Submodular Maximization

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Lecture 3

4th Cargese Workshop on Combinatorial Optimization

Recap: a continuous relaxation for maximization

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Multilinear Extension:

$$F(x) = \sum_{R \subseteq \mathcal{N}} f(R) \prod_{u_i \in R} x_i \prod_{u_i \notin R} (1 - x_i) \ , \ \forall x \in [0,1]^{\mathcal{N}}$$

- Simple probabilistic interpretation.
- x integral $\Rightarrow$ $F(x) = f(x)$.

Recap: a continuous relaxation for maximization

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Multilinear Relaxation

- What are the properties of F?
- It is neither convex nor concave.

Properties of the Multilinear Extension

Lemma

The multilinear extension F satisfies:

- If f is non-decreasing, then $\frac{\partial F}{\partial x_i} \geqslant 0$ everywhere in the cube for all i.
- If f is submodular, then $\frac{\partial^2 F}{\partial x_i \partial x_j} \leqslant 0$ everywhere in the cube for all i, j.

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Useful for proving:

Theorem

The multilinear extension F satisfies:

- If f is non-decreasing, then F is non-decreasing in every direction $\vec{d}$.
- If f is submodular, then F is concave in every direction $\vec{d} \geqslant 0$.
- If f is submodular, then F is convex in every direction $\vec{e}_i - \vec{e}_j$ for all $i, j \in \mathcal{N}$.

Properties of the Multilinear Extension

Summarizing:

$$\underbrace{f^+(x)}_{\text{concave closure}} \geqslant \underbrace{F(x)}_{\text{multilinear ext.}} \geqslant \underbrace{f^-(x)}_{\text{convex closure}} = \underbrace{f^L(x)}_{\text{Lovasz ext.}}$$

Any extension can be described as $\mathbb{E}[f(R)]$ where R is chosen from a distribution that preserves the x_i values (marginals).

- concave closure maximizes expectation but is hard to compute.
- concave closure minimizes expectation and has a nice characterization (Lovasz extension).
- Multilinear extension is somewhere in the "middle".

Continuous Relaxation

constrained submodular maximization problem

Family of allowed subsets $\mathcal{M} \subseteq 2^{\mathcal{N}}$.

$$\begin{aligned} \max \quad & f(S) \\ s.t. \quad & S \in \mathcal{M} \end{aligned}$$

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comparing linear and submodular relaxations

- optimizing a fractional solution:
  - linear: easy
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comparing linear and submodular relaxations

- optimizing a fractional solution:
  - linear: easy
  - submodular: not clear ...
- rounding a fractional solution:
  - linear: hard (problem dependent)
  - submodular: easy (pipage for matroids)

Pipage Rounding on Matroids

Work of [Ageev-Sviridenko-04],[Călinescu-Chekuri-Pál-Vondrák-08].

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For a matroid $\mathcal{M}$, the matroid polytope associated with it:

$$\mathcal{P}_{\mathcal{M}} = \{x \in [0,1]^{\mathcal{N}} : \sum_{i \in S} x_i \leqslant r_{\mathcal{M}}(S) \ \ \forall S \subseteq \mathcal{M}\}$$

where $r_{\mathcal{M}}(\cdot)$ is the rank function of $\mathcal{M}$.

The extreme points of $\mathcal{P}_{\mathcal{M}}$ correspond to characterstic vectors of indepenedent sets in $\mathcal{M}$.

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Observation: if f is linear, a point x can be rounded by writing it as a convex sum of extreme points.

Question: What do we do if f is (general) submodular?

Rounding general submodular function f:

- if x is non-integral, there are $i, j \in \mathcal{N}$ for which $0 < x_i, x_j < 1$.
- recall, F is convex in every direction $e_i - e_j$.
- hence, F is non-decreasing in one of the directions $\pm(e_i - e_j)$

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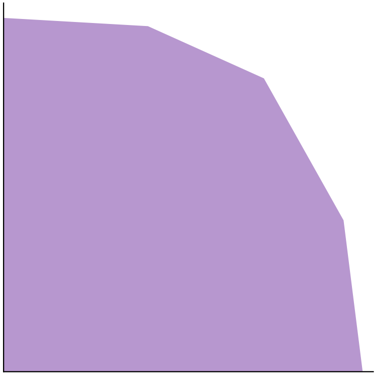
Rounding Algorithm:

