Maximization of Submodular Functions

Seffi Naor

Technion
Israel Institute of Technology

Lecture 1

4th Cargese Workshop on Combinatorial Optimization

Submodular Maximization

Optimization Problem

Family of allowed subsets $\mathcal{M} \subseteq 2^{\mathcal{N}}$.

$$\begin{aligned} \max \quad & f(S) \\ s.t. \quad & S \in \mathcal{M} \end{aligned}$$

Submodular Maximization

Optimization Problem

Family of allowed subsets $\mathcal{M} \subseteq 2^{\mathcal{N}}$.

$$\begin{aligned} \max \quad & f(S) \\ s.t. \quad & S \in \mathcal{M} \end{aligned}$$

Question - how is f given ?

Submodular Maximization

Optimization Problem

Family of allowed subsets $\mathcal{M} \subseteq 2^{\mathcal{N}}$.

$$\begin{aligned} \max \quad & f(S) \\ s.t. \quad & S \in \mathcal{M} \end{aligned}$$

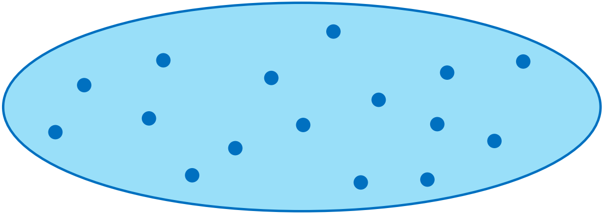
Question - how is f given ?

Value Oracle Model: Returns $f(S)$ for given $S \subseteq \mathcal{N}$.

Unconstrained Submodular Maximization

Definition - Unconstrained Submodular Maximization

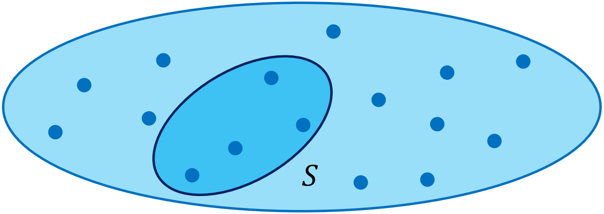
Input: Ground set $\mathcal{N}$ and non-monotone submodular $f : 2^{\mathcal{N}} \rightarrow \mathbb{R}^+$.
Goal: Find $S \subseteq \mathcal{N}$ maximizing $f(S)$.



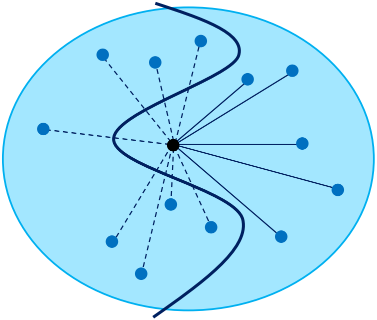
Unconstrained Submodular Maximization

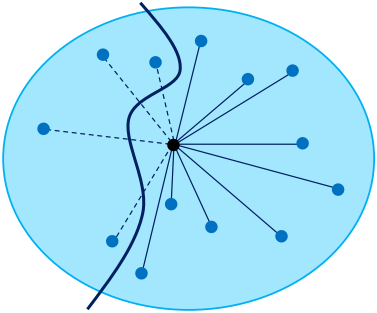
Definition - Unconstrained Submodular Maximization

Input: Ground set $\mathcal{N}$ and non-monotone submodular $f : 2^{\mathcal{N}} \to \mathbb{R}^+$.
Goal: Find $S \subseteq \mathcal{N}$ maximizing $f(S)$.



Undirected Graphs: Cut Function





$$G = (V, E) \quad \Rightarrow \quad \mathcal{N} = V \ , \ f(S) = \delta(S).$$

Unconstrained Submodular Maximization (Cont.)

Captures combinatorial optimization problems:

- **Max-Cut** in graphs and hypergraphs.
- **Max-DiCut**.
- **Max Facility-Location**.
- Variants of **Max-SAT**.

Unconstrained Submodular Maximization (Cont.)

Captures combinatorial optimization problems:

- **Max-Cut** in graphs and hypergraphs.
- **Max-DiCut**.
- **Max Facility-Location**.
- Variants of **Max-SAT**.

Additional settings:

- Marketing over social networks [Hartline-Mirrokni-Sundararajan-08]
- Utility maximization with discrete choice [Ahmed-Atamtürk-09]
- Least core value in supermodular cooperative games [Schulz-Uhan-07]
- Approximating market expansion [Dughmi-Roughgarden-Sundararajan-09]

Unconstrained Submodular Maximization (Cont.)

Operations Research:

[Cherenin-62] [Khachaturov-68] [Minoux-77] [Lee-Nemhauser-Wang-95] [Goldengorin-Sierksma-Tijssen-Tso-98] [Goldengorin-Tijssen-Tso-99] [Goldengorin-Ghosh-04] [Ahmed-Atamtürk-09]

Unconstrained Submodular Maximization (Cont.)

Operations Research:

[Cherenin-62] [Khachaturov-68] [Minoux-77] [Lee-Nemhauser-Wang-95] [Goldengorin-Sierksma-Tijssen-Tso-98] [Goldengorin-Tijssen-Tso-99] [Goldengorin-Ghosh-04] [Ahmed-Atamtürk-09]

Algorithmic Bounds:

$1/4$	random solution	[Feige-Mirrokni-Vondrak-07]
$1/3$	local search	[Feige-Mirrokni-Vondrak-07]
$2/5$	non-oblivious local search	[Feige-Mirrokni-Vondrak-07]

Unconstrained Submodular Maximization (Cont.)

Operations Research:

[Cherenin-62] [Khachaturov-68] [Minoux-77] [Lee-Nemhauser-Wang-95] [Goldengorin-Sierksma-Tijssen-Tso-98] [Goldengorin-Tijssen-Tso-99] [Goldengorin-Ghosh-04] [Ahmed-Atamtürk-09]

Algorithmic Bounds:

$1/4$	random solution	[Feige-Mirrokni-Vondrak-07]
$1/3$	local search	[Feige-Mirrokni-Vondrak-07]
$2/5$	non-oblivious local search	[Feige-Mirrokni-Vondrak-07]
≈ 0.41	simulated annealing	[Gharan-Vondrak-11]

Unconstrained Submodular Maximization (Cont.)

Operations Research:

[Cherenin-62] [Khachaturov-68] [Minoux-77] [Lee-Nemhauser-Wang-95] [Goldengorin-Sierksma-Tijssen-Tso-98] [Goldengorin-Tijssen-Tso-99] [Goldengorin-Ghosh-04] [Ahmed-Atamtürk-09]

Algorithmic Bounds:

$1/4$	random solution	[Feige-Mirrokni-Vondrak-07]
$1/3$	local search	[Feige-Mirrokni-Vondrak-07]
$2/5$	non-oblivious local search	[Feige-Mirrokni-Vondrak-07]
≈ 0.41	simulated annealing	[Gharan-Vondrak-11]
≈ 0.42	structural similarity	[Feldman-N-Schwartz-11]

Unconstrained Submodular Maximization (Cont.)

Operations Research:

[Cherenin-62] [Khachaturov-68] [Minoux-77] [Lee-Nemhauser-Wang-95] [Goldengorin-Sierksma-Tijssen-Tso-98] [Goldengorin-Tijssen-Tso-99] [Goldengorin-Ghosh-04] [Ahmed-Atamtürk-09]

Algorithmic Bounds:

$1/4$	random solution	[Feige-Mirrokni-Vondrak-07]
$1/3$	local search	[Feige-Mirrokni-Vondrak-07]
$2/5$	non-oblivious local search	[Feige-Mirrokni-Vondrak-07]
≈ 0.41	simulated annealing	[Gharan-Vondrak-11]
≈ 0.42	structural similarity	[Feldman-N-Schwartz-11]

Hardness:

- Cannot get better than $1/2$ absolute! [Feige-Mirrokni-Vondrak-07]

Unconstrained Submodular Maximization (Cont.)

