(\varepsilon)) \cdot \Pr_{S\subseteq[n]}[S \in \mathcal{S}] \end{aligned}$$

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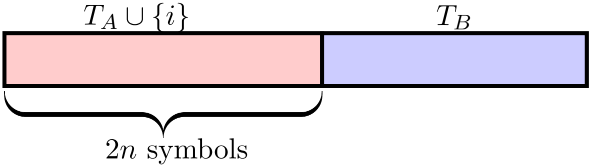

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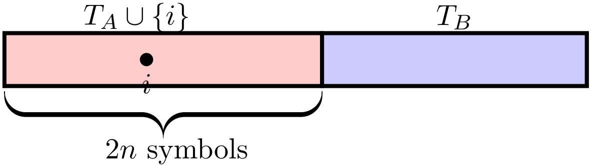

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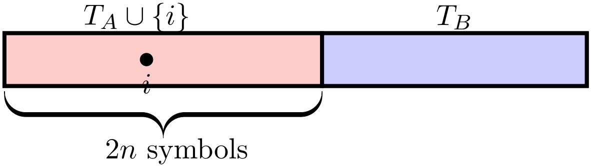

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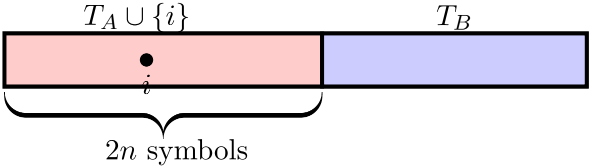

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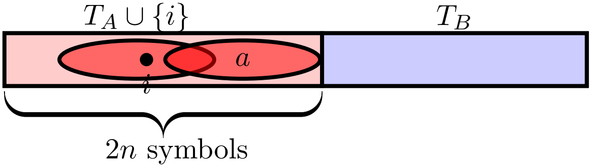

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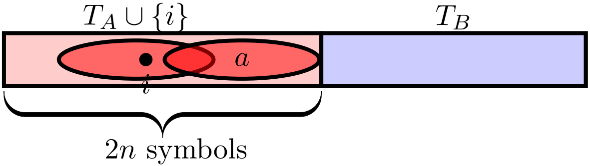

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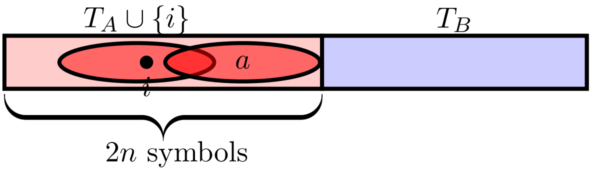

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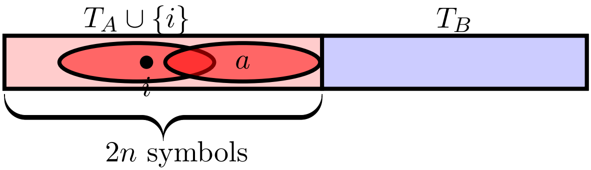

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- Equivalent to

$$\Pr_{a \subseteq T_A \cup \{i\}}[a \in A \mid i \in a] = (1 \pm O(\varepsilon)) \cdot \Pr_{a \subseteq T_A \cup \{i\}}[a \in A \mid i \notin a]$$

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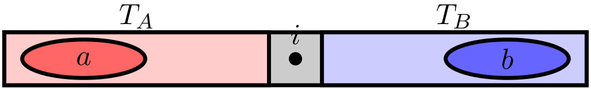

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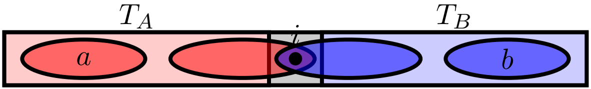

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- Hence $\mu_0(R) \lesssim \mu_1(R)$