- suppose direction $e_i - e_j$ is non-decreasing
- δ - max change (due to a tight set A)
- if either $x_i + \delta$ or $x_j - \delta$ are integral - **progress**
- else there exists a tight set $A' \subset A$, $i \in A'$, $j \notin A'$ ($|A'| < |A|$)
- recurse on A' - **progress**
- eventually: minimal tight set (contained in all tight sets) in which any pair of coordinates can be increased/decreased - **progress**

The Continuous Greedy Algorithm [Călinescu-Chekuri-Pál-Vondrák-08]

- computes an approximate fractional solution
- f is monotone (for now ...)
- $\mathcal{P}_{\mathcal{M}}$ is downward closed ($\vec{0} \in \mathcal{P}_{\mathcal{M}}$)

Continuous Greedy



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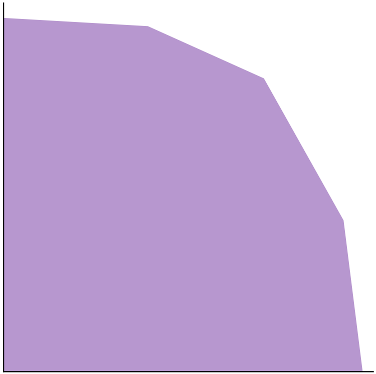
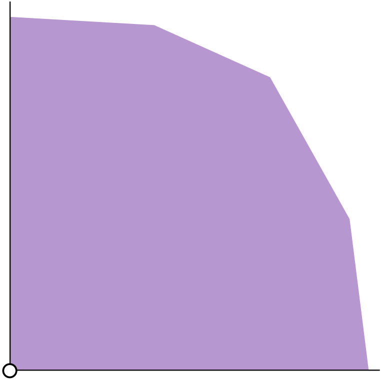


$$\vec{x}(0) = \vec{0}$$

$$\vec{y}^*(t) = \text{argmax}\left\{\sum_{i=1}^{n} \frac{\partial F(\vec{x}(t))}{\partial x_i} \cdot y_i \ : \ \vec{y} \in \mathcal{P}_{\mathcal{M}}\right\} \rightsquigarrow \frac{\partial x_i(t)}{\partial t} = y_i^*(t)$$



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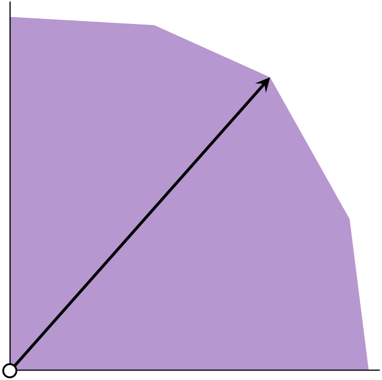

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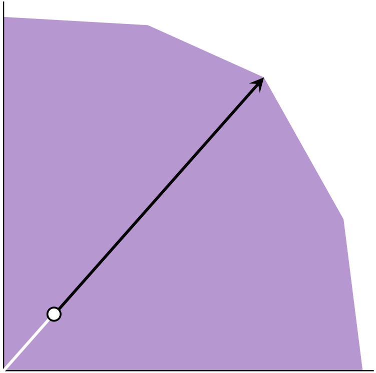

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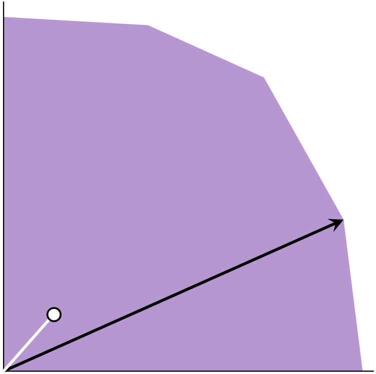

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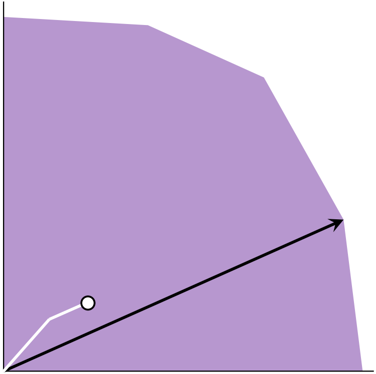

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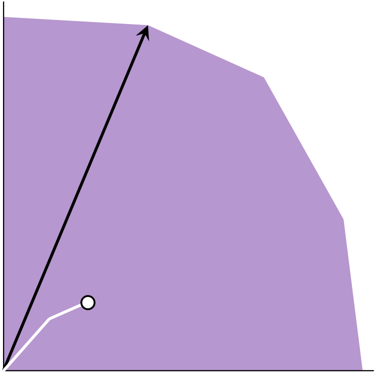

