The matching polytope has exponential extension complexity

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Extended formulation

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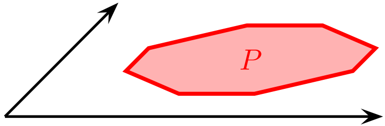
- Given polytope $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$



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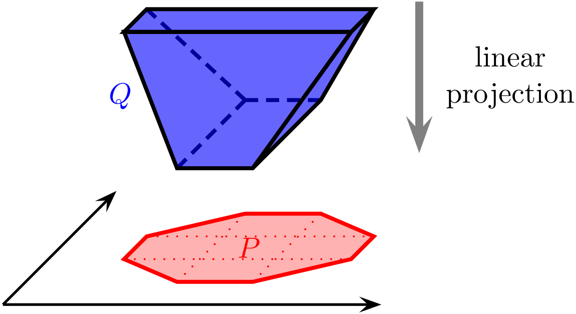
- Write $P = \{x \in \mathbb{R}^n \mid \exists y : Bx + Cy \leq d\}$



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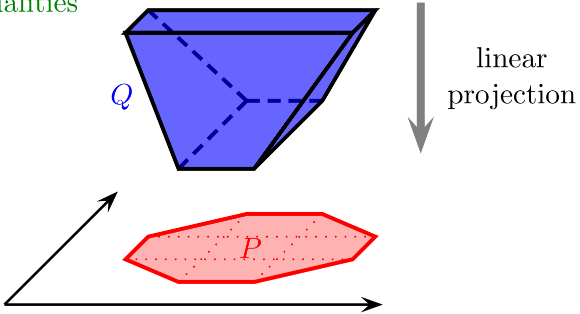

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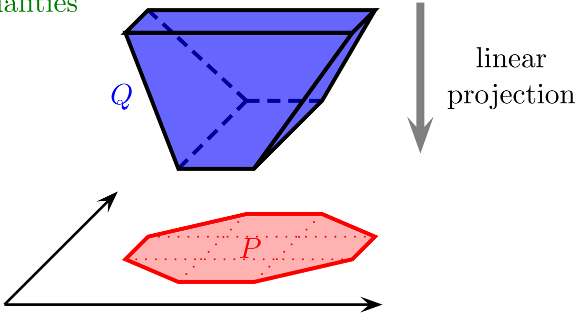

- **Extension complexity:**

$$\mathrm{xc}(P) := \min \left\{ \#\text{facets of } Q \;\middle|\; \begin{array}{c} Q \text{ polyhedron} \\ p \text{ linear map} \\ p(Q) = P \end{array} \right\}$$

What's known?

Compact formulations:

- Spanning Tree Polytope [Kipp Martin '91]
- Perfect Matching in planar graphs [Barahona '93]
- Perfect Matching in bounded genus graphs [Gerards '91]
- $O(n \log n)$-size for Permutahedron [Goemans '10] ($\rightarrow$ **tight**)
- $n^{O(1/\varepsilon)}$-size ε-apx for Knapsack Polytope [Bienstock '08]
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Here: When is the extension complexity **super polynomial**?

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Only **NP**-hard polytopes!!

What about poly-time problems?

Perfect matching polytope

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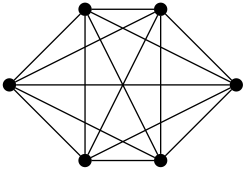
(complete)



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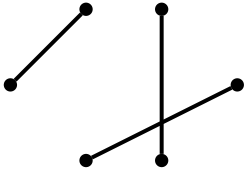

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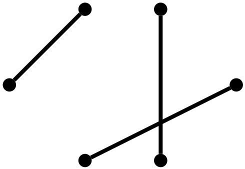

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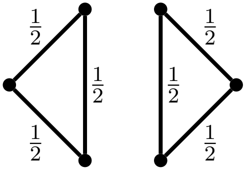

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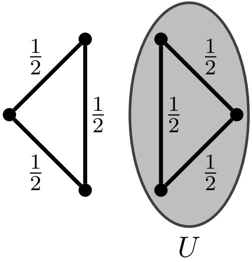

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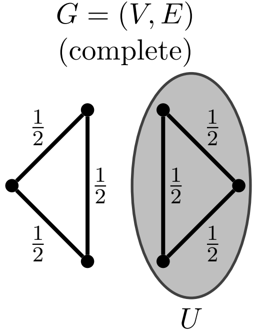

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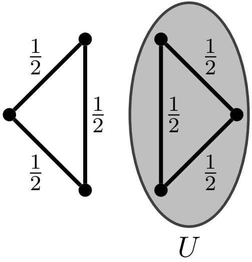

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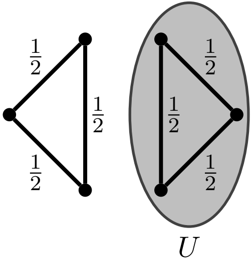

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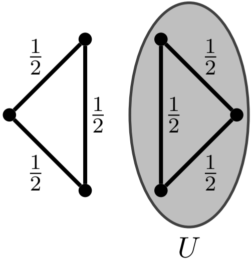

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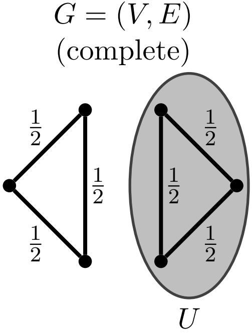

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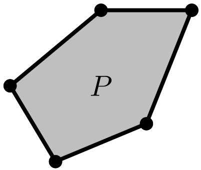
Theorem (R.13)

xc(perfect matching polytope) $\geq 2^{\Omega(n)}$.

- Previously known: $\mathrm{xc}(P) \geq \Omega(n^2)$

Slack-matrix

Write: $P = \mathrm{conv}(\{x_1, \ldots, x_v\}) = \{x \in \mathbb{R}^n \mid Ax \leq b\}$



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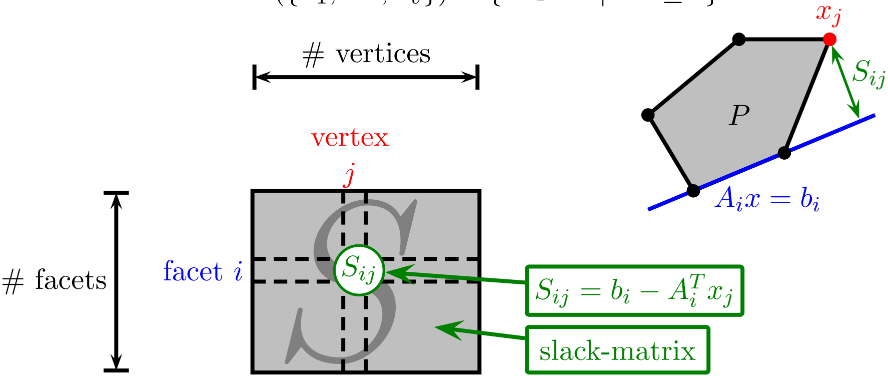