Operations Research:

[Cherenin-62] [Khachaturov-68] [Minoux-77] [Lee-Nemhauser-Wang-95] [Goldengorin-Sierksma-Tijssen-Tso-98] [Goldengorin-Tijssen-Tso-99] [Goldengorin-Ghosh-04] [Ahmed-Atamtürk-09]

Algorithmic Bounds:

$1/4$	random solution	[Feige-Mirrokni-Vondrak-07]
$1/3$	local search	[Feige-Mirrokni-Vondrak-07]
$2/5$	non-oblivious local search	[Feige-Mirrokni-Vondrak-07]
≈ 0.41	simulated annealing	[Gharan-Vondrak-11]
≈ 0.42	structural similarity	[Feldman-N-Schwartz-11]

Hardness:

- Cannot get better than $1/2$ absolute! [Feige-Mirrokni-Vondrak-07]

Question

Is $1/2$ the correct answer?

Unconstrained Submodular Maximization (Cont.)

Operations Research:

[Cherenin-62] [Khachaturov-68] [Minoux-77] [Lee-Nemhauser-Wang-95] [Goldengorin-Sierksma-Tijssen-Tso-98] [Goldengorin-Tijssen-Tso-99] [Goldengorin-Ghosh-04] [Ahmed-Atamtürk-09]

Algorithmic Bounds:

$1/4$	random solution	[Feige-Mirrokni-Vondrak-07]
$1/3$	local search	[Feige-Mirrokni-Vondrak-07]
$2/5$	non-oblivious local search	[Feige-Mirrokni-Vondrak-07]
≈ 0.41	simulated annealing	[Gharan-Vondrak-11]
≈ 0.42	structural similarity	[Feldman-N-Schwartz-11]

Hardness:

- Cannot get better than $1/2$ absolute! [Feige-Mirrokni-Vondrak-07]

Question

Is $1/2$ the correct answer?
Yes! [Buchbinder, Feldman, N., Schwartz FOCS 2012]

Failure of Greedy Approach

Greedy: Useful for monotone f (discrete and continuous setting).
[Fisher-Nemhauser-Wolsey-78] [Calinescu-Chekuri-Pal-Vondrak-07]

Failure of Greedy Approach

Greedy: Useful for monotone f (discrete and continuous setting).
[Fisher-Nemhauser-Wolsey-78] [Calinescu-Chekuri-Pal-Vondrak-07]

Fails for Unconstrained Submodular Maximization

Greedy is unbounded!

Failure of Greedy Approach

Greedy: Useful for monotone f (discrete and continuous setting).
[Fisher-Nemhauser-Wolsey-78] [Calinescu-Chekuri-Pal-Vondrak-07]

Fails for Unconstrained Submodular Maximization

Greedy is unbounded!

Key Insight

$$f \text{ is submodular} \quad \Rightarrow \quad \overline{f}(S) \triangleq f(\mathcal{N} \setminus S) \text{ is submodular}$$

Optimal solution of $\overline{f}$ is $\mathcal{N} \setminus OPT$.
Both optima have the same value.

Failure of Greedy Approach

Greedy: Useful for monotone f (discrete and continuous setting).
[Fisher-Nemhauser-Wolsey-78] [Calinescu-Chekuri-Pal-Vondrak-07]

Fails for Unconstrained Submodular Maximization

Greedy is unbounded!

Key Insight

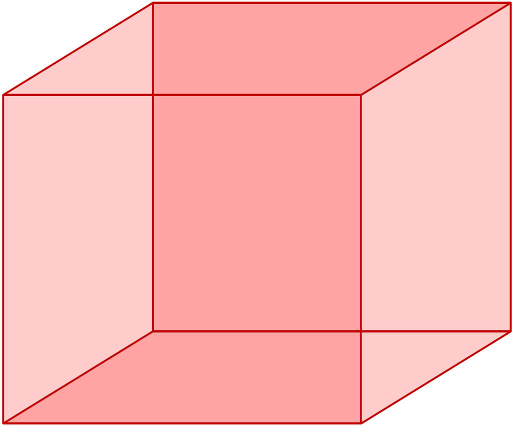
$$f \text{ is submodular} \quad \Rightarrow \quad \overline{f}(S) \triangleq f(\mathcal{N} \setminus S) \text{ is submodular}$$

Optimal solution of $\overline{f}$ is $\mathcal{N} \setminus OPT$.
Both optima have the same value.

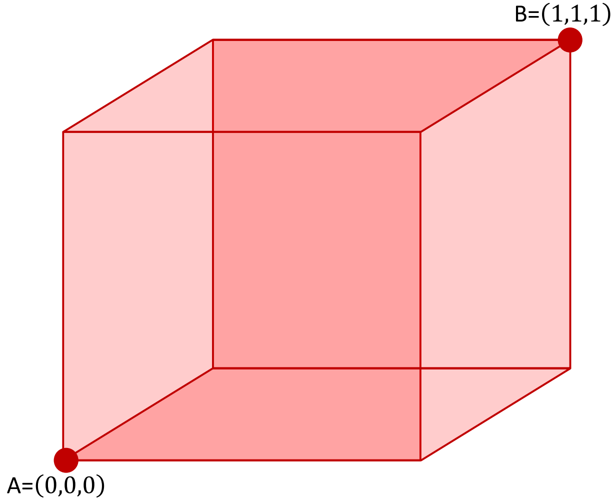
Questions:

- Why start with an empty solution and add elements?
- Why not start with $\mathcal{N}$ and remove elements?

