The matching polytope has exponential extension complexity

Thomas Rothvoss
Department of Mathematics, UW Seattle
Extended formulation
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- Given polytope $P = \{ x \in \mathbb{R}^n \mid Ax \leq b \}$
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- Given polytope $P = \{ x \in \mathbb{R}^n \mid Ax \leq b \}$

- Write $P = \{ x \in \mathbb{R}^n \mid \exists y : Bx + Cy \leq d \}$

![Diagram showing linear projection from $Q$ to $P$]
Extended formulation

- Given polytope $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$
  $\rightarrow$ many inequalities
- Write $P = \{x \in \mathbb{R}^n \mid \exists y: Bx + Cy \leq d\}$
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Extended formulation

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**Extension complexity:**

$$xc(P) := \min \left\{ \# \text{facets of } Q \mid \begin{array}{l} Q \text{ polyhedron} \\ p \text{ linear map} \\ p(Q) = P \end{array} \right\}$$
What’s known?

Compact formulations:

- **Spanning Tree Polytope** [Kipp Martin ’91]
- **Perfect Matching** in planar graphs [Barahona ’93]
- **Perfect Matching** in bounded genus graphs [Gerards ’91]
- $O(n \log n)$-size for **Permutahedron** [Goemans ’10] ($\rightarrow$ **tight**)
- $n^{O(1/\varepsilon)}$-size $\varepsilon$-apx for **Knapsack Polytope** [Bienstock ’08]
- ...
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Here: When is the extension complexity **super polynomial**?
Lower bounds
Lower bounds

- No symmetric compact form. for TSP [Yannakakis ’91]
  Compact formulation for $\log n$ size matchings, but no symmetric one [Kaibel, Pashkovich & Theis ’10]
Lower bounds

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- \( xc(\text{random } 0/1 \text{ polytope}) \geq 2^{\Omega(n)} \) [R. ’11]
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- **Breakthrough**: $\text{xc}(\text{TSP}) \geq 2^{\Omega(\sqrt{n})}$
  [Fiorini, Massar, Pokutta, Tiwary, de Wolf ’12]
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- \( n^{1/2-\varepsilon} \)-apx for clique polytope needs super-poly size [Braun, Fiorini, Pokutta, Steuer ’12] Improved to \( n^{1-\varepsilon} \) [Braverman, Moitra ’13], [Braun, P. ’13]
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- $\chi_c$ (random 0/1 polytope) $\geq 2^{\Omega(n)}$ [R. ’11]

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- $(2 - \epsilon)$-apx LPs for MaxCut have size $n^{\Omega(\log n / \log \log n)}$
  [Chan, Lee, Raghavendra, Steurer ’13]
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Only \textbf{NP}-hard polytopes!!

What about poly-time problems?
Perfect matching polytope
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\[ G = (V, E) \]
(complete)
Perfect matching polytope

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\[ x(\delta(v)) = 1 \quad \forall v \in V \]

\[ x_e \geq 0 \quad \forall e \in E \]

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\[ U \]
Perfect matching polytope

\[ x(\delta(v)) = 1 \quad \forall v \in V \]

\[ x(\delta(U)) \geq 1 \quad \forall U \subseteq V : |U| \text{ odd} \]

\[ x_e \geq 0 \quad \forall e \in E \]

Quick facts:

- Description by [Edmonds ’65]
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- Can optimize \( c^T x \) in strongly poly-time [Edmonds ’65]
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▷ \( 2^{\Theta(n)} \) facets
Perfect matching polytope

\begin{align*}
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  x(\delta(U)) &\geq 1 \quad \forall U \subseteq V : |U| \text{ odd} \\
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Quick facts:
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- \( 2^{\Theta(n)} \) facets

Theorem (R.13)

\[ xc(\text{perfect matching polytope}) \geq 2^{\Omega(n)}. \]