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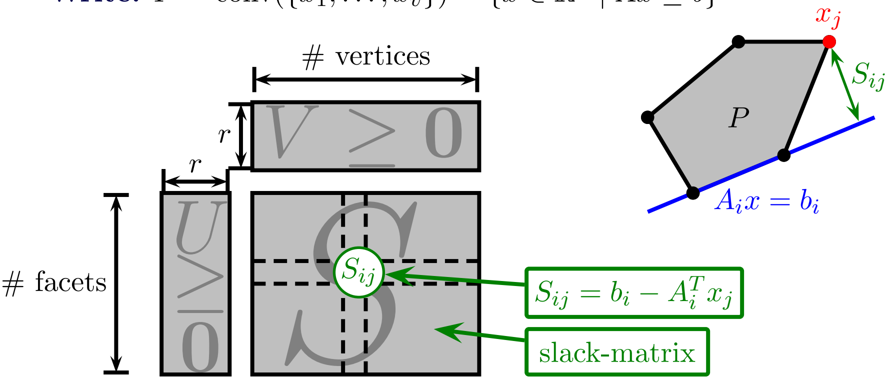


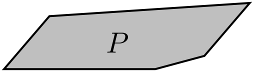
Non-negative rank:

$$\mathrm{rk}_+(S) = \min\{r \mid \exists U \in \mathbb{R}^{f \times r}_{\geq 0}, V \in \mathbb{R}^{r \times v}_{\geq 0} : S = UV\}$$

Yannakakis' Theorem

Theorem (Yannakakis '91)

If S is the **slack-matrix** for $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$, then $\mathrm{xc}(P) = \mathrm{rk}_+(S)$.



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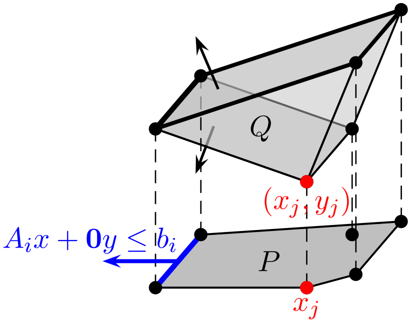
Idea: Factor $S = UV$ with

$$\begin{aligned} U &= (\text{conic comb. to derive constraint } i)_i \\ V &= (\text{slack vector of } (x_j, v_j))_j \end{aligned}$$



Hyperplane separation lower bound [Fiorini]

$$\mathrm{rk}_+(S) \quad = \min\left\{ r : S = \sum_{i=1}^{r} R_i \text{ and } R_i \geq \mathbf{0} \text{ rank-1 matrix} \right\}$$

$$\boxed{S} = \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 2 \end{bmatrix}}_{R_1} + \ldots + \underbrace{\begin{bmatrix} 3 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 3 \end{bmatrix}}_{R_r}$$

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$$\boxed{S} = \lambda_1 \underset{R_1}{\boxed{\begin{matrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & 1 \\ \frac{1}{2} & 1 & 1 \end{matrix}}} + \ldots + \lambda_r \underset{R_r}{\boxed{\begin{matrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{matrix}}}$$

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$$\mathrm{rk}_+(S) \quad \gtrsim \min\left\{\|\lambda\|_1 : S = \sum_{i=1}^{r} \lambda_i R_i \text{ and } \mathbf{0} \le R_i \le \mathbf{1} \text{ rank-1 matrix}\right\}$$

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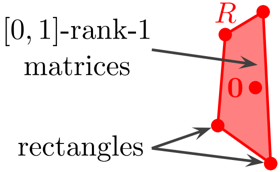
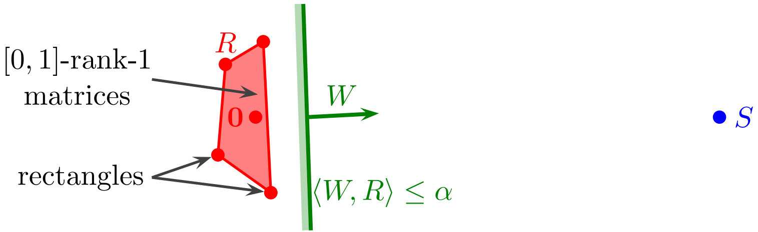


$$\boxed{S} = \lambda_1 \underbrace{\begin{array}{|ccc|}\hline 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ \hline\end{array}}_{R_1} + \ldots + \lambda_r \underbrace{\begin{array}{|ccc|}\hline 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \\ \hline\end{array}}_{R_r}$$

Hyperplane separation lower bound [Fiorini]

$$\mathrm{rk}_+(S) \gtrsim \min\left\{\|\lambda\|_1 : \langle W, S\rangle = \sum_{i=1}^{r} \lambda_i \langle W, R_i\rangle \text{ and } R_i \text{ rect.}\right\}$$



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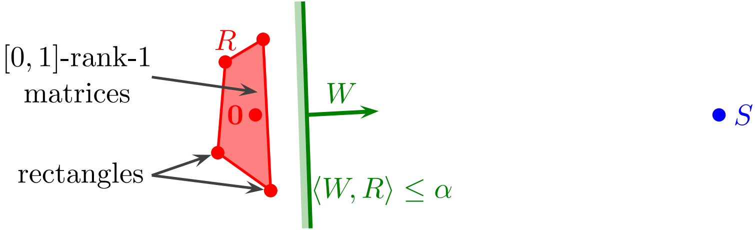

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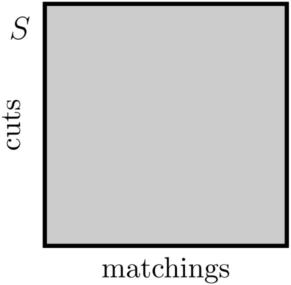
Applying the Hyperplane bound

Goal: Find W with $\frac{\langle W,S\rangle}{\langle W,R\rangle}$ large for each rectangle.

- Slack matrix $S_{UM} = |\delta(U) \cap M| - 1$



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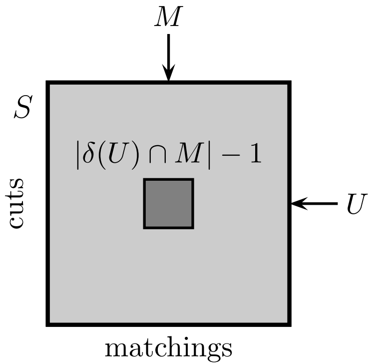

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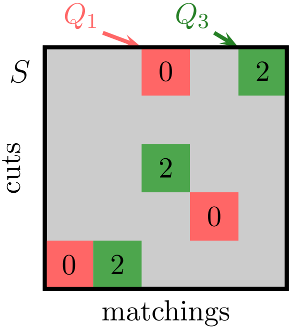
- **Uniform measure**: $\mu_\ell(R) := \frac{|R \cap Q_\ell|}{|Q_\ell|}$



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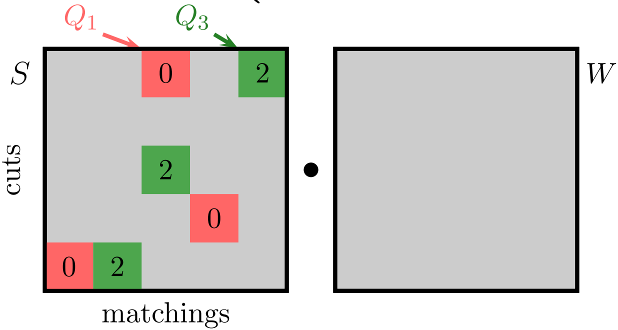
$$W_{U,M} = \begin{cases} \\ 0 & \text{otherwise.} \end{cases}$$