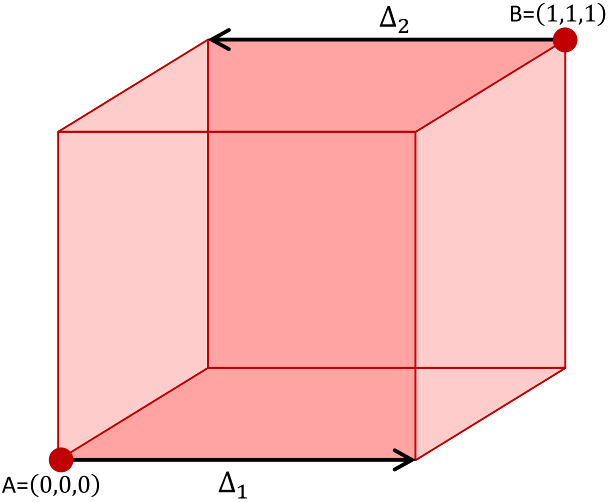
Attempt I - Geometric Interpretation



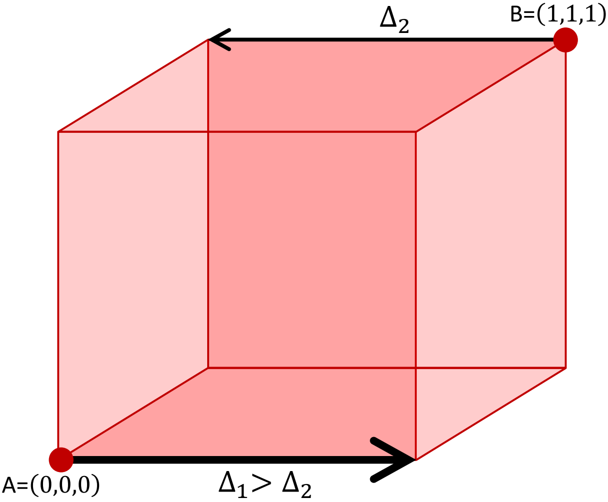
Attempt I - Geometric Interpretation



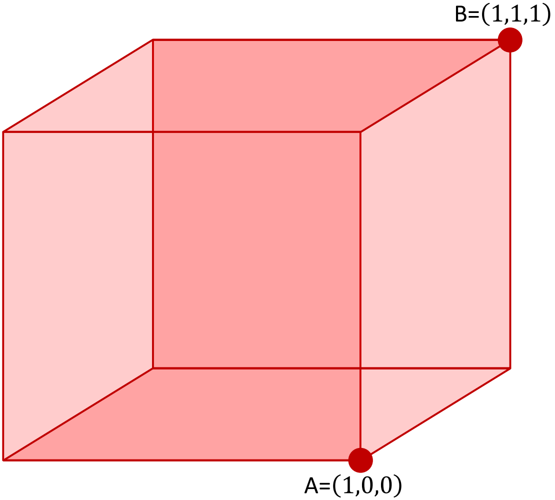
Attempt I - Geometric Interpretation



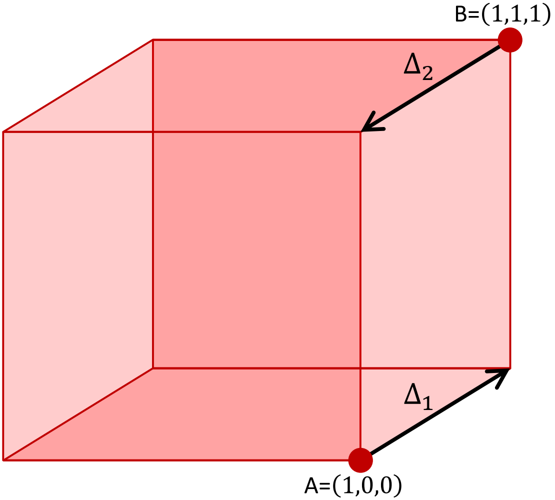
Attempt I - Geometric Interpretation



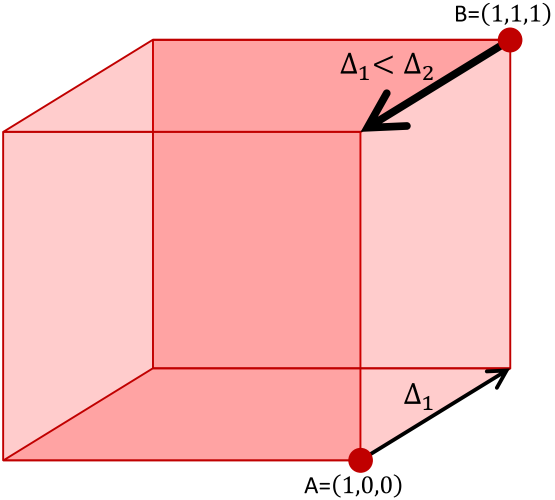
Attempt I - Geometric Interpretation



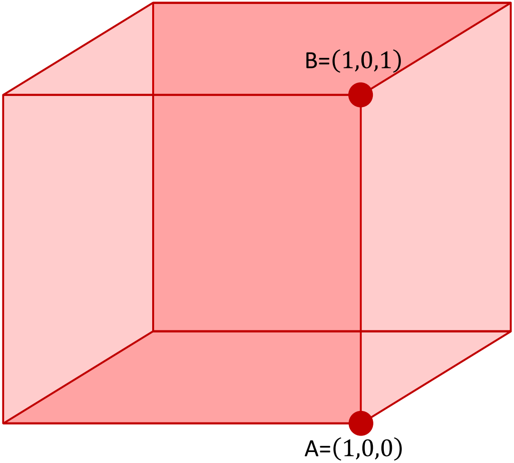
Attempt I - Geometric Interpretation



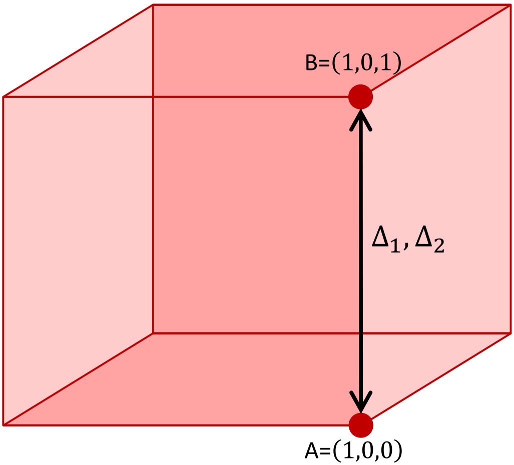
Attempt I - Geometric Interpretation



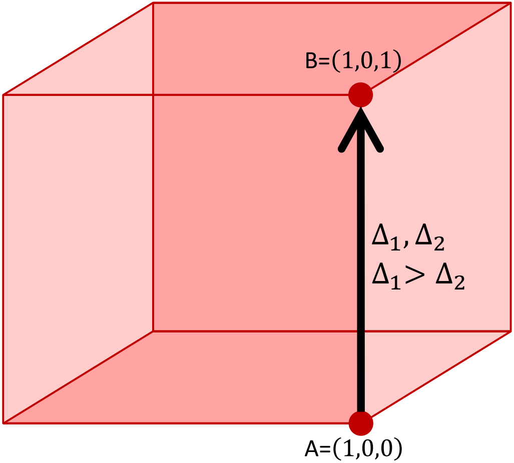
Attempt I - Geometric Interpretation



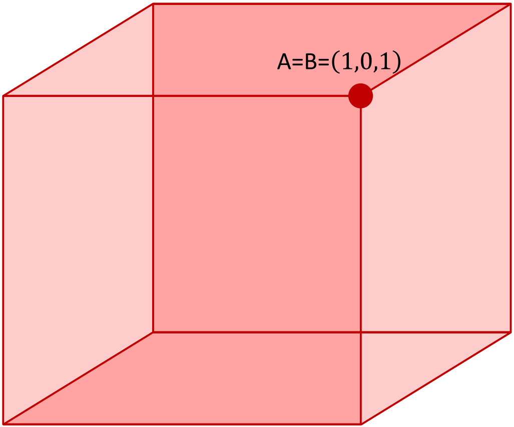
Attempt I - Geometric Interpretation



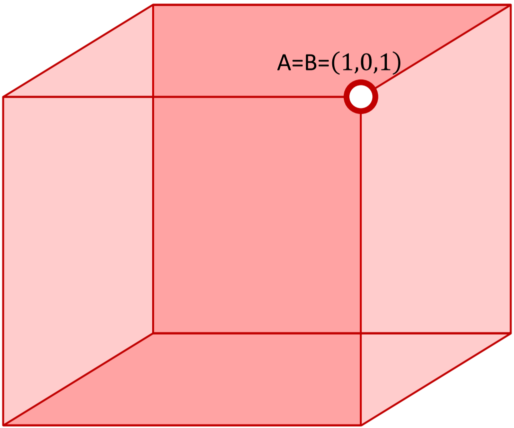
Attempt I - Geometric Interpretation



Attempt I - Geometric Interpretation



Attempt I - Geometric Interpretation



Attempt I - Algorithm

Notation: $\mathcal{N} = \{u_1, u_2, \ldots, u_n\}$

Algorithm I

1. $A \leftarrow \emptyset$, $B \leftarrow \mathcal{N}$.
2. **for** $i = 1$ **to** n **do**:
   $\Delta_1 \leftarrow f(A \cup \{u_i\}) - f(A)$.
   $\Delta_2 \leftarrow f(B \setminus \{u_i\}) - f(B)$.
   if $\Delta_1 \geqslant \Delta_2$ **then** $A \leftarrow A \cup \{u_i\}$.
   else $B \leftarrow B \setminus \{u_i\}$.
3. **Return** A.