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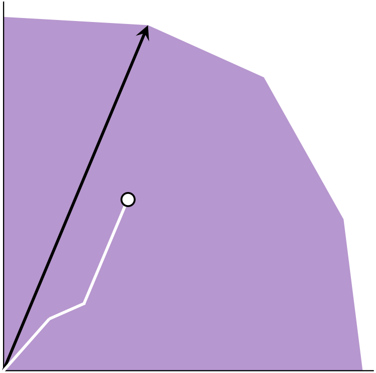

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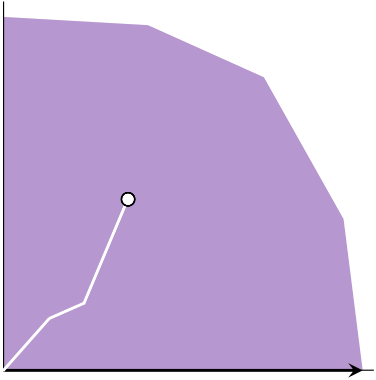

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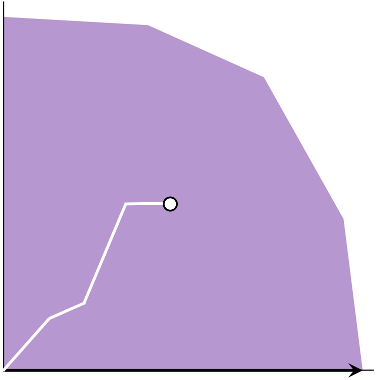

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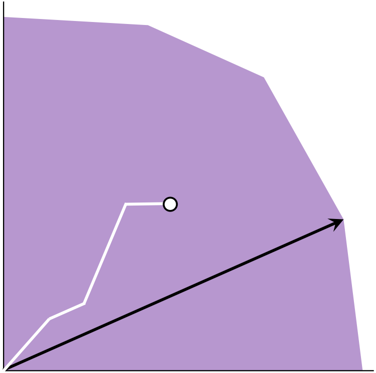

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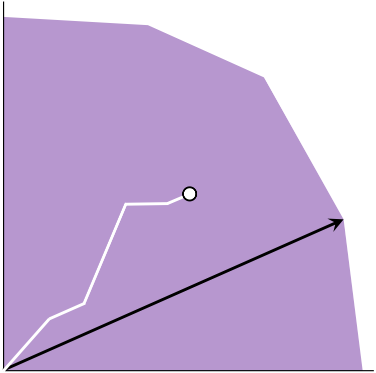

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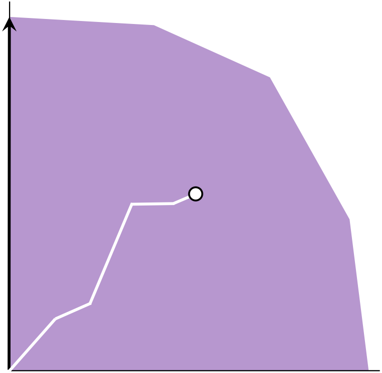

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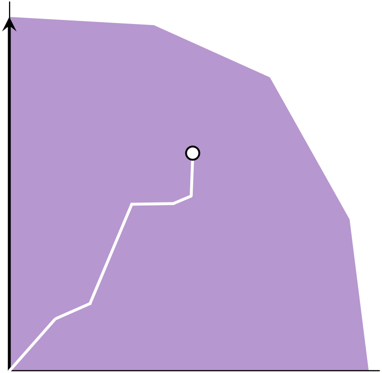

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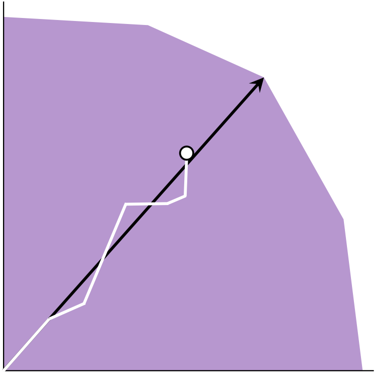

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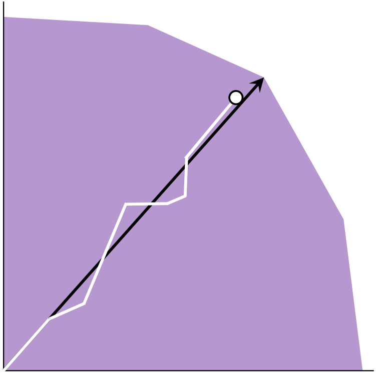

