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Abstract

The cut polyhedron $cut(G)$ of an undirected graph $G = (V, E)$ is the dominant of the convex hull of all its nonempty edge cutsets. After examining various compact extended formulations for $cut(G)$, we study some of its polyhedral properties. In particular, we characterize all the facets induced by inequalities with right-hand side at most 2. These include all the rank facets of the polyhedron.

1. Introduction

Let $G = (V, E)$ be an undirected graph. For $S \subseteq V$, the *cut* $\delta(S)$ of G is the set of all the edges of G having exactly one endnode in S. For a subset U of a set W, $\chi(U) \in \mathbb{R}^W$ denotes the incidence vector of U in W, where the notation $\mathbb{R}^W$ ($\mathbb{R}^W_+$) stands for the set of all real valued (nonnegative) functions with domain W.

The *cut polyhedron* is the dominant of the convex hull of the incidence vectors of all the nonempty cuts of G, i.e.,

$$cut(G) = Conv \left\{ x \in \mathbb{R}^E_+ \mid x \geq \chi(\delta(S)) \text{ for some } \emptyset \neq S \subset V \right\}.$$

The polyhedron

$$syn(G) = \left\{ x \in \mathbb{R}^E_+ \mid x(\delta(S)) \geq 1 \text{ for all } \emptyset \neq S \subset V \right\},$$

called the *network synthesis polyhedron*, whose facets are given by the extreme points of $cut(G)$, is the blocking polyhedron of $cut(G)$. Note that with a right hand side vector of 2's, $syn(G)$ provides a relaxation of problems like the k-Connected Subgraph, the Traveling Salesman, and the Graphical Traveling Salesman problems (see [5]).

With a blocking pair of polyhedra, it is natural to consider four closely linked problems:

Problem 1 (Separation for $cut(G)$**)** *Given* $\bar{x} \in \mathbb{R}^E_+$*, find an inequality* $ax \geq b$ *valid for* $cut(G)$ *such that* $a\bar{x} < b$ *or prove that no such an inequality exists.*

Problem 2 (Optimization over $cut(G)$**)** *Given a cost function* $c \in \mathbb{R}^E_+$*, solve*

$$\min \{cx \mid x \in cut(G)\}.$$

Problem 3 (Separation for $syn(G)$**)** *Given* $\bar{x} \in \mathbb{R}^E_+$*, find a set* $\emptyset \subset S \subset V$ *such that*

$$\bar{x}(\delta(S)) < 1$$

or prove that all such inequalities are satisfied.

Problem 4 (Optimization over $syn(G)$ **- network synthesis)** *Given a cost function* $c \in \mathbb{R}^E_+$*, solve*

$$\min \{cx \mid x \in syn(G)\}.$$

This last problem amounts to allocating (fractional) capacities $x \in \mathbb{R}^E_+$ such that at least one unit of flow can be sent between every pair of nodes at minimum c-cost.

As we have a blocking pair of polyhedra, problems 1 and 4 are equivalent and problems 2 and 3 are equivalent. What is more, all four problems are polynomially solvable provided that one of them is polynomially solvable. In fact fast combinatorial algorithms are known for Problem 2 (minimum cut), and for Problem 3 (cut inequality violation), and the open question is whether there is a fast combinatorial algorithm for separation over the cut polytope (Problem 1) and for Problem 4. Incidentally, observe how, contrary to what usually happens, the integral version of Problem 4 is much easier, as it amounts to finding a minimum c-cost spanning tree of G.

In this note we make several simple observations about the blocking pair $cut(G)$ and $syn(G)$ motivated by this open question. Specifically, in Section 2 we point out how to construct a variety of extended formulations of polynomial size for the two polyhedra. Clearly any such formulation and a fast linear programming algorithm lead to algorithms to solve all four problems 1–4 in polynomial time.

In Section 3 we make some observations about the facets/extreme points of $cut(G)/syn(G)$, respectively. These two polyhedra have been studied earlier. For small graphs the facets of $cut(G)$ have been enumerated and classified by Alevras [1]. Various polynomial size formulations are presented in Tamir [11]. The polyhedron $syn(G)$ has been studied in [5], and a formulation of polynomial size is presented (for directed graphs) in [4] and [10].

2. Extended Formulations

Given a graph $G = (V, E)$ with n nodes, we choose node $r = n$ to be the *root* node. The following trivial and well-known observation is important in what follows. We say that a cut $\delta(S)$ is an (r,t)-*cut* if $r \in S$ and $t \in V \setminus S$.