Attempt I - Algorithm

Notation: $\mathcal{N} = \{u_1, u_2, \ldots, u_n\}$

Algorithm I

1. $A \leftarrow \emptyset$, $B \leftarrow \mathcal{N}$.
2. **for** $i = 1$ **to** n **do**:
   $\Delta_1 \leftarrow f(A \cup \{u_i\}) - f(A)$.
   $\Delta_2 \leftarrow f(B \setminus \{u_i\}) - f(B)$.
   if $\Delta_1 \geqslant \Delta_2$ **then** $A \leftarrow A \cup \{u_i\}$.
   else $B \leftarrow B \setminus \{u_i\}$.
3. **Return** A.

Theorem [Buchbinder-Feldman-N-Schwartz]

Algorithm I is a $(1/3)$-approximation for **Unconstrained Submodular Maximization**.

Attempt I - Analysis

Observation

By submodularity, $\Delta_1 + \Delta_2 \geqslant 0$.

Attempt I - Analysis

Observation

By submodularity, $\Delta_1 + \Delta_2 \geqslant 0$.

$$A_{i-1} = \text{elements in } A \text{ at the end of iteration } i-1.$$
$$B_{i-1} = \text{elements in } B \text{ at the end of iteration } i-1.$$

$$\Delta_1 + \Delta_2 = (f(A_{i-1} \cup u_i) - f(A_{i-1})) + (f(B_{i-1} \setminus u_i) - f(B_{i-1}))$$

$$= (f(A_{i-1} \cup u_i) + f(B_{i-1} \setminus u_i)) - (f(A_{i-1}) + f(B_{i-1}))$$

$$\geqslant (f(\underbrace{(A_{i-1} \cup u_i) \cup (B_{i-1} \setminus u_i)}_{B_{i-1}}) + f(\underbrace{(A_{i-1} \cup u_i) \cap (B_{i-1} \setminus u_i)}_{A_{i-1}})$$
$$- (f(A_{i-1}) + f(B_{i-1})) = 0$$

□

Attempt I - Analysis

Intuition: Bound loss in terms of gain.

Attempt I - Analysis

Intuition: Bound loss in terms of gain.

Define: $OPT_i \triangleq (OPT \cup A_i) \cap B_i$

- agrees with A_i and B_i on first i elements
- agrees with OPT on last $n - i$ elements

Attempt I - Analysis

Intuition: Bound loss in terms of gain.

Define: $OPT_i \triangleq (OPT \cup A_i) \cap B_i$

- agrees with A_i and B_i on first i elements
- agrees with OPT on last $n - i$ elements

Evolution of OPT_i:

$$f(OPT_0) = f(OPT \cup A_0) \cap B_0)) = f(OPT)$$

$$\vdots$$

$$f(OPT_n) = f(OPT \cup A_n) \cap B_n)) = f(ALG)$$

Attempt I - Analysis

Lemma

$$f(OPT_{i-1}) - f(OPT_i) \leqslant \begin{cases} f(A_i) - f(A_{i-1}) = \Delta_1 & u_i \in A_i \\ f(B_i) - f(B_{i-1}) = \Delta_2 & u_i \notin B_i \end{cases}$$

Attempt I - Analysis

Lemma

$$f(OPT_{i-1}) - f(OPT_i) \leqslant \begin{cases} f(A_i) - f(A_{i-1}) = \Delta_1 & u_i \in A_i \\ f(B_i) - f(B_{i-1}) = \Delta_2 & u_i \notin B_i \end{cases}$$

Case (1) $u_i \in A_i \quad (\Rightarrow \quad \Delta_1 \geqslant \Delta_2)$.
Thus, $OPT_i = OPT_{i-1} \cup u_i$, $A_i = A_{i-1} \cup u_i$, $B_i = B_{i-1}$.

Attempt I - Analysis

Lemma

$$f(OPT_{i-1}) - f(OPT_i) \leqslant \begin{cases} f(A_i) - f(A_{i-1}) = \Delta_1 & u_i \in A_i \\ f(B_i) - f(B_{i-1}) = \Delta_2 & u_i \notin B_i \end{cases}$$

Case (1) $u_i \in A_i \qquad (\Rightarrow \quad \Delta_1 \geqslant \Delta_2)$.
Thus, $OPT_i = OPT_{i-1} \cup u_i$, $A_i = A_{i-1} \cup u_i$, $B_i = B_{i-1}$.

If $u_i \in OPT_{i-1}$: $\;f(OPT_{i-1}) - f(OPT_i) = 0 \leqslant \Delta_1$,
since $\Delta_1 \geqslant \Delta_2$ and $\Delta_1 \geqslant -\Delta_2$.

Attempt I - Analysis

Lemma

$$f(OPT_{i-1}) - f(OPT_i) \leqslant \begin{cases} f(A_i) - f(A_{i-1}) = \Delta_1 & u_i \in A_i \\ f(B_i) - f(B_{i-1}) = \Delta_2 & u_i \notin B_i \end{cases}$$

Case (1) $u_i \in A_i \quad (\Rightarrow \quad \Delta_1 \geqslant \Delta_2)$.
Thus, $OPT_i = OPT_{i-1} \cup u_i$, $A_i = A_{i-1} \cup u_i$, $B_i = B_{i-1}$.

If $u_i \in OPT_{i-1}$: $\quad f(OPT_{i-1}) - f(OPT_i) = 0 \leqslant \Delta_1$,
since $\Delta_1 \geqslant \Delta_2$ and $\Delta_1 \geqslant -\Delta_2$.

Else $u_i \notin OPT_{i-1}$:

$$\begin{aligned} f(OPT_{i-1}) - f(OPT_i) &= f(OPT_{i-1}) - f(OPT_{i-1} \cup u_i) \\ &\leqslant f(B_{i-1} \setminus u_i) - f(B_{i-1}) = \Delta_2 \leqslant \Delta_1 \end{aligned}$$

By submodularity, since $OPT_{i-1} \subseteq B_{i-1} \setminus u_i$

Attempt I - Analysis

Lemma

$$f(OPT_{i-1}) - f(OPT_i) \leqslant \begin{cases} f(A_i) - f(A_{i-1}) = \Delta_1 & u_i \in A_i \\ f(B_i) - f(B_{i-1}) = \Delta_2 & u_i \notin B_i \end{cases}$$

Case (1) $u_i \in A_i \quad (\Rightarrow \quad \Delta_1 \geqslant \Delta_2)$.
Thus, $OPT_i = OPT_{i-1} \cup u_i$, $A_i = A_{i-1} \cup u_i$, $B_i = B_{i-1}$.

If $u_i \in OPT_{i-1}$: $\ f(OPT_{i-1}) - f(OPT_i) = 0 \leqslant \Delta_1$,
since $\Delta_1 \geqslant \Delta_2$ and $\Delta_1 \geqslant -\Delta_2$.

Else $u_i \notin OPT_{i-1}$:

$$\begin{aligned} f(OPT_{i-1}) - f(OPT_i) &= f(OPT_{i-1}) - f(OPT_{i-1} \cup u_i) \\ &\leqslant f(B_{i-1} \setminus u_i) - f(B_{i-1}) = \Delta_2 \leqslant \Delta_1 \end{aligned}$$

By submodularity, since $OPT_{i-1} \subseteq B_{i-1} \setminus u_i$

Case (2): analogous ($u_i \notin A_i$)

Potential Function:

Attempt I - Analysis Overview (Cont.)