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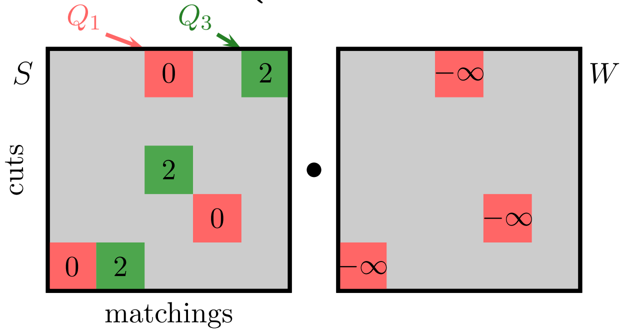

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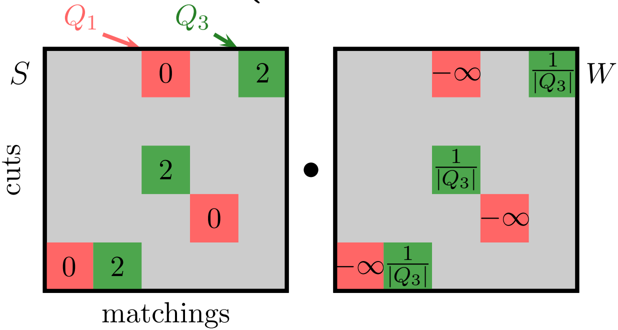

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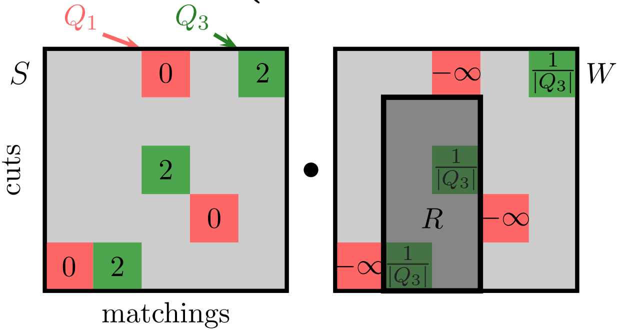
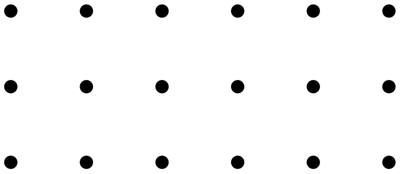


Rectangle covering for matching

Claim: There is a rectangle with $\langle W, R\rangle = \Theta(\frac{1}{n^4})$.



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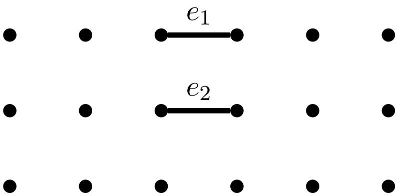

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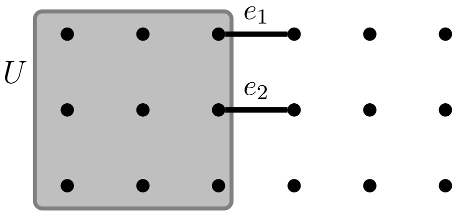

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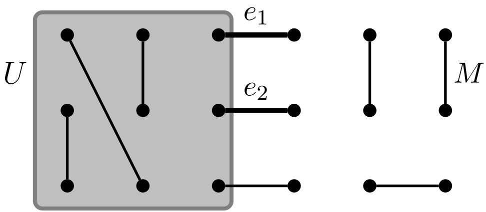

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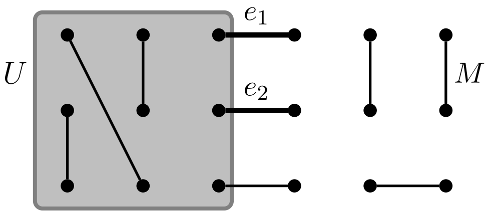


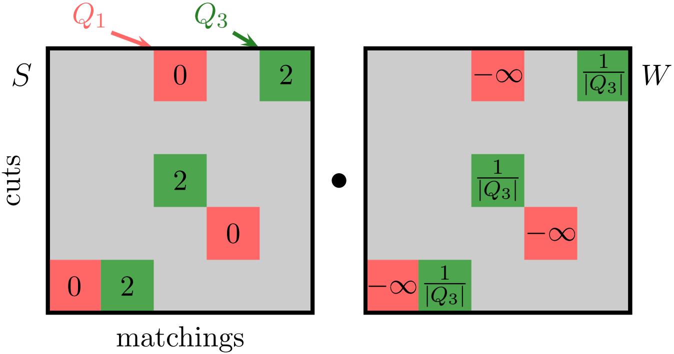
- For $e_1, e_2 \in E$: take $\{U \mid e_1, e_2 \in \delta(U)\} \times \{M \mid e_1, e_2 \in M\}$
- But $\mu_k(R) = \Theta(\frac{k^2}{n^4})$

Applying the Hyperplane bound (II)

Goal: Find W with $\frac{\langle W,S\rangle}{\langle W,R\rangle}$ large for each rectangle.

- Choose

$$W_{U,M} = \begin{cases} -\infty & |\delta(U) \cap M| = 1 \\ \frac{1}{|Q_3|} & |\delta(U) \cap M| = 3 \\ 0 & \text{otherwise.} \end{cases}$$



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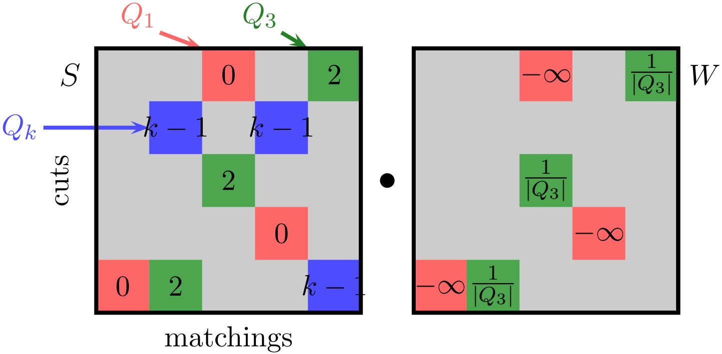

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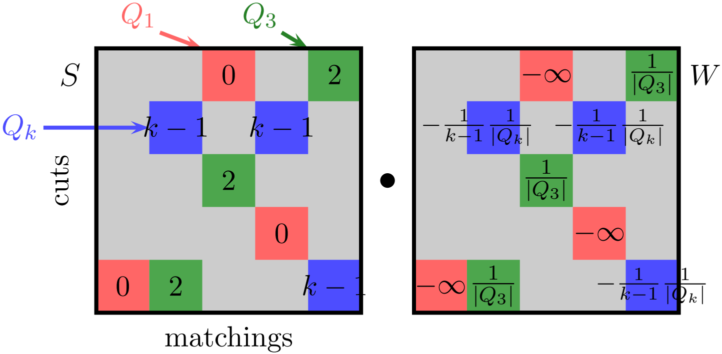

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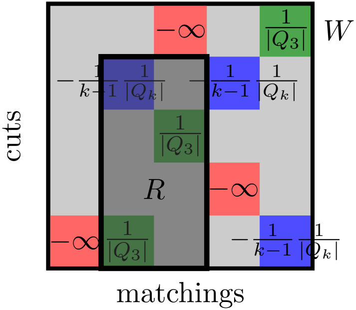
- Then
  $\langle W,S\rangle = 0 + 2 - 1 = 1$

Lemma

For k large, any rectangle R has $\langle W,R\rangle \le 2^{-\Omega(n)}$.