Observation 1. *Every cut $\delta(S)$ with $\emptyset \subset S \subset V$ is an (r,t)-cut for some $t = 1, \dots, n-1$.*

2.1. Formulations for $cut(G)$

Let $cut_{r,t}(G) = Conv\left\{x \in \mathbb{R}^E \mid x \geq \chi(\delta(S)) \text{ for some } \{r\} \subseteq S \subset V \setminus \{t\}\right\}$. We now use Observation 1.

Proposition 1. $cut(G) = Conv(\bigcup_{t=1}^{n-1} cut_{r,t}(G))$.

Proposition 2 ([2]) *Let $P^i = \left\{x \in \mathbb{R}^d_+ \mid A^i x \geq b^i\right\}$ for $i = 1, \dots, k$ be polyhedra with the same recession cone, i.e., $R^i = \left\{x \in \mathbb{R}^d_+ \mid A^i x \geq 0\right\}$ are identical for all i. Then*

$$Conv(\bigcup_{i=1}^{k} P^t) = proj_x\Big\{(x, z^1, \dots, z^k, \lambda) \in \mathbb{R}^d \times \mathbb{R}^{kd}_+ \times \mathbb{R}^k_+ \;\Big|\; x = \sum_{i=1}^{k} z^i,\ A^i z^i \geq b^i \lambda_i \text{ for } i = 1, \dots, k,\ \sum_{i=1}^{k} \lambda_i = 1\Big\}.$$

Proof. The inclusion "$\subseteq$" is straightforward. Conversely, suppose a feasible point $(x, z^1, \dots, z^k, \lambda)$ is given. Let $I = \{i : \lambda_i > 0\}$, $v = \sum_{i \notin I} z^i$, and $x^i = z^i/\lambda_i$ for $i \in I$. Now for $i \in I$, $x^i + v \in P^i$ as $x^i \in P^i$ and $v \in R^i$. But now $x = \sum_{i=1}^{k}(x^i + v)\lambda_i$ with $\sum_{i=1}^{k} \lambda_i = 1$ and $\lambda_i \geq 0$ for all i, and thus $x \in Conv(\bigcup_{i=1}^{k} P^i)$. □

This shows that if compact formulations are available for all the polyhedra P^i, k is not too large, and the above condition is satisfied, then we have a compact formulation for $Conv(\bigcup_{i=1}^{k} P^i)$ (albeit in a higher dimensional space).

As the recession cone of $cut_{r,t}(G)$ is the nonnegative orthant, it suffices to take any formulation of $cut_{r,t}(G)$ and apply propositions 1 and 2 to obtain an extended formulation for $cut(G)$.

One extended formulation for $cut_{r,t}(G)$ is obtained by forming a digraph $D = (V, A)$ and putting arcs ij and ji in A if and only if the edge $e = (i, j) \in E$. We take the classical formulation consisting of the dual of the linear program that defines the flows circulations in D, where the flow in the arc tr is maximized. Dropping the superscript t, we obtain:

$$\begin{aligned} & z_e = y_{ij} + y_{ji} && \text{for all } e = (i,j) \in E \\ & \pi_j - \pi_i \leq y_{ij} && \text{for all } ij \in A \setminus \{tr\} \\ & \pi_t - \pi_r = 1 && \\ & y_{ij} \geq 0 && \text{for all } ij \in A. \end{aligned}$$

Applying Proposition 2 and projecting out the z and λ variables, immediately leads to the extended formulation

$$\begin{array}{ll} x_e = \sum_{t=1}^{n-1}(y^t_{ij}+y^t_{ji}) & \text{for all } e=(i,j)\in E \\ \pi^t_j - \pi^t_i \le y^t_{ij} & \text{for all } ij \in A\setminus\{tr\},\ t=1,\dots,n-1 \\ \sum_{t=1}^{n-1}(\pi^t_t-\pi^t_r)=1 & \\ \pi^t_t-\pi^t_r \ge 0 & \text{for } t=1,\dots,n-1 \\ y^t_{ij}\ge 0 & \text{for all } ij\in A,\ t=1,\dots,n-1. \end{array}$$

Also without loss of generality we can set $\pi^t_r = 0$ for $t = 1,\dots,n-1$.