Potential Function:

$$\Phi_i = f(A_i) + f(B_i) + f(OPT_i)$$

Attempt I - Analysis Overview (Cont.)

Potential Function:

$$\Phi_i = f(A_i) + f(B_i) + f(OPT_i)$$

Note that:

$$\Phi_0 \geqslant f(OPT)$$
$$\Phi_n = 3 \cdot f(ALG)$$

Attempt I - Analysis Overview (Cont.)

Potential Function:

$$\Phi_i = f(A_i) + f(B_i) + f(OPT_i)$$

Note that:

$$\Phi_0 \geqslant f(OPT)$$
$$\Phi_n = 3 \cdot f(ALG)$$

Potential is non-decreasing:

$$\Phi_i - \Phi_{i-1} = f(A_i) + f(B_i) + f(OPT_i) - f(A_{i-1}) - f(B_{i-1}) - f(OPT_{i-1}) \geqslant 0$$

Attempt I - Analysis Overview (Cont.)

Potential Function:

$$\Phi_i = f(A_i) + f(B_i) + f(OPT_i)$$

Note that:

$$\Phi_0 \geqslant f(OPT)$$
$$\Phi_n = 3 \cdot f(ALG)$$

Potential is non-decreasing:

$$\Phi_i - \Phi_{i-1} = f(A_i) + f(B_i) + f(OPT_i) - f(A_{i-1}) - f(B_{i-1}) - f(OPT_{i-1}) \geqslant 0$$

$$\Rightarrow \quad 3 \cdot f(ALG) = \Phi_n \geqslant \Phi_0 \geqslant f(OPT) \quad \Rightarrow \quad f(ALG) \geqslant \frac{1}{3} \cdot f(OPT)$$

Comment: analysis is tight.

Attempt II

Reminder

$$\Delta_1 = f(A \cup \{u_i\}) - f(A)$$
$$\Delta_2 = f(B \setminus \{u_i\}) - f(B)$$

Attempt II

Reminder

$$\Delta_1 = f(A \cup \{u_i\}) - f(A)$$
$$\Delta_2 = f(B \setminus \{u_i\}) - f(B)$$

- $\Delta_1 + \Delta_2 \geqslant 0$ always (submodularity).
- $\Delta_1 \geqslant 0$ and $\Delta_2 < 0 \quad \Rightarrow \quad$ add u_i to A.
- $\Delta_1 < 0$ and $\Delta_2 \geqslant 0 \quad \Rightarrow \quad$ remove u_i from B.

Attempt II

Reminder

$$\Delta_1 = f(A \cup \{u_i\}) - f(A)$$
$$\Delta_2 = f(B \setminus \{u_i\}) - f(B)$$

- $\Delta_1 + \Delta_2 \geqslant 0$ always (submodularity).
- $\Delta_1 \geqslant 0$ and $\Delta_2 < 0 \quad \Rightarrow \quad$ add u_i to A.
- $\Delta_1 < 0$ and $\Delta_2 \geqslant 0 \quad \Rightarrow \quad$ remove u_i from B.

Question: What should we do when $\Delta_1, \Delta_2 \geqslant 0$? or $\Delta_1 = \Delta_2$?

Attempt II

Reminder

$$\Delta_1 = f(A \cup \{u_i\}) - f(A)$$
$$\Delta_2 = f(B \setminus \{u_i\}) - f(B)$$

- $\Delta_1 + \Delta_2 \geqslant 0$ always (submodularity).
- $\Delta_1 \geqslant 0$ and $\Delta_2 < 0 \quad \Rightarrow \quad$ add u_i to A.
- $\Delta_1 < 0$ and $\Delta_2 \geqslant 0 \quad \Rightarrow \quad$ remove u_i from B.

Question: What should we do when $\Delta_1, \Delta_2 \geqslant 0$? or $\Delta_1 = \Delta_2$?

There is no reason a-priori to prefer one choice over the other!

Solution

When $\Delta_1, \Delta_2 \geqslant 0$, choose at random in proportion to their values.

Attempt II

Algorithm II

1. $A \leftarrow \emptyset$, $B \leftarrow \mathcal{N}$.
2. **for** $i = 1$ **to** n **do**:
   $\Delta_1 \leftarrow \max\{f(A \cup \{u_i\}) - f(A), 0\}$.
   $\Delta_2 \leftarrow \max\{f(B \setminus \{u_i\}) - f(B), 0\}$.
   With probability $\Delta_1/(\Delta_1 + \Delta_2)$: $A \leftarrow A \cup \{u_i\}$.
   With the complement probability: $B \leftarrow B \setminus \{u_i\}$.
3. **Return** A.

Attempt II

Algorithm II

1. $A \leftarrow \emptyset$, $B \leftarrow \mathcal{N}$.
2. **for** $i = 1$ **to** n **do**:
   $\Delta_1 \leftarrow \max\{f(A \cup \{u_i\}) - f(A), 0\}$.
   $\Delta_2 \leftarrow \max\{f(B \setminus \{u_i\}) - f(B), 0\}$.
   With probability $\Delta_1/(\Delta_1 + \Delta_2)$: $A \leftarrow A \cup \{u_i\}$.
   With the complement probability: $B \leftarrow B \setminus \{u_i\}$.
3. **Return** A.

Theorem [Buchbinder-Feldman-N-Schwartz]

Algorithm II is a linear-time tight $(1/2)$-approximation for **Unconstrained Submodular Maximization**.

Attempt II (Analysis)

Potential Function:

$$\Phi_i = f(A_i) + f(B_i) + 2 \cdot OPT_i$$

A_i, B_i and OPT_i are random variables.

Attempt II (Analysis)

Potential Function:

$$\Phi_i = f(A_i) + f(B_i) + 2 \cdot OPT_i$$

A_i, B_i and OPT_i are random variables.

Note that:

$$\Phi_0 \geqslant 2 \cdot f(OPT)$$
$$\Phi_n = 4 \cdot f(ALG)$$

New main observation:

$$\mathbb{E}[OPT_{i-1} - OPT_i] \leqslant \frac{1}{2} \cdot \mathbb{E}[f(A_i) - f(A_{i-1})] + \frac{1}{2} \cdot \mathbb{E}[f(B_i) - f(B_{i-1})]$$

Attempt II (Analysis)

Potential Function:

$$\Phi_i = f(A_i) + f(B_i) + 2 \cdot OPT_i$$

A_i, B_i and OPT_i are random variables.

Note that:

$$\Phi_0 \geqslant 2 \cdot f(OPT)$$
$$\Phi_n = 4 \cdot f(ALG)$$

New main observation:

$$\mathbb{E}[OPT_{i-1} - OPT_i] \leqslant \frac{1}{2} \cdot \mathbb{E}[f(A_i) - f(A_{i-1})] + \frac{1}{2} \cdot \mathbb{E}[f(B_i) - f(B_{i-1})]$$

$$\mathbb{E}[\Phi_{i+1} - \Phi_i] \geqslant 0 \quad \Rightarrow \quad 4 \cdot \mathbb{E}[f(ALG)] = \mathbb{E}[\Phi_n] \geqslant \Phi_0 \geqslant 2 \cdot f(OPT)$$

Attempt II - Analysis

Lemma

$$\mathbb{E}[OPT_{i-1} - OPT_i] \leqslant \frac{1}{2} \cdot \mathbb{E}[f(A_i) - f(A_{i-1})] + \frac{1}{2} \cdot \mathbb{E}[f(B_i) - f(B_{i-1})]$$

Attempt II - Analysis

Lemma

$$\mathbb{E}[OPT_{i-1} - OPT_i] \leqslant \frac{1}{2} \cdot \mathbb{E}[f(A_i) - f(A_{i-1})] + \frac{1}{2} \cdot \mathbb{E}[f(B_i) - f(B_{i-1})]$$

Suffices to condition on A_{i-1}, B_{i-1}, OPT_{i-1} to prove the lemma.