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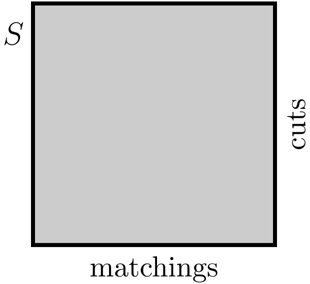
$\mu_1(R) = 0 \implies \mu_3(R) \leq O(\frac{1}{k^2}) \cdot \mu_k(R) + 2^{-\Omega(n)}$



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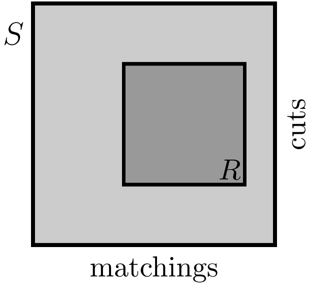

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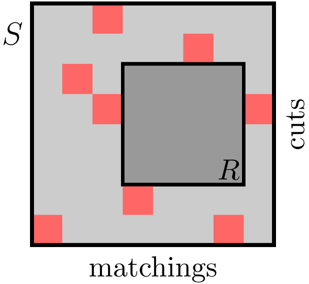

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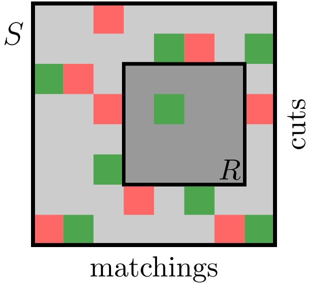

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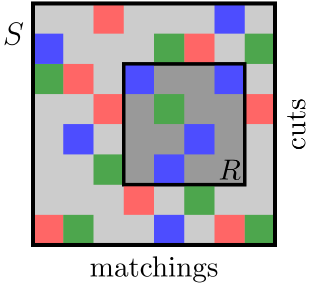

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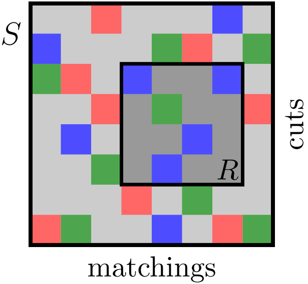

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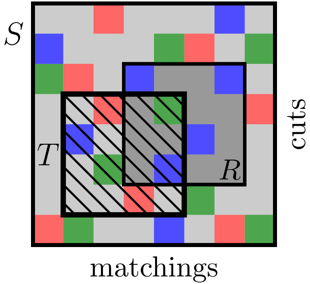

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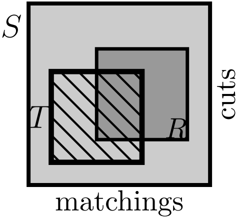

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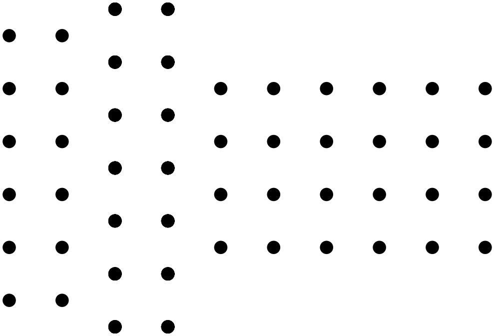

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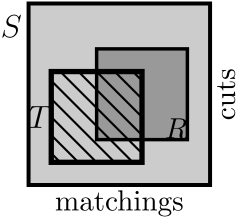

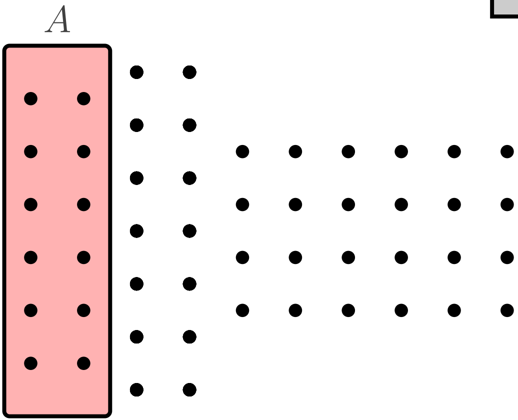

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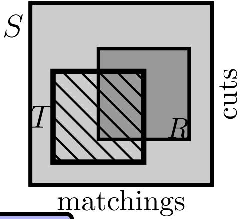

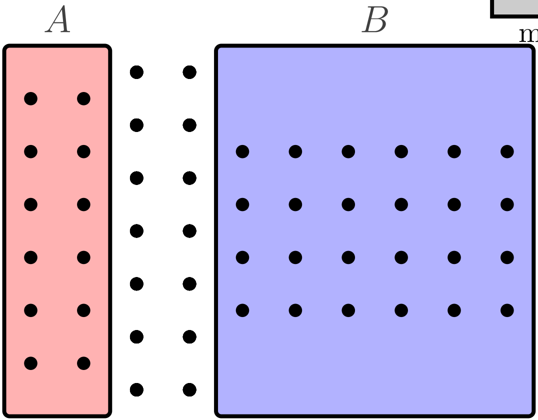

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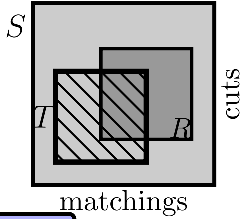

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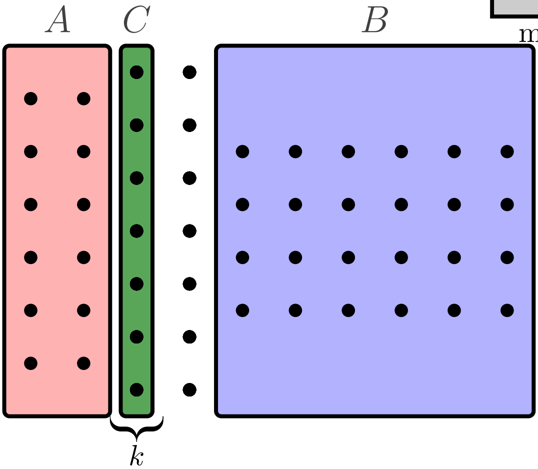

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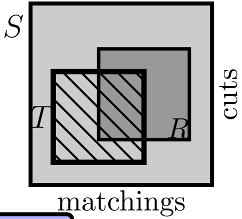