A second extended formulation for $cut_{r,t}(G)$ is obtained by introducing edge variables y for all edges of the complete graph K_n along with the triangle inequalities. Again t is fixed and the superscript t has been dropped.

$$\begin{array}{ll} x_e \ge y_e & \text{for } e=(i,j)\in E \\ y_e+y_f\ge y_g & \text{for all triangles } \{e,f,g\} \text{ in } K_n \\ y_e = 1 & \text{for } e=(r,t) \\ y_e\ge 0 & \text{for all } e \text{ in } K_n. \end{array}$$

Observe that the triangle inequalities and $y_{rt} = 1$ imply that $\sum_{e\in P} y_e \ge 1$ for every $r-t$ path P, and it is well-known that these constraints generate $cut_{r,t}(G)$.

Using Proposition 2 and eliminating the variables z and λ gives

$$\begin{array}{ll} x_e \ge \sum_{t=1}^{n-1} y^t_e & \text{for } e=(i,j)\in E,\ t=1,\dots,n-1 \\ y^t_e+y^t_f\ge y^t_g & \text{for all triangles } \{e,f,g\} \text{ in } K_n,\ t=1,\dots,n-1 \\ \sum_{t=1}^{n-1} y^t_{rt}=1 & \\ y^t_e\ge 0 & \text{for all } e \text{ in } K_n,\ t=1,\dots,n-1. \end{array}$$

These formulations are from Tamir [11].

2.2. Formulations for $syn(G)$

Now we consider formulations for $syn(G)$. We fix the root r as before. Let

$$syn_{r,t}(G) = \left\{x\in\mathbb{R}^E_+ \mid x(\delta(S))\ge 1 \text{ for all } \{r\}\subseteq S\subseteq V\setminus\{t\}\right\}.$$

Again we use Observation 1.

Observation 2. $syn(G) = \bigcap_{t=1}^{n-1} syn_{r,t}(G)$.

So now it suffices to take any formulation for $syn_{r,t}(G)$. One possibility is to again bidirect the graph G, and then let f^t_{ij} be the flow on arc $ij \in A$. The resulting formulation is

$$\begin{array}{ll} f^t_{uv}\le x_e, & \text{for all } e=(u,v)\in E,\ uv\ne tr \\ f^t_{vu}\le x_e, & \text{for all } e=(u,v)\in E,\ vu\ne tr \\ f^t_{tr}\ge 1 & \\ \sum_{vu\in A} f^t_{vu} = \sum_{uv\in A} f^t_{uv} & \text{for all } v\in V \\ f^t\ge 0. & \end{array}$$

Observe that for any value $\bar{x}$ of the vector x, the system associated with node t has a feasible solution if and only if the $\bar{x}$-value of a minimum (r,t)-cut is at least 1. Therefore the resulting polyhedron has a feasible solution if and only if the $\bar{x}$ value of every (r,t)-cut is at least 1, i.e., if and only if $\bar{x} \in syn_{r,t}(G)$. So this polyhedron provides a compact formulation for $syn_{r,t}(G)$.

3. Some facial properties of $cut(G)$

We go back now to the cut polyhedron $cut(G)$ in its "natural" space $\mathbb{R}^E$. This polyhedron is of the dominant type and is obviously full-dimensional. So the facet-inducing inequalities are uniquely defined, up to positive scaling factors. Therefore, from now on, we assume that a facet-inducing inequality $ax \geq b$ is given in its (unique) *minimum integer form*, i.e., the coefficients of the integer vector (a, b) are relatively prime.

A facet-inducing inequality $ax \geq b$ can be represented by a weighted subgraph $G_a = (V, E_a)$ of G, made with the edges $e \in E$ with $a_e \neq 0$ and where a is its weight vector. The following facts are easy to prove for a facet-inducing inequality $ax \geq b$:

a) the right hand side b is nonnegative; if $b > 0$, then a is a nonnegative vector; if $b = 0$, then $ax \geq b$ coincides with $x_e \geq 0$, for some $e \in E$;

b) if $b > 0$, then b is the minimum weight of a cut in G_a;

c) if $b > 0$, then the inequality $ax \geq b$ is facet-inducing for $cut(G)$ if and only if G_a contains a family of $|E_a|$ linearly independent minimum weight cuts (a set of cuts is said linear independent if so is the set of the corresponding incidence vectors);

d) if $b = 1$ then $ax \geq b$ coincides with $x(F) \geq 1$, where F is a spanning tree of G.