- Previously known: \( xc(P) \geq \Omega(n^2) \)
Slack-matrix

**Write:** \( P = \text{conv}(\{x_1, \ldots, x_v\}) = \{x \in \mathbb{R}^n \mid Ax \leq b\} \)
**Slack-matrix**

Write: \( P = \text{conv}(\{x_1, \ldots, x_v\}) = \{x \in \mathbb{R}^n \mid Ax \leq b\} \)

- **# facets**
- **# vertices**
- **facet** \( i \)
- **vertex** \( j \)
- **slack-matrix**
  \( S_{ij} = b_i - A_i^T x_j \)
Slack-matrix

Write: $P = \text{conv}(\{x_1, \ldots, x_v\}) = \{x \in \mathbb{R}^n \mid Ax \leq b\}$

Non-negative rank:

$$\text{rk}_+(S) = \min\{r \mid \exists U \in \mathbb{R}^{f \times r}_{\geq 0}, V \in \mathbb{R}^{r \times v}_{\geq 0} : S = UV\}$$
Yannakakis’ Theorem

Theorem (Yannakakis ’91)

If $S$ is the slack-matrix for $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$, then $xc(P) = rk_+(S)$. 
Yannakakis’ Theorem

**Theorem (Yannakakis ’91)**

If $S$ is the **slack-matrix** for $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$, then $xc(P) = \text{rk}_+(S)$.

**Idea:** Factor $S = UV$ with

- $U = (\text{conic comb. to derive constraint } i)_i$
- $V = (\text{slack vector of } (x_j, v_j))_j$

![Diagram of a polyhedron with constraints and slack matrices](image-url)
Hyperplane separation lower bound [Fiorini]

\[ \text{rk}_+ (S) = \min \left\{ r : S = \sum_{i=1}^{r} R_i \text{ and } R_i \geq 0 \text{ rank-1 matrix} \right\} \]
Hyperplane separation lower bound [Fiorini]

\[
\text{rk}_+(S) = \min \left\{ r : S = \sum_{i=1}^{r} \lambda_i \begin{pmatrix} R_i \end{pmatrix} \text{ and } 0 \leq R_i \leq 1 \text{ rank-1 matrix} \right\}
\]

\[
S = \lambda_1 \begin{pmatrix} \frac{1}{2} & 1 & 1 \\ \frac{1}{2} & 2 & 1 \\ \frac{1}{2} & 1 & 2 \end{pmatrix} + \ldots + \lambda_r \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}
\]
Hyperplane separation lower bound [Fiorini]

\[
\operatorname{rk}_+(S) \geq \min \left\{ \|\lambda\|_1 : S = \sum_{i=1}^{r} \lambda_i R_i \text{ and } 0 \leq R_i \leq 1 \text{ rank-1 matrix} \right\}
\]

\[
S = \lambda_1 \begin{bmatrix}
\frac{1}{2} & 1 & 1 \\
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\frac{1}{2} & 1 & 1 \\
\end{bmatrix} + \ldots + \lambda_r \begin{bmatrix}
1 & 0 & 1 \\
0 & 0 & 0 \\
1 & 0 & 1 \\
\end{bmatrix}
\]
Hyperplane separation lower bound [Fiorini]

$$\text{rk}_+(S) \gtrsim \min \left\{ \| \lambda \|_1 : S = \sum_{i=1}^{r} \lambda_i R_i \text{ and } R_i \in \{0, 1\}^{f \times v} \text{ rank-1} \right\}$$

[0, 1]-rank-1 matrices

rectangles

$$S = \lambda_1 R_1 + \ldots + \lambda_r R_r$$
Hyperplane separation lower bound [Fiorini]

\[ \text{rk}_+(S) \gtrsim \min \left\{ \| \lambda \|_1 : \langle W, S \rangle = \sum_{i=1}^{r} \lambda_i \langle W, R_i \rangle \text{ and } R_i \text{ rect.} \right\} \]
Hyperplane separation lower bound [Fiorini]

\[ \text{rk}_+(S) \gtrsim \min \left\{ \frac{\langle W, S \rangle}{\langle W, R \rangle} : R \text{ rectangle} \right\} \]

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 1 \\
0 & 1 & 1 \\
\end{bmatrix}
= \lambda_1 \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \ldots + \lambda_r \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}
\]
Applying the Hyperplane bound

Goal: Find \( W \) with \( \frac{\langle W, S \rangle}{\langle W, R \rangle} \) large for each rectangle.