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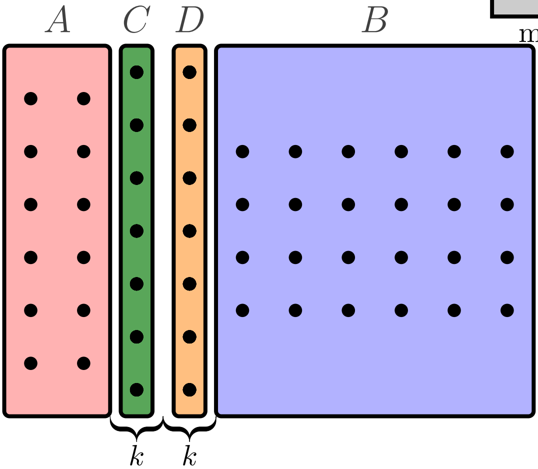

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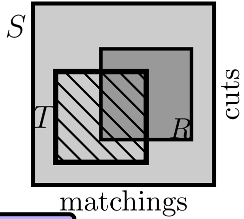

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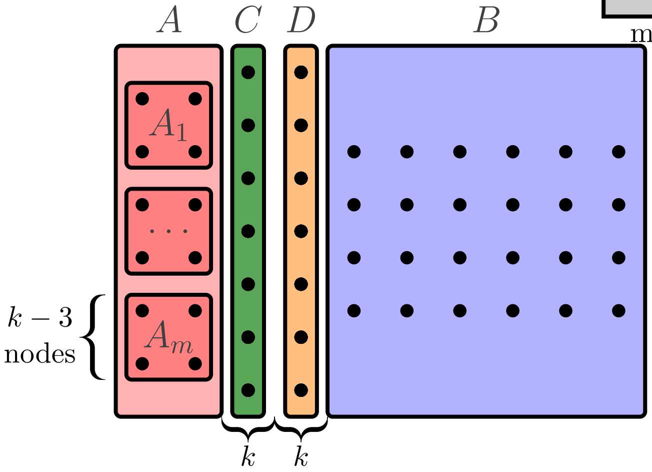

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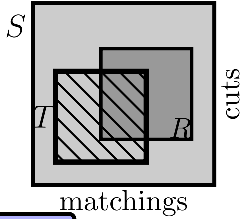

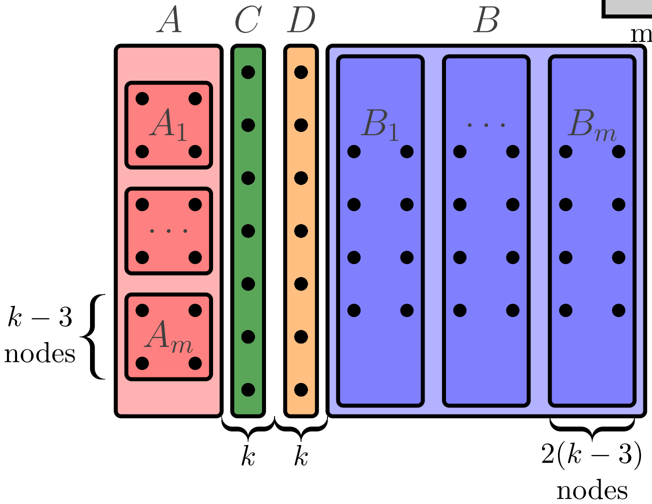

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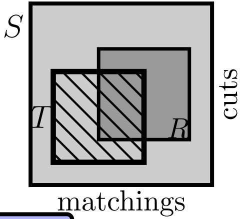

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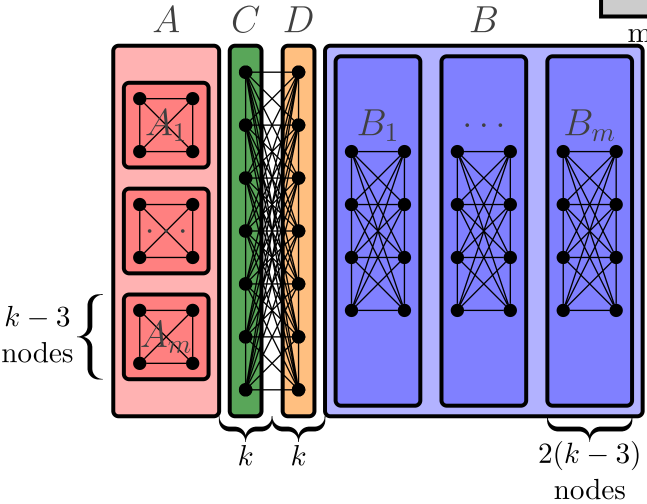

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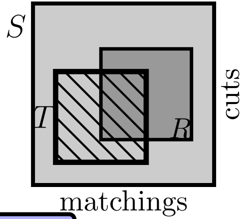

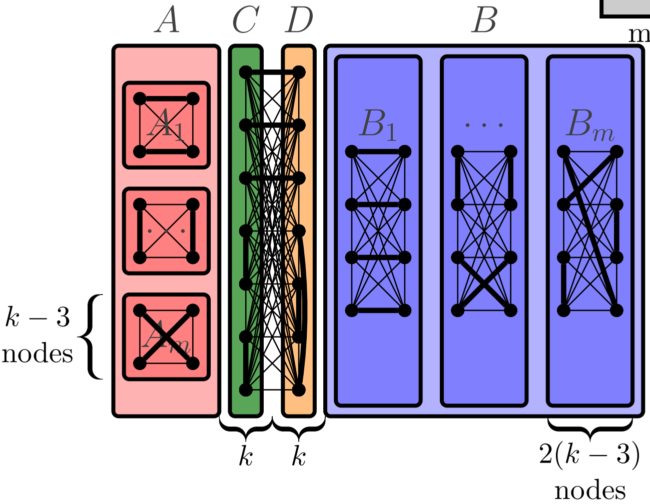

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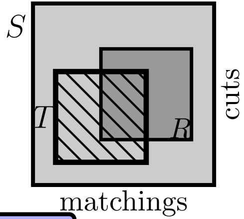

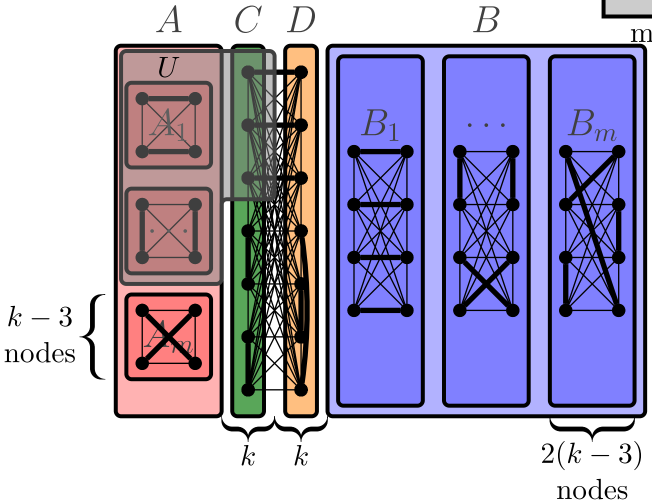

Pseudo-random behaviour of large set systems

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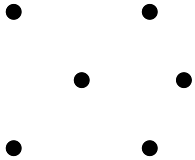

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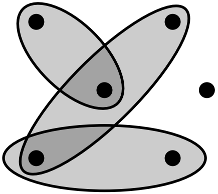