Before giving two general properties of the facet-inducing inequalities for $cut(G)$, we introduce some definitions that are needed to state an important result in polyhedral combinatorics, that combines results of Edmonds, Johnson and of Lehman.

Let T be a subset of nodes in an undirected graph $G = (V, E)$ of even cardinality. A subset E' of E is a *T-join* if T is the set of nodes of odd degree in $G' = (V, E')$. A cut $\delta(S)$ is a *T-cut* if $S \cap T$ has odd cardinality. It is easy to see that the family of all the minimal T-joins of G and the family of all the T-cuts of G are blocking families, in the sense that the minimal T-joins are the minimal set that intersect every T-cut and vice versa. Let $X(TJ)$ and $X(TC)$ be the incidence matrices of all the minimal T-joins and all the T-cuts, respectively, versus the edges of G.

A $0,1$-matrix M is *ideal* if the set covering polyhedron $Q(M) = \{x \in \mathbb{R}^n_+ \mid Mx \geq 1\}$ has all integer vertices.

Theorem 1 ([7],[8]) *Let T be a subset of the nodes of an undirected graph $G = (V, E)$ of even cardinality. Then both the $0,1$-matrices $X(TJ)$ and $X(TC)$ are ideal.*

Proof. Edmonds and Johnson [7] give an efficient algorithm that finds a T-join of minimum weight. If the weights of G are nonnegative, their algorithm constructs a corresponding (fractional) packing of T-cuts of the same value, thus proving that $X(TC)$ is an ideal matrix.

Given a $0,1$ matrix M, its *blocking matrix* $\mathcal{B}(M)$ is the $0,1$ matrix whose rows are all the $0,1$ vectors of minimal support in $Q(M)$. Lehman [8] shows that a $0,1$ matrix M with no dominated row is ideal if and only if $\mathcal{B}(M)$ is also ideal. Since $X(TJ) = \mathcal{B}(X(TC))$, then $X(TJ)$ is ideal as well. □

Padberg and Rao [9] give an efficient algorithm to compute a T-cut of minimum weight, when all the weight are nonnegative. To our knowledge, their algorithm does not explicitly provide a corresponding fractional packing of T-joins of the same value and therefore it does not give a direct proof that $X(TJ)$ is an ideal matrix. It would be nice to have an algorithm that explicitly computes such a dual solution.

We now can state our first general property of the inequalities (in minimum integer form) that are facet-inducing for $cut(G)$:

Theorem 2. *If an inequality $ax \geq b$ is facet-inducing for $cut(G)$ and b is greater than one, then b is even.*

Proof. Assume $ax \geq b$ is facet-inducing for $cut(G)$ with $b > 1$ and odd. In G_a, let T be the set of nodes v such that $a(\delta(v))$ is odd. Then T is a nonempty set of even cardinality since the weight b of a cut of G_a is odd.

Let $\delta(S) \subseteq E_a$ be any T-cut of minimum weight in G_a. Since all weights of G_a are positive, by Theorem 1 the incidence vector $\chi(\delta(S))$ of $\delta(S)$ is the solution of the linear program:

$$\min \{ax \mid x \in Q(X(TJ))\}. \tag{1}$$

Let $E' \subseteq E$ be a T-join of G_a whose associated constraint has positive dual variable in some optimal dual solution of the above linear program. Such a variable certainly exists, since by assumption the optimal value of (1) is strictly positive. Then E' intersects every T-cut of G_a at least once and, by complementary slackness, E' intersects every minimum weight T-cut $\delta(S)$ *exactly* once (since $\chi(\delta(S))$ is an optimal solution of the linear program (1)).

Let $\chi(E')$ be the incidence vector of E' and define $a' = a + \chi(E')$ and $b' = b + 1$. Since the cuts of weight b in G_a are exactly the minimum weight T-cuts, by the above argument, b' is the minimum value of a cut in $G_{a'}$ and every cut of weight b in G_a has weight b' in $G_{a'}$. So $a'x \geq b'$ is a valid inequality that is satisfied with equality by the incidence vectors of all the cuts that satisfy $ax \geq b$ at equality. Since $ax \geq b$ is in minimum integer form and $b > 1$, this inequality cannot be obtained by multiplying the other by a positive scaling factor, a contradiction to the assumption that $ax \geq b$ is facet-inducing. □