- Slack matrix \( S_{UM} = |\delta(U) \cap M| - 1 \)
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- Abbreviate $Q_\ell := \{(U, M) : |\delta(U) \cap M| = \ell\}$
- Uniform measure: $\mu_\ell(R) := \frac{|R \cap Q_\ell|}{|Q_\ell|}$
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- Choose $W_{U,M} = \begin{cases} 0 & \text{otherwise.} \end{cases}$
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W_{U,M} = \begin{cases} 
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Rectangle covering for matching

Claim: There is a rectangle with $\langle W, R \rangle = \Theta \left( \frac{1}{n^4} \right)$. 
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For $e_1, e_2 \in E$: take $\{U \mid e_1, e_2 \in \delta(U)\}$
Claim: There is a rectangle with $\langle W, R \rangle = \Theta(\frac{1}{n^4})$.

For $e_1, e_2 \in E$: take $\{U \mid e_1, e_2 \in \delta(U)\} \times \{M \mid e_1, e_2 \in M\}$
Claim: There is a rectangle with $\langle W, R \rangle = \Theta(\frac{1}{n^4})$. 

- For $e_1, e_2 \in E$: take $\{U \mid e_1, e_2 \in \delta(U)\} \times \{M \mid e_1, e_2 \in M\}$ 
- But $\mu_k(R) = \Theta(\frac{k^2}{n^4})$
Applying the Hyperplane bound (II)

**Goal:** Find $W$ with $\frac{\langle W, S \rangle}{\langle W, R \rangle}$ large for each rectangle.

- Choose

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-\frac{1}{k-1} \cdot \frac{1}{|Q_k|} & |\delta(U) \cap M| = k \\
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\end{cases}$$

- Then

$$\langle W, S \rangle = 0 + 2 - 1 = 1$$

**Lemma**

For $k$ large, any rectangle $R$ has $\langle W, R \rangle \leq 2^{-\Omega(n)}$. 
Applying the Hyperplane bound (III)
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**Main lemma**

\[ \mu_1(R) = 0 \implies \mu_3(R) \leq O\left(\frac{1}{k^2}\right) \cdot \mu_k(R) + 2^{-\Omega(n)} \]
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➤ **Technique:** Partition scheme [Razborov ’91]
Applying the Hyperplane bound (III)

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- **Technique:** Partition scheme [Razborov ’91]
Partitions

- Partition $T = (A, C, D, B)$
Partitions

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The diagram illustrates a partition $T = (A, C, D, B)$ with $k - 3$ nodes, $k$ nodes, and $k$ nodes in each of the sets $A$, $C$, and $D$, respectively. The set $B$ is a large rectangular area.
PARTITIONS

- Partition $T = (A, C, D, B)$

$k - 3$ nodes

$k$ nodes

$2(k - 3)$ nodes
Partitions

- Partition $T = (A, C, D, B)$
- Edges $E(T)$

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$k$ nodes

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Partitions

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$\mathcal{P}$
Pseudo-random behaviour of large set systems

Imagine the following setting:
Pseudo-random behaviour of large set systems

Imagine the following setting:

- $n$ elements
Pseudo-random behaviour of large set systems

Imagine the following setting:

- $n$ elements
- set system $S$ with $2^{(1-o(1))n}$ sets
Pseudo-random behaviour of large set systems

Imagine the following setting:

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Questions:

- Is it possible that $\geq 1\%$ of elements are in no set at all?
Pseudo-random behaviour of large set systems

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- Is it possible that $\geq 1\%$ elements are in $\leq 49\%$ of sets?
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- \( n \) elements
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Questions:

- Is it possible that \( \geq 1\% \) of elements are in \textbf{no} set at all? **NO!** The 0.99\( n \) active elements form at most \( 2^{0.99n} \) sets
- Is it possible that \( \geq 1\% \) elements are in \( \leq 49\% \) of sets? **NO!**

Proof:

- Take a random set from \( S \)
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- Is it possible that $\geq 1\%$ elements are in $\leq 49\%$ of sets? NO!