Attempt II - Analysis

Lemma

$$\mathbb{E}[OPT_{i-1} - OPT_i] \leqslant \frac{1}{2} \cdot \mathbb{E}[f(A_i) - f(A_{i-1})] + \frac{1}{2} \cdot \mathbb{E}[f(B_i) - f(B_{i-1})]$$

Suffices to condition on A_{i-1}, B_{i-1}, OPT_{i-1} to prove the lemma.

Interesting case: $\Delta_1, \Delta_2 > 0$

With probability $\frac{\Delta_1}{\Delta_1+\Delta_2}$, $A_i = A_{i-1} \cup u_i$, $B_i = B_{i-1}$.

With probability $\frac{\Delta_2}{\Delta_1+\Delta_2}$, $A_i = A_{i-1}$, $B_i = B_{i-1} \setminus u_i$.

Attempt II - Analysis

Left Handside:

$$\mathbb{E}[OPT_{i-1} - OPT_i] = \frac{\Delta_1}{\Delta_1 + \Delta_2} \cdot (f(OPT_{i-1}) - f(OPT_{i-1} \cup u_i))$$

$$+ \frac{\Delta_2}{\Delta_1 + \Delta_2} \cdot (f(OPT_{i-1}) - f(OPT_{i-1} \setminus u_i)) \underbrace{\leqslant}_{?} \frac{\Delta_1 \cdot \Delta_2}{\Delta_1 + \Delta_2}$$

Attempt II - Analysis

Left Handside:

$$\mathbb{E}[OPT_{i-1} - OPT_i] = \frac{\Delta_1}{\Delta_1 + \Delta_2} \cdot (f(OPT_{i-1}) - f(OPT_{i-1} \cup u_i))$$
$$+ \frac{\Delta_2}{\Delta_1 + \Delta_2} \cdot (f(OPT_{i-1}) - f(OPT_{i-1} \setminus u_i)) \underbrace{\leqslant}_{?} \frac{\Delta_1 \cdot \Delta_2}{\Delta_1 + \Delta_2}$$

Right Handside:

$$\mathbb{E}[f(A_i) - f(A_{i-1})] + \mathbb{E}[f(B_i) - f(B_{i-1})] =$$
$$\frac{\Delta_1}{\Delta_1 + \Delta_2} \cdot \underbrace{(f(A_{i-1} \cup u_i) - f(A_i))}_{\Delta_1} + \frac{\Delta_2}{\Delta_1 + \Delta_2} \cdot \underbrace{(f(B_{i-1} \setminus u_i) - f(B_{i-1}))}_{\Delta_2}$$
$$= \frac{\Delta_1^2 + \Delta_2^2}{\Delta_1 + \Delta_2}$$

Attempt II - Analysis

Clearly:

$$\frac{\Delta_1 \cdot \Delta_2}{\Delta_1 + \Delta_2} \leqslant \frac{1}{2} \cdot \frac{\Delta_1^2 + \Delta_2^2}{\Delta_1 + \Delta_2}$$

Attempt II - Analysis

Clearly:

$$\frac{\Delta_1 \cdot \Delta_2}{\Delta_1 + \Delta_2} \leqslant \frac{1}{2} \cdot \frac{\Delta_1^2 + \Delta_2^2}{\Delta_1 + \Delta_2}$$

It remains to be convinced that:

$$\mathbb{E}[OPT_{i-1} - OPT_i] = \frac{\Delta_1}{\Delta_1 + \Delta_2} \cdot (f(OPT_{i-1}) - f(OPT_{i-1} \cup u_i))$$
$$+ \frac{\Delta_2}{\Delta_1 + \Delta_2} \cdot (f(OPT_{i-1}) - f(OPT_{i-1} \setminus u_i)) \underbrace{\leqslant}_{?} \frac{\Delta_1 \cdot \Delta_2}{\Delta_1 + \Delta_2}$$

Attempt II - Analysis

If $u_i \in OPT_{i-1}$:

$$f(OPT_{i-1}) - f(OPT_{i-1} \cup u_i) = 0$$

$$f(OPT_{i-1}) - f(OPT_{i-1} \setminus u_i) \leqslant f(A_{i-1} \cup u_i) - f(A_{i-1}) = \Delta_1$$

(By submodularity, since $A_{i-1} \subseteq OPT_{i-1} \setminus u_i$)

$$\Rightarrow \mathbb{E}[OPT_{i-1} - OPT_i] \leqslant \frac{\Delta_2}{\Delta_1 + \Delta_2} \cdot \Delta_1$$

Attempt II - Analysis

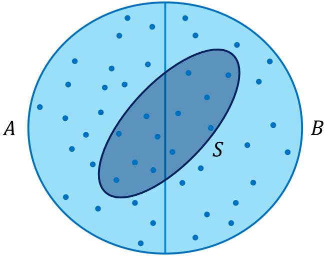
If $u_i \in OPT_{i-1}$:

$$f(OPT_{i-1}) - f(OPT_{i-1} \cup u_i) = 0$$
$$f(OPT_{i-1}) - f(OPT_{i-1} \setminus u_i) \leqslant f(A_{i-1} \cup u_i) - f(A_{i-1}) = \Delta_1$$
(By submodularity, since $A_{i-1} \subseteq OPT_{i-1} \setminus u_i$)
$$\Rightarrow \mathbb{E}[OPT_{i-1} - OPT_i] \leqslant \frac{\Delta_2}{\Delta_1 + \Delta_2} \cdot \Delta_1$$

Else $u_i \notin OPT_{i-1}$:

$$f(OPT_{i-1}) - f(OPT_{i-1} \setminus u_i) = 0$$
$$f(OPT_{i-1}) - f(OPT_{i-1} \cup u_i) \leqslant f(B_{i-1} \setminus u_i) - f(B_{i-1}) = \Delta_2$$
(By submodularity, since $OPT_{i-1} \subseteq B_{i-1} \setminus u_i$)
$$\Rightarrow \mathbb{E}[OPT_{i-1} - OPT_i] \leqslant \frac{\Delta_1}{\Delta_1 + \Delta_2} \cdot \Delta_2$$