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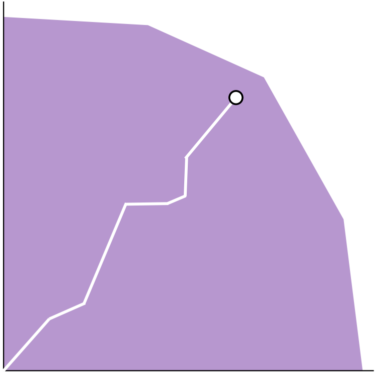

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[Călinescu-Chekuri-Pál-Vondrák-08]

$$F\left(\vec{x}(t)\right) \geqslant \left(1 - e^{-t}\right) F(\mathbf{1}_{\mathsf{OPT}})$$

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- **When to stop the algorithm?**

[Călinescu-Chekuri-Pál-Vondrák-08]

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- **When to stop the algorithm?**

$$t = 1 \quad \Rightarrow \quad \begin{cases} \vec{x}(1) \text{ feasible (convex combination of feasible vectors)} \\ F\left(\vec{x}(1)\right) \geqslant \left(1 - \frac{1}{e}\right) F(\mathbf{1}_{\mathsf{OPT}}) \end{cases}$$

Proof:

Continuous Greedy - Analysis (Cont.)

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$$\frac{\partial F\left(\vec{x}(t)\right)}{\partial t}$$

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$$\frac{\partial F\left(\vec{x}(t)\right)}{\partial t} = \sum_{i=1}^{n} \frac{\partial F\left(\vec{x}(t)\right)}{\partial x_i} \cdot \frac{\partial x_i(t)}{\partial t}$$

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$$\geqslant \sum_{i \in \mathrm{OPT}} \frac{\partial F(\vec{x}(t))}{\partial x_i} = \sum_{i \in \mathrm{OPT}} \frac{F\left(\vec{x}(t) \vee \mathbf{1}_{\{i\}}\right) - F(\vec{x}(t))}{1 - x_i(t)}$$

Proof:

$$\begin{aligned}\frac{\partial F\left(\vec{x}(t)\right)}{\partial t} &= \sum_{i=1}^{n} \frac{\partial F\left(\vec{x}(t)\right)}{\partial x_i} \cdot \frac{\partial x_i(t)}{\partial t} = \sum_{i=1}^{n} \frac{\partial F\left(\vec{x}(t)\right)}{\partial x_i} \cdot y_i^*(t) \\ &\geqslant \sum_{i \in \mathrm{OPT}} \frac{\partial F\left(\vec{x}(t)\right)}{\partial x_i} = \sum_{i \in \mathrm{OPT}} \frac{F\left(\vec{x}(t) \vee \mathbf{1}_{\{i\}}\right) - F\left(\vec{x}(t)\right)}{1 - x_i(t)} \\ &\geqslant \sum_{i \in \mathrm{OPT}} \left[F\left(\vec{x}(t) \vee \mathbf{1}_{\{i\}}\right) - F\left(\vec{x}(t)\right)\right]\end{aligned}$$

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We obtain a differential equation:

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Continuous Greedy - Tight?

[Nemhauser-Wolsey-78]

Maximizing a monotone submodular f over a matroid is $\left(1 - \frac{1}{e}\right)$-hard.

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1. Submodular Welfare:

$1-\frac{1}{e}$	[Călinescu-Chekuri-Pál-Vondrák-08]
$\frac{k}{2k-1}$	[Dobzinski-Schapira-06]
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   Is the case of two players special?

2. Greedy and Continuous Greedy fail for **non-monotone** f.

Continuous Greedy:

$$\begin{cases} \vec{y}^*(t) & = \text{argmax}\left\{\sum_{i=1}^{n} \frac{\partial F(\vec{x}(t))}{\partial x_i} \cdot y_i \; : \; \vec{y} \in \mathcal{P}_{\mathcal{M}}\right\} \\ \frac{\partial x_i(t)}{\partial t} & = y_i^*(t) \end{cases}$$

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Intuition:

$$\frac{\partial F(\vec{x}(t))}{\partial x_i} = \frac{F\left(\vec{x}(t) \vee \mathbf{1}_{\{i\}}\right) - F(\vec{x}(t))}{1 - x_i(t)}$$

Continuous greedy ignores the current position $x_i(t)$.