Let $G = (V, E)$ be a graph with nonnegative edge-weights w_e and let λ be the minimum weight of a cut in G. Two cuts $\delta(S_1)$ and $\delta(S_2)$ of G are *crossing* if none of the four sets $V_1 = S_1 \setminus S_2$, $V_2 = S_2 \setminus S_1$, $V_3 = S_1 \cap S_2$, and $V_4 = V \setminus (S_1 \cup S_2)$ is empty. Now let S_1 and S_2 be disjoint subsets of V. We indicate with $\delta(S_1, S_2)$ the set of edges with one endnode in S_1 and the other in S_2 and with $w(\delta(S_1, S_2))$ the sum of the weights of such edges.

Lemma 1 ([3],[6]) *Let $G = (V, E)$ be a graph with nonnegative edge-weights w_e and let λ be the minimum weight of a cut in G. Let $\delta(S_1)$, $\delta(S_2)$ be two minimum cuts of G that are crossing. Then*

$$\begin{aligned} w(\delta(V_1, V_2)) &= w(\delta(V_3, V_4)) = 0 \\ w(\delta(V_1, V_3)) &= w(\delta(V_1, V_4)) = w(\delta(V_2, V_3)) \\ &= w(\delta(V_2, V_4)) = \frac{\lambda}{2}\,. \end{aligned}$$

Proof. Since $w(\delta(V_i)) \geq \lambda$ for $1 \leq i \leq 4$, then

$$2\lambda \leq \frac{1}{2} \sum_{i=1}^{4} w(\delta(V_i)) = \sum_{1 \leq i < j \leq 4} w(\delta(V_i, V_j))$$

However,

$$2\lambda = w(\delta(S_1)) + w(\delta(S_2)) = \sum_{1\leq i<j\leq 4} w(\delta(V_i,V_j)) + w(\delta(V_1,V_3)) + w(\delta(V_2,V_4)).$$

Subtracting the second from the first equation, we get $w(\delta(V_1,V_3)) + w(\delta(V_2,V_4)) \leq 0$ which proves the first result.

From $2\lambda = \frac{1}{2}\sum_{i=1}^{4} w(\delta(V_i))$ it follows that $w(\delta(V_i)) = \lambda$ for $1 \leq i \leq 4$, which implies immediately $w(\delta(V_1,V_4)) = w(\delta(V_4,V_2)) = w(\delta(V_2,V_3)) = w(\delta(V_3,V_1)) = \frac{\lambda}{2}$. □

The following theorem was proven by Cornuéjols, Fonlupt, and Naddef in their study of the structure of the vertices of $syn(G)$. We now translate their proof in the context of our problem.

A family $\mathcal{F}$ of cuts is *laminar* if, for every pair of cuts $\delta(S_i)$, $\delta(S_j)$ in $\mathcal{F}$, either $S_i \cap S_j = \emptyset$, or $S_i \subset S_j$, or $S_j \subset S_i$.

Theorem 3 ([5]) *Let $ax \geq b$ be a facet-inducing inequality for $cut(G)$, then G_a contains $|E_a|$ linearly independent cuts of (minimum) weight b that induce a laminar family.*

Proof. Let $|E_a| = k$ and $\mathcal{F} = \{\delta(S_1), \dots, \delta(S_k)\}$ be a family of linearly independent minimum cuts of G_a and $M = \sum_{i=1}^{k} |S_i|$ is as small as possible.

Assume that $\mathcal{F}$ contains two cuts, say $\delta(S_1)$ and $\delta(S_2)$, such that all three sets $S_1 \setminus S_2$, $S_2 \setminus S_1$, and $S_1 \cap S_2$ are nonempty. Then $V \setminus (S_1 \cup S_2)$ is also nonempty, else $\delta(S_1) = \delta(S_2 \setminus S_1)$ and $\delta(S_2) = \delta(S_1 \setminus S_2)$, thus contradicting the minimality of M. So the cuts $\delta(S_1)$ and $\delta(S_2)$ are crossing and, by Lemma 1, both $\delta(S_1 \setminus S_2)$ and $\delta(S_2 \setminus S_1)$ are minimum cuts of G_a.

Since the incidence vectors of the cuts in $\mathcal{F}$ are a basis for $\mathbb{R}^k$, then both systems

$$\begin{aligned}
\chi(\delta(S_1 \setminus S_2)) &= \sum_{i=1}^{k} \alpha_i \chi(\delta(S_i)) \\
\chi(\delta(S_2 \setminus S_1)) &= \sum_{i=1}^{k} \beta_i \chi(\delta(S_i))
\end{aligned}$$

have a unique solution.