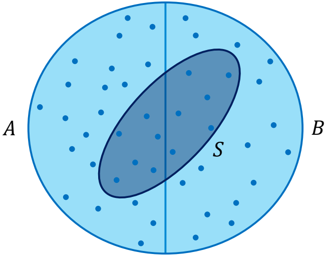
Unconstrained Submodular Maximization - Hardness



$$|A| = |B| = \frac{n}{2}$$

$$\begin{cases} k = |S \cap A| \\ \ell = |S \cap B| \end{cases}$$

Unconstrained Submodular Maximization - Hardness

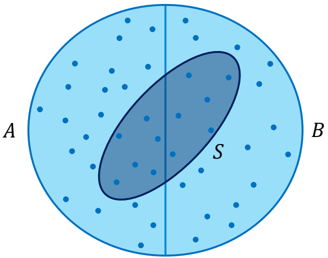


$$f(S) \approx \begin{cases} |S|\,(n - |S|) & |k - \ell| \leqslant \varepsilon n \\ k(n - 2\ell) + \ell(n - 2k) & \text{otherwise} \end{cases}$$

$$|A| = |B| = \frac{n}{2}$$

$$\begin{cases} k = |S \cap A| \\ \ell = |S \cap B| \end{cases}$$

Unconstrained Submodular Maximization - Hardness



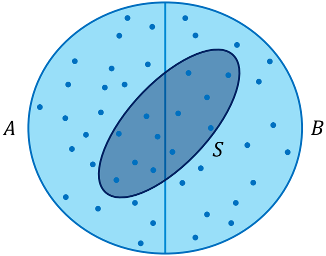
$$|A| = |B| = \frac{n}{2}$$

$$\begin{cases} k = |S \cap A| \\ \ell = |S \cap B| \end{cases}$$

$$f(S) \approx \begin{cases} |S|\,(n - |S|) & |k - \ell| \leqslant \varepsilon n \\ k(n - 2\ell) + \ell(n - 2k) & \text{otherwise} \end{cases}$$

$$\max\{f(S)\} \approx \begin{cases} n^2/4 & |k - \ell| \leqslant \varepsilon n \\ n^2/2 & \text{otherwise} \end{cases}$$

Unconstrained Submodular Maximization - Hardness



$$|A| = |B| = \frac{n}{2}$$

$$\begin{cases} k = |S \cap A| \\ \ell = |S \cap B| \end{cases}$$

$$f(S) \approx \begin{cases} |S|\,(n - |S|) & |k - \ell| \leqslant \varepsilon n \\ k(n-2\ell) + \ell(n-2k) & \text{otherwise} \end{cases}$$

$$\max\{f(S)\} \approx \begin{cases} n^2/4 & |k - \ell| \leqslant \varepsilon n \\ n^2/2 & \text{otherwise} \end{cases}$$

Observation

(A, B) random

$$\Downarrow$$

$$\Pr\left[|k - \ell| > \varepsilon n\right] \leqslant 2e^{-\varepsilon^2 n/4}$$

The case $|k - \ell| > \varepsilon n$ is missed!

Continuous Relaxations of Submodular Functions

Minimization: Convex closure f^-

- maximum (pointwise) convex function that lower bounds f:

$$f^-(x) = \min_{\mathcal{D}} \ \mathbb{E}_{U \in \mathcal{D}}[f(U)] \ , \ \forall x \in [0,1]^{\mathcal{N}}$$

- $\mathcal{D}$ preserves marginals: $\Pr[u_i \in U] = x_i$
- if x is integral then $f(x) = f^-(x)$

Continuous Relaxations of Submodular Functions

Minimization: Convex closure f^-

- maximum (pointwise) convex function that lower bounds f:

$$f^-(x) = \min_{\mathcal{D}} \ \mathbb{E}_{U \in \mathcal{D}}[f(U)] \ , \ \forall x \in [0,1]^{\mathcal{N}}$$

- $\mathcal{D}$ preserves marginals: $\Pr[u_i \in U] = x_i$
- if x is integral then $f(x) = f^-(x)$

Submodular Functions:

- $f^- = f^L$ (Lovasz extension):

$$f^L(x) = \mathbb{E}_\theta[f(\{i : \ x_i > \theta\})], \quad \theta \in [0,1] \ u.r., \quad \forall x \in [0,1]^{\mathcal{N}}$$

- Nice probabilistic interpretation: correlated rounding of elements with respect to uniform choice of $\theta \in [0,1]$.
- Submodular objective function value is preserved in expectation.

Continuous Relaxations of Submodular Functions

Maximization: Concave closure f^+

- minimum (pointwise) concave function that upper bounds f:

$$f^+(x) = \max_{\mathcal{D}} \ \mathbb{E}_{U \in \mathcal{D}}[f(U)] \ , \ \forall x \in [0,1]^{\mathcal{N}}$$

- $\mathcal{D}$ preserves marginals: $\Pr[u_i \in U] = x_i$
- if x is integral then $f(x) = f^+(x)$

Continuous Relaxations of Submodular Functions

Maximization: Concave closure f^+

- minimum (pointwise) concave function that upper bounds f:

$$f^+(x) = \max_{\mathcal{D}} \ \mathbb{E}_{U \in \mathcal{D}}[f(U)] \ , \ \forall x \in [0,1]^{\mathcal{N}}$$

- $\mathcal{D}$ preserves marginals: $\Pr[u_i \in U] = x_i$
- if x is integral then $f(x) = f^+(x)$

Submodular Functions:

- no compact representation, even for submodular functions
- could be helpful for (undirected) max cut ...

Useful Relaxation of Submodular Maximization

Multilinear Extension:

$$F(x) = \sum_{R \subseteq \mathcal{N}} f(R) \prod_{u_i \in R} x_i \prod_{u_i \notin R} (1 - x_i) \quad , \quad \forall x \in [0,1]^{\mathcal{N}}$$

Note:

- Simple probabilistic interpretation: independent rounding of elements to $\{0,1\}$
- If x is integral then $f(x) = F(x)$
- F is neither convex nor concave
- Rounding in the **unconstrained** case is easy - sample independently from the distribution.

Continuous Algorithm

Continuous counterpart of Algorithm II

- The sets $A, B \subseteq 2^{\mathcal{N}}$ are replaced by vectors $a, b \in [0,1]^{\mathcal{N}}$
- Initially: $a \leftarrow 0^{\mathcal{N}}$ (empty solution), $b \leftarrow 1^{\mathcal{N}}$ (full solution)
- In each step:
    - uses multilinear extension F instead of the submodular function f
    - assigns a **fractional** value to the elements (when $\Delta_1, \Delta_2 \geqslant 0$) instead of a **randomized** choice.

Continuous Algorithm

Continuous counterpart of Algorithm II

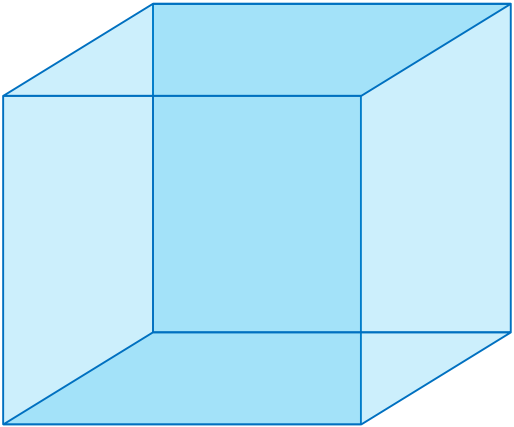
- The sets $A, B \subseteq 2^{\mathcal{N}}$ are replaced by vectors $a, b \in [0,1]^{\mathcal{N}}$
- Initially: $a \leftarrow 0^{\mathcal{N}}$ (empty solution), $b \leftarrow 1^{\mathcal{N}}$ (full solution)
- In each step:
  - uses multilinear extension F instead of the submodular function f
  - assigns a **fractional** value to the elements (when $\Delta_1, \Delta_2 \geqslant 0$) instead of a **randomized** choice.