Measured Continuous Greedy

[Feldman-N-Schwartz-11]

The measured continuous greedy algorithm achieves:

1. monotone f: $F(\vec{x}(t)) \geqslant \left(1 - e^{-t}\right) F(\mathbf{1}_{\mathsf{OPT}})$.
2. non-monotone f: $F(\vec{x}(t)) \geqslant te^{-t} \cdot F(\mathbf{1}_{\mathsf{OPT}})$.

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Non-Monotone f:

- Stopping at $t = 1 \;\Rightarrow\; (1/e)$-approximation.
- All known rounding procedures work for non-monotone f as well. (matroid and $O(1)$ knapsack)
- Greedy methods **fail** in the discrete setting.

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Measured Continuous Greedy - Non-Monotone f

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monotone: $\geqslant F(\mathbf{1}_{\text{OPT}}) - F(\vec{x}(t))$ yielding same factor as continuous greedy

Measured Continuous Greedy - Non-Monotone f

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non-monotone: how to lower bound $F(\vec{x}(t) \vee \mathbf{1}_{\mathrm{OPT}})$?

Measured Continuous Greedy - Non-Monotone f (Cont.)

1. $x_i(t)$ cannot be too large:

$$\begin{cases} \frac{\partial x_i(t)}{\partial t} \leqslant 1 - x_i(t) \\ x_i(0) = 0 \end{cases}$$

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2. $\forall S \subseteq \mathcal{N}$ and $\vec{y} \in [0,1]^{\mathcal{N}}$ $s.t.$ $\max_{i \in \mathcal{N}} \{y_i\} = y_{\max}$:

$$F\left(\mathbf{1}_S \vee \vec{y}\right) \geqslant \left(1 - y_{\max}\right) F(\mathbf{1}_S).$$

Intuition: by submodularity (decreasing marginals), when “0” coordinates are increased to $y_{\max}$, loss to $F(\mathbf{1}_S)$ is at most a $y_{\max}$-fraction

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$$F\left(\vec{x}(t) \vee \mathbf{1}_{\mathsf{OPT}}\right) \geqslant e^{-t} \cdot F(\mathbf{1}_{\mathsf{OPT}}).$$

Measured Continuous Greedy - Non-Monotone f (Cont.)

$$\begin{aligned}\frac{\partial F\left(\vec{x}(t)\right)}{\partial t} &\geqslant F\left(\vec{x}(t) \vee \mathbf{1}_{\mathrm{OPT}}\right) - F\left(\vec{x}(t)\right) \\ &\geqslant e^{-t} \cdot F(\mathbf{1}_{\mathrm{OPT}}) - F\left(\vec{x}(t)\right)\end{aligned}$$

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Solving the differential equation with $F(\vec{x}(0)) \geqslant 0$ gives:

$$F(\vec{x}(t)) \geqslant te^{-t} \cdot F(\mathbf{1}_{\mathsf{OPT}}).$$

■

Non-Monotone f Guarantee

$$F(\vec{x}(1)) \geqslant \left(\frac{1}{e}\right) \cdot F(\mathbf{1}_{\mathsf{OPT}})$$

Measured Continuous Greedy - Monotone f

Monotone f Guarantee

$$F\left(\vec{x}(t)\right) \geqslant F(\mathbf{1}_{\mathsf{OPT}})\left(1 - e^{-t}\right)$$

Measured Continuous Greedy - Monotone f

Monotone f Guarantee

$$F\left(\vec{x}(t)\right) \geqslant F(\mathbf{1}_{\mathsf{OPT}})\left(1 - e^{-t}\right)$$

Note: $\vec{x}(t)$ gains the same value but advances less:

$$x_i(t) \leqslant 1 - e^{-t}$$

$\Rightarrow$ one might possibly stop at times $t > 1$ and still be feasible!

Measured Continuous Greedy - Monotone f

Submodular MAX-SAT

- CNF formula and a monotone submodular function f defined over the clauses
- Goal: find an assignment ϕ maximizing f (over clauses satisfied by ϕ)

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- each x_i is replaced by a group $\{(x_i, 0), (x_i, 1)\}$
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- $C_{x,v}$ - clauses satisfied by setting $x \leftarrow v$
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submodular MAX SAT can be represented as a monotone submodular maximization problem over a matroid

Question: for which $T > 1$ can we run the measured continuous greedy algorithm and stay feasible?