Proof:

- Take a random set from $S$
- Denote char. vector as $x \in \{0, 1\}^n$
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\[ \log |S| = H(x) \]
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- Is it possible that $\geq 1\%$ elements are in $\leq 49\%$ of sets?
  NO!

Proof:

- Take a random set from $S$
- Denote char. vector as $x \in \{0, 1\}^n$

$$\log |S| = H(x) \overset{\text{subadd}}{\leq} \sum_{i=1}^{n} H(x_i)$$
Pseudo-random behaviour of large set systems

Imagine the following setting:

- $n$ elements
- set system $\mathcal{S}$ with $2^{(1-o(1))n}$ sets

Questions:

- Is it possible that $\geq 1\%$ of elements are in no set at all? NO! The $0.99n$ active elements form at most $2^{0.99n}$ sets
- Is it possible that $\geq 1\%$ elements are in $\leq 49\%$ of sets? NO!

Proof:

- Take a random set from $\mathcal{S}$
- Denote char. vector as $x \in \{0, 1\}^n$

\[
\log |\mathcal{S}| = H(x) \leq \sum_{i=1}^{n} H(x_i) \leq n - \Omega(n)
\]
Pseudo-random behaviour of large set systems

Imagine the following setting:

- $n$ elements
- set system $S$ with $2^{(1-o(1))n}$ sets

Questions:

- Is it possible that $\geq 1\%$ of elements are in no set at all?
  NO! The $0.99n$ active elements form at most $2^{0.99n}$ sets
- Is it possible that $\geq 1\%$ elements are in $\leq 49\%$ of sets?
  NO!

Lemma

If $S$ large, for most elements $i$,

$$\Pr_{S \subseteq [n]} [S \in S] \approx \Pr_{S \subseteq [n]} [S \in S \mid i \in S]$$
Rewriting $\mu_k(R)$

Randomly generate $(U, M) \sim Q_k$:

$$\mu_k(R) =$$
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1. Choose $T$

$$\mu_k(R) = \mathbb{E}_T \left[ \right]$$
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Randomly generate $(U, M) \sim Q_k$:
1. Choose $T$
2. Choose $k$ edges $F \subseteq C \times D$

\[
\mu_k(R) = \mathbb{E}_T \left[ \mathbb{E}_{|F|=k} \left[ \ldots \right] \right]
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Rewriting $\mu_k(R)$

Randomly generate $(U, M) \sim Q_k$:

1. Choose $T$
2. Choose $k$ edges $F \subseteq C \times D$
3. Choose $M \supseteq F$

$$\mu_k(R) = \mathbb{E}_T \left[ \mathbb{E}_{|F|=k} \left[ \Pr[M \in R \mid T, H] \right] \right]$$
Rewriting $\mu_k(R)$

Randomly generate $(U, M) \sim Q_k$:
1. Choose $T$
2. Choose $k$ edges $F \subseteq C \times D$
3. Choose $M \supseteq F$
4. Choose $U \supseteq C$ (not cutting any $A_i$)

$$\mu_k(R) = \mathbb{E}_T \left[ \mathbb{E}_{|F|=k} \left[ \Pr[M \in R \mid T, H] \cdot \Pr[U \in R \mid T, H] \right] \right]$$
How does an average partition look like