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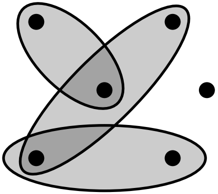

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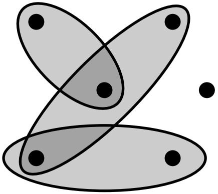

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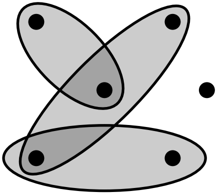

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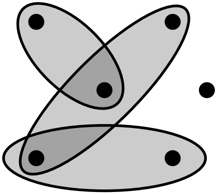

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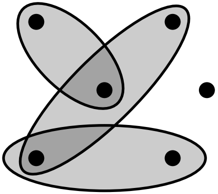

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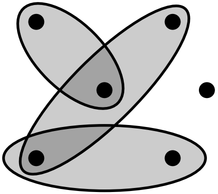

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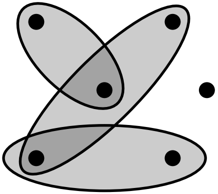

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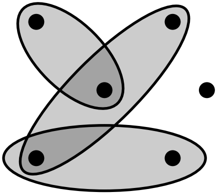

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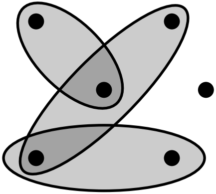

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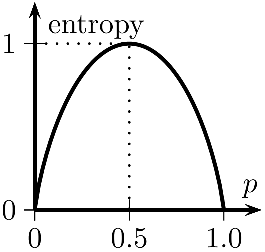
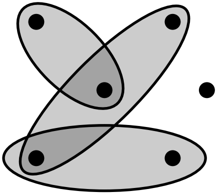


Pseudo-random behaviour of large set systems

Imagine the following setting:



- n elements
- set system $\mathcal{S}$ with $2^{(1-o(1))n}$ sets

Questions:

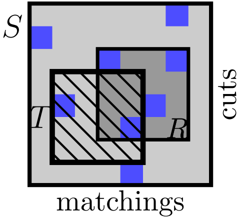
- Is it possible that $\geq$ **1**% of elements are in **no** set at all?
  NO! The $0.99n$ active elements form at most $2^{0.99n}$ sets
- Is it possible that $\geq$ **1**% elements are in $\leq$ **49**% of sets?
  NO!

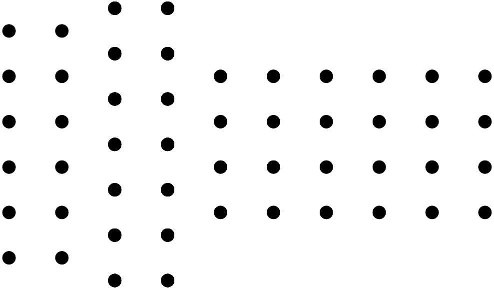
Lemma

If $\mathcal{S}$ large, for most elements i,

$$\Pr_{S \subseteq [n]}[S \in \mathcal{S}] \approx \Pr_{S \subseteq [n]}[S \in \mathcal{S} \mid i \in S]$$

Rewriting $\mu_k(R)$

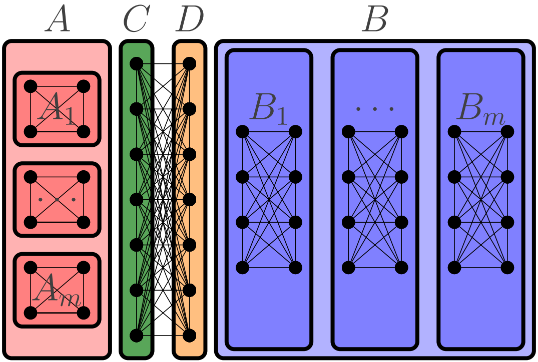


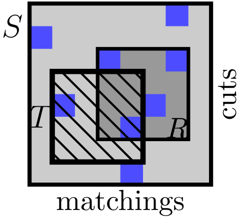


Randomly generate $(U, M) \sim Q_k$:

$$\mu_k(R) =$$

Rewriting $\mu_k(R)$



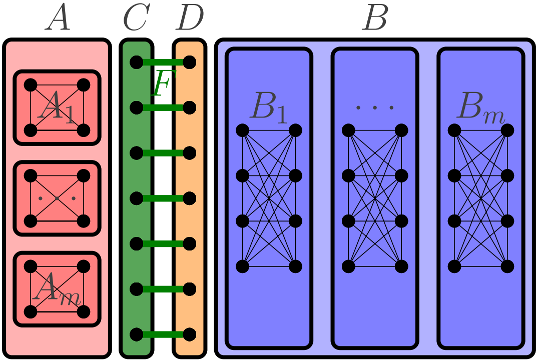


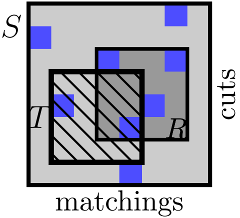
Randomly generate $(U, M) \sim Q_k$:

1. Choose T

$$\mu_k(R) = \mathop{\mathbb{E}}_{T} \Big[\qquad \Big]$$

Rewriting $\mu_k(R)$



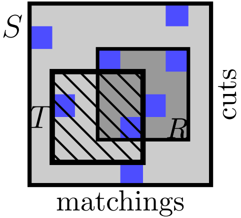


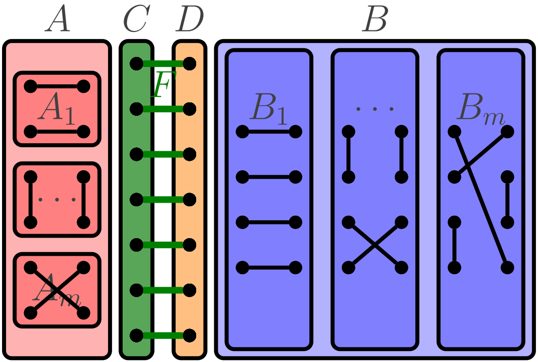
Randomly generate $(U, M) \sim Q_k$:

1. Choose T
2. Choose k edges $F \subseteq C \times D$

$$\mu_k(R) = \mathop{\mathbb{E}}_{T}\left[\mathop{\mathbb{E}}_{|F|=k} \left[\qquad \right]\right]$$

Rewriting $\mu_k(R)$





Randomly generate $(U, M) \sim Q_k$:

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3. Choose $M \supseteq F$

$$\mu_k(R) = \mathop{\mathbb{E}}_{T}\left[\mathop{\mathbb{E}}_{|F|=k}\left[\Pr[M \in R \mid T, H] \qquad\qquad \right]\right]$$

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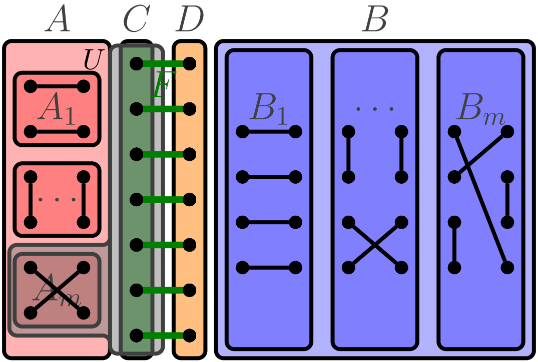
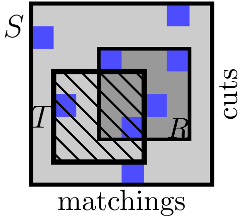