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For variable x:

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Clearly, $T_0 + T_1 \leqslant T$

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$$(x, 1) + (x, 1) \leqslant (1 - e^{-T_0}) + (1 - e^{-T_1}) \leqslant 2 - e^{-T_0} - e^{T_0 - T} \qquad (T_0 + T_1 \leqslant T)$$

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Clearly, $T_0 + T_1 \leqslant T$

By properties of measured greedy:

$$(x, 0) \leqslant 1 - e^{-T_0}, \quad (x, 1) \leqslant 1 - e^{-T_1}$$

$$(x, 1) + (x, 1) \leqslant (1 - e^{-T_0}) + (1 - e^{-T_1}) \leqslant 2 - e^{-T_0} - e^{T_0 - T} \qquad (T_0 + T_1 \leqslant T)$$

When is $2 - e^{-T_0} - e^{T_0 - T}$ maximized? $T_0 = T/2$

Question: for which $T > 1$ can we run the measured continuous greedy algorithm and stay feasible?

For variable x:

- T_0 - total time in which $(x, 0)$ is increased
- T_1 - total time in which $(x, 1)$ is increased

Clearly, $T_0 + T_1 \leqslant T$

By properties of measured greedy:

$$(x,0) \leqslant 1 - e^{-T_0}, \quad (x,1) \leqslant 1 - e^{-T_1}$$

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When is $2 - e^{-T_0} - e^{T_0 - T}$ maximized? $T_0 = T/2$

Matroid constraints need to be satisfied:

$$2 - e^{-T_0} - e^{T_0 - T} \leqslant 2(1 - e^{-T/2}) \leqslant 1$$

Yielding: $T \leqslant 2 \ln 2$

Measured Continuous Greedy - Monotone f

Question: for which $T > 1$ can we run the measured continuous greedy algorithm and stay feasible?

For variable x:

- T_0 - total time in which $(x, 0)$ is increased
- T_1 - total time in which $(x, 1)$ is increased

Clearly, $T_0 + T_1 \leqslant T$

By properties of measured greedy:

$$(x,0) \leqslant 1 - e^{-T_0}, \quad (x,1) \leqslant 1 - e^{-T_1}$$

$$(x,1) + (x,1) \leqslant (1 - e^{-T_0}) + (1 - e^{-T_1}) \leqslant 2 - e^{-T_0} - e^{T_0 - T} \qquad (T_0 + T_1 \leqslant T)$$

When is $2 - e^{-T_0} - e^{T_0 - T}$ maximized? $T_0 = T/2$

Matroid constraints need to be satisfied:

$$2 - e^{-T_0} - e^{T_0 - T} \leqslant 2(1 - e^{-T/2}) \leqslant 1$$

Yielding: $T \leqslant 2 \ln 2$

Approximation factor is $(1 - e^{-T})$: $\frac{3}{4}$ for $T = 2 \ln 2$.

Measured Continuous Greedy - Monotone f (Cont.)

$$\mathcal{P} = \left\{ x \;\middle|\; \sum_{i \in \mathcal{N}} a_{i,j} x_i \leqslant b_j, 1 \leqslant j \leqslant m \;,\; 0 \leqslant x_i \leqslant 1, \forall i \in \mathcal{N} \right\}$$

$$d(\mathcal{P}) \triangleq \min_{1 \leqslant j \leqslant m} \left\{ \frac{b_j}{\sum_{i \in \mathcal{N}} a_{i,j}} \right\}$$

[Feldman-N-Schwartz-11]

$\vec{x}(t) \in \mathcal{P}$ if

$$t \leqslant \frac{\ln\left(\frac{1}{1-d(\mathcal{P})}\right)}{d(\mathcal{P})} \quad (*).$$

Note: $(*) \geqslant 1$ since $d\left(\mathcal{P}\right) > 0$.

Measured Continuous Greedy - Results

Problem	**Result**	**Previous**	**Hardness**
Submodular Welfare k players	$1 - \left(1 - \frac{1}{k}\right)^k$	$\max\left\{1 - {}^1/_e, \frac{k}{2k-1}\right\}$	$1 - \left(1 - \frac{1}{k}\right)^k$
Submodular Max-SAT	${}^3/_4$	${}^2/_3$	${}^3/_4$
non-monotone f matroid	${}^1/_e$	≈ 0.325	≈ 0.478
non-monotone f $O(1)$ knapsack	${}^1/_e$	≈ 0.325	≈ 0.491