If $\alpha_1 \neq 0$, then $\mathcal{F}' = \mathcal{F} \cup \{\delta(S_1 \setminus S_2)\} \setminus \{\delta(S_1)\}$ is a family of minimum cuts of G_a whose incidence vectors are linearly independent, a contradiction to the minimality of M. So $\alpha_1 = 0$ and, by the same argument, $\beta_2 = 0$. Again, by Lemma 1,

$$\chi(\delta(S_2 \setminus S_1)) + \chi(\delta(S_1 \setminus S_2)) = \chi(\delta(S_1)) + \chi(\delta(S_2)).$$

Since $\chi(\delta(S_1))$ and $\chi(\delta(S_2))$ cannot be expressed as linear combination of the incidence vectors of the other cuts in $\mathcal{F}$, α_2 and β_1 are both nonzero in the above systems and therefore $\mathcal{F}' = \mathcal{F} \cup \{\delta(S_1 \setminus S_2), \delta(S_2 \setminus S_1)\} \setminus \{\delta(S_1), \delta(S_2)\}$ is a family of minimum cuts of G_a whose incidence vectors are linearly independent, again a contradiction to the minimality of M. □

A laminar family of subsets of V that does not contain $\emptyset$, V, and both a subset S and its complement $V \setminus S$ has at most $2|V| - 3$ subsets. So Theorem 3 implies that if $ax \geq b$ is a facet-inducing inequality for $cut(G)$, then G_a is a sparse graph, for $|E_a| \leq 2|V| - 3$ and this bound is tight (take, e.g., the complete graph $K_3 = (V,E)$ and the inequality $x(E) \geq 2$, which is facet-inducing for $cut(K_3)$).

A facet induced by an inequality $ax \geq b$ in minimum integer form is a *rank facet* if a is a $0,1$ vector.

Theorem 4. *If $ax \geq b$ induces a rank facet of $cut(G)$, then $b \leq 2$.*

Proof. If $ax \geq b$ induces a rank facet of $cut(G)$, then every edge of G_a has unit weight and b is the minimum cardinality of a cut of G_a. Therefore, every node of V has degree at least b in G_a. This implies that $2|E_a| \geq b|V|$. Since $|E_a| \leq 2|V| - 3$, we have that $b \leq 3$ and, by Theorem 2, we have that $b \leq 2$. □

Let B be the set of bridges of G_a. Then a facet-inducing inequality $ax \geq b$ in minimum integer form with $b = 2$ is of the following type:

$$x(E_a \setminus B) + 2x(B) \geq 2 \tag{2}$$

Remark 1. *In a connected graph $G = (V, E)$, let E_2 be the subset of E containing the edges that are not bridges of G but belong to a cut of cardinality 2 (2-cut). Then E_2 can be partitioned into classes so that every 2-cut is contained in a class and every pair of edges in the same class is a 2-cut.*

We now characterize the inequalities $ax \geq b$ with $b = 2$ that are valid for $cut(G)$ and are facet-inducing.

Theorem 5. *An inequality (2) is facet-inducing for $cut(G)$ if and only if $E_2 = E \setminus B$ and no class of the partition of E_2 contains exactly two edges.*

Proof. Let G_a be associated to the inequality (2) and let M be the incidence matrix of edges of G_a versus cuts of weight 2 in G_a. Now M has full column rank, so $E_2 = E \setminus B$. Therefore, M has a block-diagonal structure, where the blocks are the bridges of G_a and the classes of the partition of E_2 given in Remark 1. Now M must have full column rank if and only if each block has full column rank. If a block corresponds to a bridge e, there exists a unique cut of weight 2 in G_a that contains e, so this is obviously true. If a block corresponds to a class of the partition of E_2, then its corresponding submatrix of M is the incidence matrix of all the 2-element subsets of a set with at least two elements. Obviously, this matrix has full column rank if and only if the corresponding class contains more than two edges. □

A question that we find interesting is:

> *Given a graph $G = (V, E)$, what is the largest value of b in a facet-inducing inequality $ax \geq b$ for $cut(G)$?*

It is known [5] that if G is a series-parallel graph, then $b \leq 2$.

10.

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