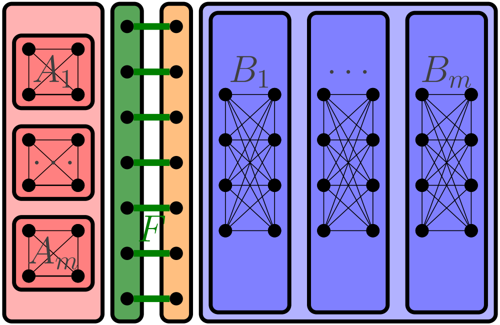
Randomly generate $(U, M) \sim Q_k$:

1. Choose T
2. Choose k edges $F \subseteq C \times D$
3. Choose $M \supseteq F$
4. Choose $U \supseteq C$ (not cutting any A_i)

$$\mu_k(R) = \mathop{\mathbb{E}}_{T}\left[\mathop{\mathbb{E}}_{|F|=k}\left[\Pr[M \in R \mid T, H] \cdot \Pr[U \in R \mid T, H]\right]\right]$$

How does an average partition look like

- Suppose for a fixed (T, F):
  $\mu_k(R) \approx \Pr[(U, M) \in R \mid T, F] =: p$



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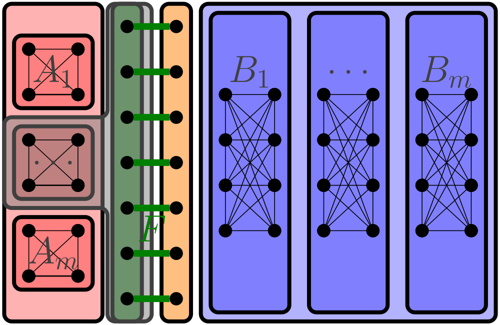

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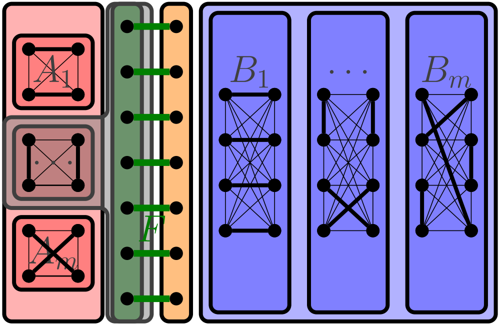

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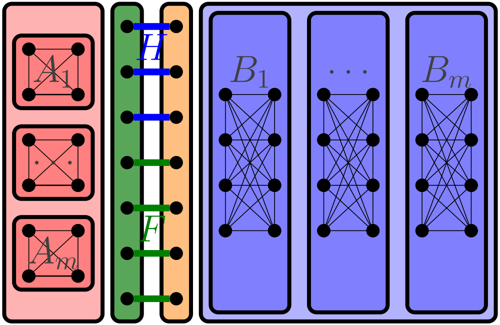
$$\mu_3(R) \approx \mathop{\mathbb{E}}_{H \sim \binom{F}{3}} \left[\; \Pr[(U, M) \in R \mid T, H] \right]$$



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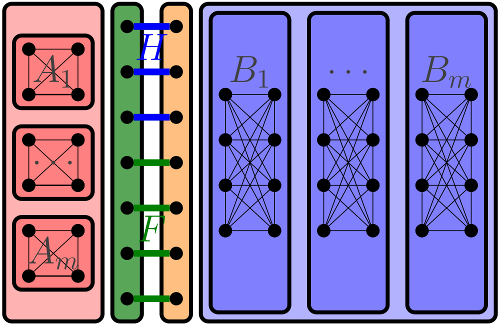

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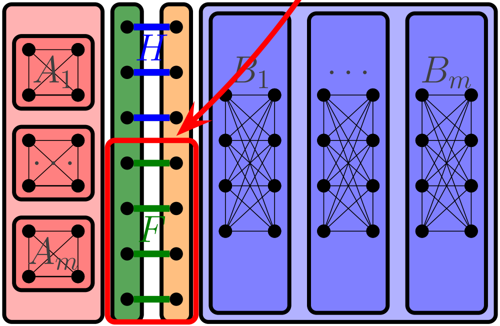
- `GOOD` means it doesn't matter what condition on here



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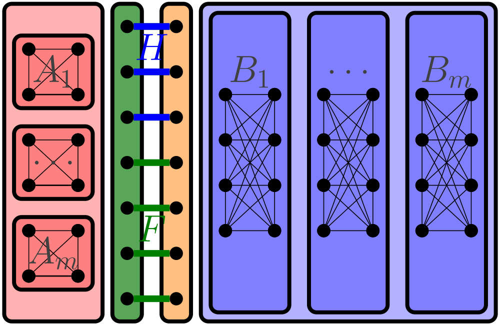

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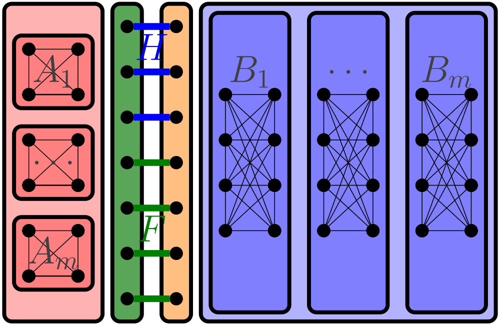

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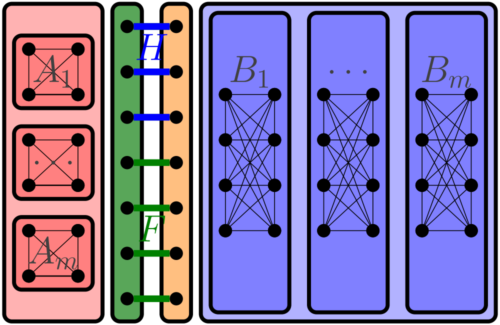
- Suffices to show: $H, H^* \subseteq F$ good $\Rightarrow |H \cap H^*| \geq 2$



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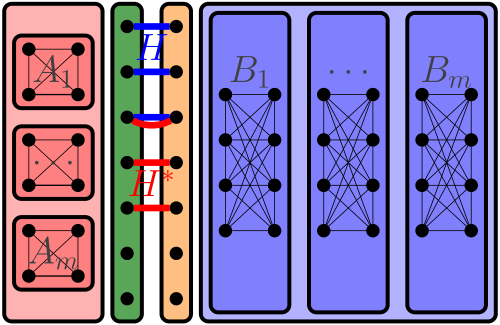
- Suppose $|H \cap H^*| \leq 1$



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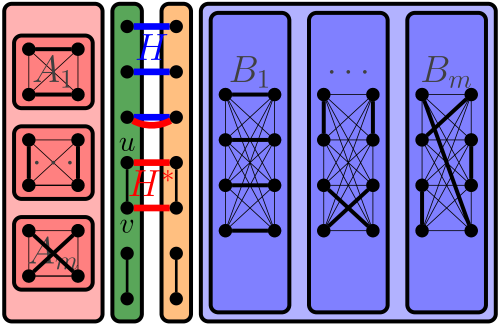
  $\Rightarrow \exists M : \{u, v\} \in M$