- Suppose for a fixed \((T, F)\):
  \[
  \mu_k(R) \approx \Pr[(U, M) \in R \mid T, F] =: p
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- Then
  \[ \mu_3(R) \approx \mathbb{E}_{H \sim \binom{F}{3}} \left[ \Pr[(U, M) \in R \mid T, H] \right] \]
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  \[
  \mu_3(R) \approx \mathbb{E}_{H \sim (F^3)} \left[ \text{GOOD}(T, H) \cdot \Pr[(U, M) \in R \mid T, H] \right] \leq O(1/k^2)
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- \text{GOOD} means it doesn’t matter what condition on here

- Suffices to show: \( H, H^* \subseteq F \text{ good } \Rightarrow |H \cap H^*| \geq 2 \)
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  \[ \leq O(1/k^2) \]
- \text{GOOD} means it doesn’t matter what condition on \( H \) here
- Suffices to show: \( H, H^* \subseteq F \) good \( \Rightarrow |H \cap H^*| \geq 2 \)
- Suppose \( |H \cap H^*| \leq 1 \)
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  \( \Rightarrow \exists M : \{u, v\} \in M \)
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- Suppose \(|H \cap H^*| \leq 1\)
  - \((T, H)\) good
    \(\Rightarrow \exists M : \{u, v\} \in M\)
  - \((T, H^*)\) good
    \(\Rightarrow \exists U : u, v \in U\)
How does an average partition look like

- Suppose for a fixed \((T, F)\):
  \[ \mu_k(R) \approx \Pr[(U, M) \in R \mid T, F] =: p \]
- Then
  \[ \mu_3(R) \approx \mathbb{E}_{H \sim \binom{F}{3}} \left[ \text{GOOD}(T, H) \cdot \Pr[(U, M) \in R \mid T, H] \right] \approx p \]
  \[ \leq O\left(\frac{1}{k^2}\right) \]
- \text{GOOD means it doesn’t matter what condition on here}
- Suffices to show: \(H, H^* \subseteq F\) good \(\Rightarrow |H \cap H^*| \geq 2\)

- Suppose \(|H \cap H^*| \leq 1\)
  - \((T, H)\) good
    \(\Rightarrow \exists M : \{u, v\} \in M\)
  - \((T, H^*)\) good
    \(\Rightarrow \exists U : u, v \in U\)
  - \(|\delta(U) \cap M| = 1\)
    Contradiction!
Most partitions are good

Lemma

\[ \Pr[(T, H) \text{ is } M\text{-bad}] \leq \varepsilon \]
Most partitions are good

Lemma

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- Pick \( H \)
Most partitions are good

Lemma
$\Pr[\{T, H\} \text{ is } M\text{-bad}] \leq \varepsilon$

- Pick $H$, $A$
Most partitions are good

**Lemma**

\[ \Pr[(T, H) \text{ is } M\text{-bad}] \leq \varepsilon \]

- Pick \( H, A, \tilde{B}_1, \ldots, \tilde{B}_{m+1} \).
**Most partitions are good**

**Lemma**

\[ \Pr[(T, H) \text{ is } M\text{-bad}] \leq \varepsilon \]

- Pick \( H, A, \tilde{B}_1, \ldots, \tilde{B}_{m+1} \). Split \( \tilde{B}_i = C_i \cup D_i \).
Most partitions are good

Lemma

\[ \Pr[(T, H) \text{ is } M\text{-bad}] \leq \varepsilon \]

- Pick \( H, A, \tilde{B}_1, \ldots, \tilde{B}_{m+1} \). Split \( \tilde{B}_i = C_i \cup D_i \).
- Pick randomly \( i \in \{1, \ldots, m\} \)
Most partitions are good

Lemma

\[ \Pr[(T, H) \text{ is } M\text{-bad}] \leq \varepsilon \]

- Pick \( H, A, \tilde{B}_1, \ldots, \tilde{B}_{m+1} \). Split \( \tilde{B}_i = C_i \cup D_i \).
- Pick randomly \( i \in \{1, \ldots, m\} \) and let \( C := C_i, D := D_i \)
## Open problems

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Thanks for your attention