The continuous counterpart of Algorithm II yields:

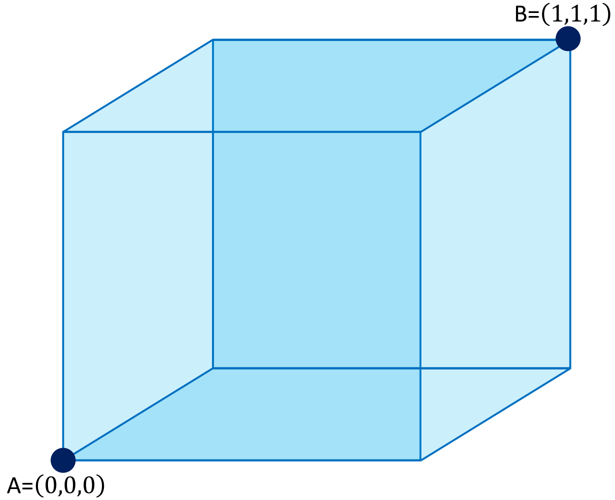
$$F(ALG) \geqslant \frac{f(OPT)}{2}.$$

▪

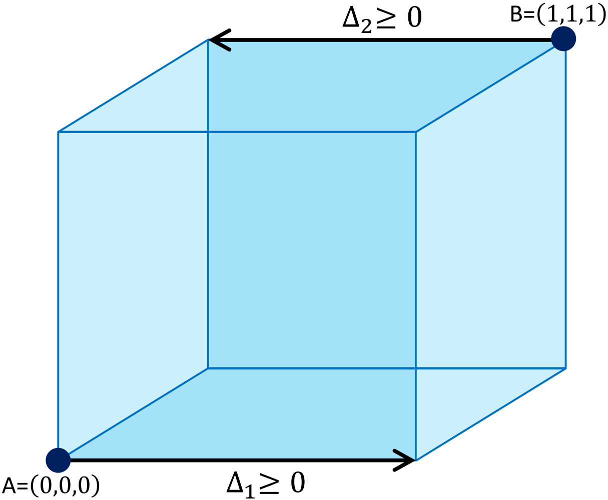
Attempt II - Continuous Approach



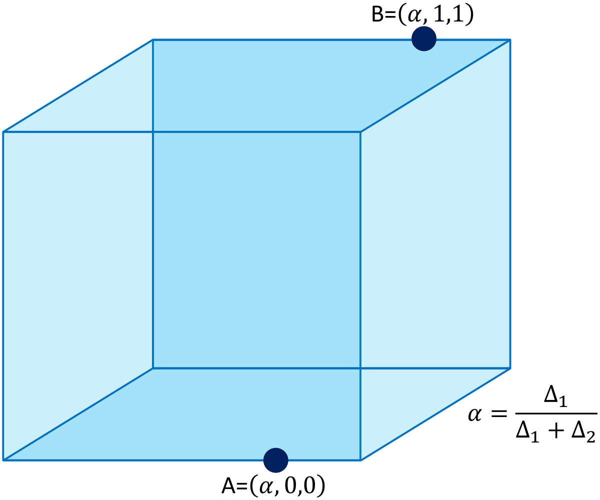
Attempt II - Continuous Approach



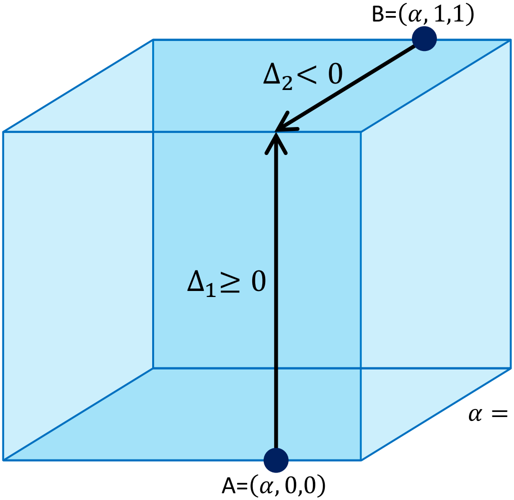
Attempt II - Continuous Approach



Attempt II - Continuous Approach

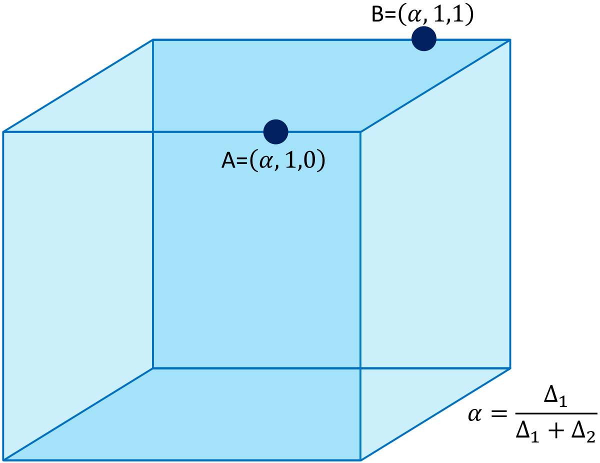


Attempt II - Continuous Approach

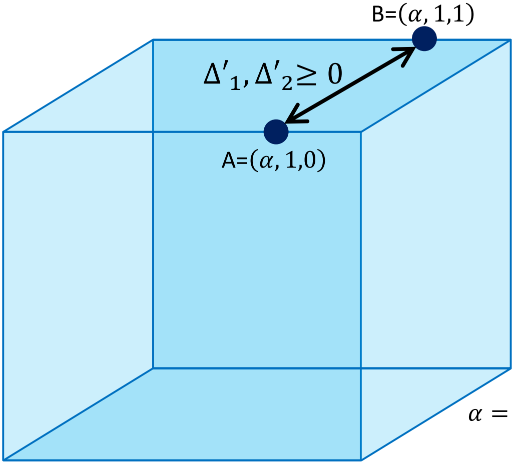


$$\alpha = \frac{\Delta_1}{\Delta_1 + \Delta_2}$$

Attempt II - Continuous Approach

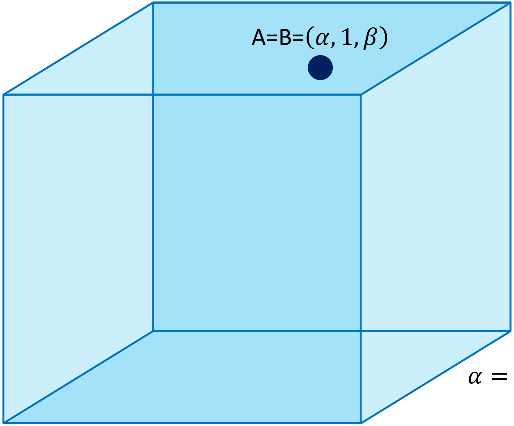


Attempt II - Continuous Approach



$$\alpha = \frac{\Delta_1}{\Delta_1 + \Delta_2}$$

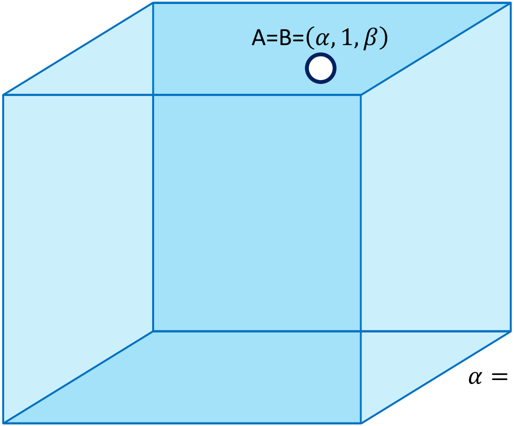
Attempt II - Continuous Approach



$$\alpha = \frac{\Delta_1}{\Delta_1 + \Delta_2}$$

$$\beta = \frac{\Delta'_1}{\Delta'_1 + \Delta'_2}$$

Attempt II - Continuous Approach



$$\alpha = \frac{\Delta_1}{\Delta_1 + \Delta_2}$$

$$\beta = \frac{\Delta'_1}{\Delta'_1 + \Delta'_2}$$

Open Questions

Unconstrained Submodular Maximization:

- Can the tight $(1/2)$-approximation be derandomized?
- Is there a $(1/3)$-hardness for any deterministic algorithm?