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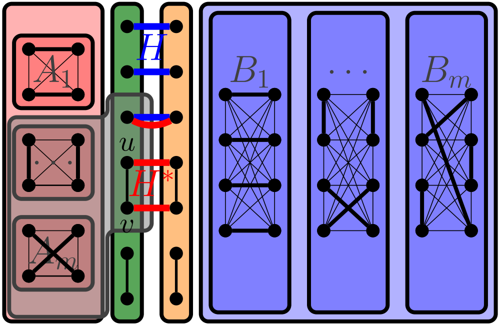
- (T, H^*) good
  $\Rightarrow \exists U : u, v \in U$



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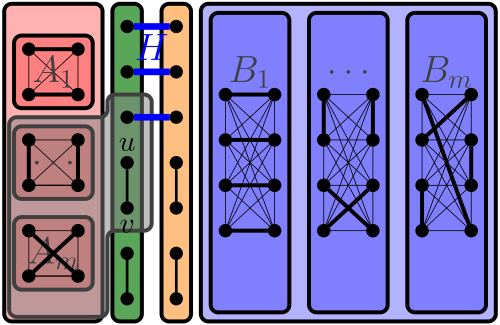
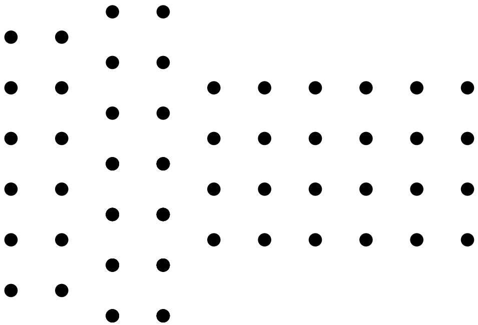
- (T, H) good
  $\Rightarrow \exists M : \{u, v\} \in M$
- (T, H^*) good
  $\Rightarrow \exists U : u, v \in U$
- $|\delta(U) \cap M| = 1$
  Contradiction!



Most partitions are good

Lemma

$\Pr[(T, H) \text{ is } M\text{-bad}] \leq \varepsilon$



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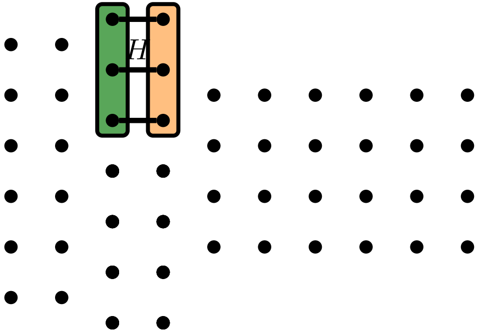

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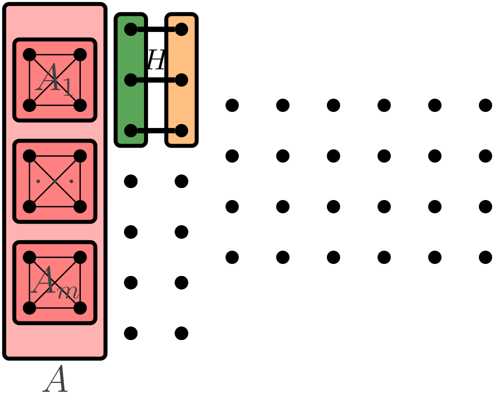
- Pick H, A



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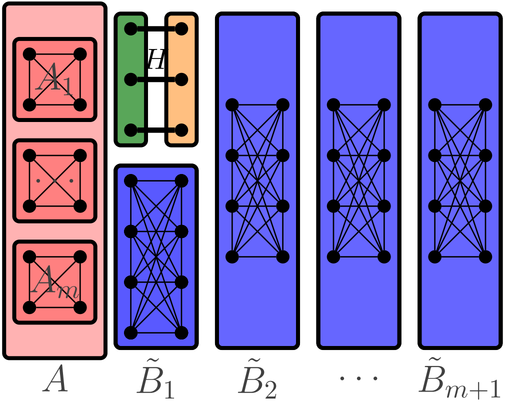
- Pick H, A, $\tilde{B}_1, \ldots, \tilde{B}_{m+1}$.



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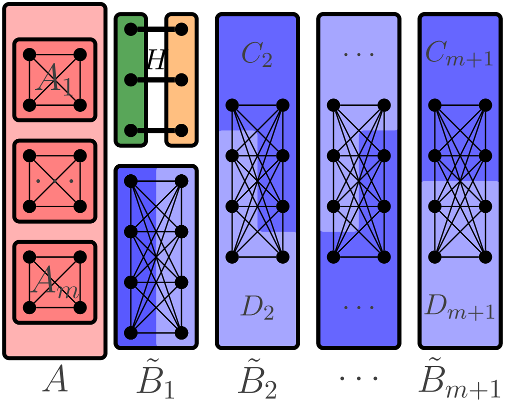
- Pick H, A, $\tilde{B}_1, \ldots, \tilde{B}_{m+1}$. Split $\tilde{B}_i = C_i \dot\cup D_i$.



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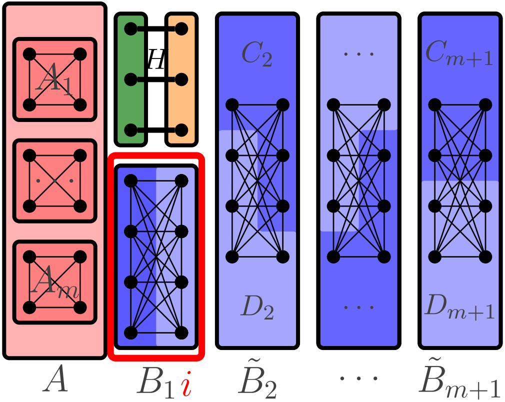
- Pick randomly $i \in \{1, \ldots, m\}$



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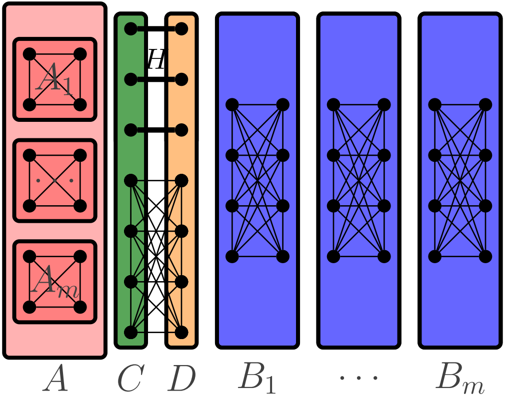
Lemma

$\Pr[(T, H) \text{ is } M\text{-bad}] \le \varepsilon$

- Pick H, A, $\tilde{B}_1, \ldots, \tilde{B}_{m+1}$. Split $\tilde{B}_i = C_i \dot{\cup} D_i$.
- Pick randomly $i \in \{1, \ldots, m\}$ and let $C := C_i$, $D := D_i$



Open problems

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Show that there is no small **SDP** representing the Correlation/TSP/matching polytope!

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Thanks for your attention