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ON THE CIRCULAR CHROMATIC NUMBER OF PARTITIONABLE GRAPHS

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Abstract

A graph is star-extremal if its fractional chromatic number equals its circular chromatic number. In Zhu's survey [10], only one example of a circulant graph of prime order which is not star-extremal is mentioned, and it was found by a computer search [4]. In this paper, we investigate the circular chromatic number of partitionable graphs. Our results entail that every circulant partitionable graph, which is not a variant of a web, is not star-extremal, exhibiting thereby an infinite sequence of circulant graphs of prime order, which are not star-extremal.

Keywords: *partitionable graph, homomorphism, circular chromatic number, fractional chromatic number.*

1. Introduction

A graph G is said to be *partitionable* if there exist two integers p and q (with $p \geq 2$ and $q \geq 2$) such that G has $pq+1$ vertices and for every vertex v of G, the induced subgraph $G \setminus \{v\}$ admits a partition into p cliques of cardinality q and also admits a partition into q stable sets of cardinality p.

Due to results of Lovász (1972) [5] and Padberg (1974) [6], the partitionable graphs contain all minimal imperfect graphs, and they received a lot of attention in the past decades when looking for counterexamples to the famous Strong Perfect Graph Conjecture of Berge (proved by Chudnovsky and Seymour in 2002).

In 1979, Chvátal, Graham, Perold and Whitesides introduced two constructions for partitionable graphs [3]. Due to the initials of these four authors, we call CGPW_2 graphs the graphs given by the second construction.

In 1996, Gao and Zhu started the study of chromatic properties of circulant graphs [4]. They provided several classes of circulant graphs which are star-extremal, i.e., whose circular chromatic number equals their fractional chromatic number. They also exhibit circulant graphs which are not star-extremal, but, apart from one obtained by a computer search, all the others are constructed using lexicographic product.

In this paper we study the circular chromatic number of partitionable graphs and we provide an infinite class of non-star-extremal graphs.

The paper is organized as follows: the first section is devoted to definitions needed for proofs.

In the beginning of the second section, we introduce a basic lemma, which is the starting point of the paper: it suggests that to study the circular chromatic number of a partitionable graph, one may restrict its attention to homomorphisms from G into *partitionable antiwebs*, with the *same maximum clique size* of G. We first study a more general setting, by providing some necessary conditions for the existence of a homomorphism between two partitionable graphs. Then we introduce the notion of rank of a partitionable graph G which gives the set of *partitionable antiwebs* with same maximum clique size than G, that G is homomorphic to.

The results on the structure of partitionable graphs of rank $\alpha(G)$ will allow us to prove that many of the CGPW_2 graphs are not star-extremal.

2. Definitions

Let $G=(V,E)$ be a simple, undirected graph. A *clique* is a set of pairwise adjacent vertices and a *stable set* is a set of pairwise non-adjacent vertices. We denote by $\omega=\omega(G)$ the maximum cardinality of a clique of Γ and $\alpha=\alpha(G)$ the maximum cardinality of a stable set of Γ. A *maximum clique* is a clique of size $\omega(G)$ and a *maximum stable set* is a stable set of size $\alpha(G)$.

A *homomorphism* from a graph G into a graph H is a map from the vertex set of G into the vertex set of H such that if ij is an edge of G then $f(i)f(j)$ is an edge of H.

The *complement graph* $\overline{G}$ of $G=(V,E)$ is the graph with vertex set V and edge set $\{ij \mid i \in V,\ j \in V,\ ij \notin E\}$. The graph G with vertex set V is *isomorphic* to the graph G' with vertex set V' if there exists a bijective map f from V onto V' such that ij is an edge of G if and only if $f(i)f(j)$ is an edge of G'.

A *determined edge* is an edge ij such that there exists a maximum clique containing both i and j. A graph is *normalized* if every of its edges is a determined edge. The *normalized graph* of a graph G is the graph with vertex set $V(G)$ and with the set of determined edges of G for edge set. A *variant* of G is a partitionable graph with the same vertices, the same maximum cliques and the same maximum stable sets.

A *web* W_k^n is a graph with vertex set $\{0,\dots,n-1\}$ and edge set $\{ij|\ \min(|i-j|,|j-i|)<k\}$. An *antiweb* is the complement of a web. We denote by C_k^n the antiweb $\overline{W_k^n}$.

A well-known class of partitionable graphs is the class of (α,ω)-*webs*, where an (α,ω)-web (with $\alpha\geq 2$ and $\omega\geq 2$) is the web $W_\omega^{\alpha\omega+1}$. Notice that $\omega(W_\omega^{\alpha\omega+1})=\omega$ and $\alpha(W_\omega^{\alpha\omega+1})=\alpha$. We call *partitionable webs* these webs, and *partitionable antiwebs* their complements.

We denote by $I(G)$ the *intersection graph* of a graph G, that is the graph whose vertices are the maximum cliques of G, and two vertices Q_i and Q_j are adjacent if and only if Q_i and Q_j are not two disjoint maximum cliques of G. The intersection graph $I(G)$ of a graph partitionable graph G is partitionable and such that $\omega(I(G))=\omega(G)$ and $\alpha(I(G))=\alpha(G)$ [8].

Let n be any positive integer and $\mathbb{Z}_n$ denote the cyclic group. Let S be any symmetric subset of $\mathbb{Z}_n\setminus\{0\}$. The *circulant graph* $G(n,S)$ is the graph with n vertices whose vertices are the elements of the cyclic group $\mathbb{Z}_n$ and two vertices i and j are adjacent if and only if $i-j$ is an element of S.

Let $m_1,\dots,m_{2r}$ be a set of $2r$ integers greater than or equal to 2. Let $\mu_0=1$. For every integer $1\leq i\leq 2r-1$, let

$$\mu_i = m_1m_2\cdots m_i$$
$$M_i=\{0,\mu_{i-1},2\mu_{i-1},\dots,(m_i-1)\mu_{i-1}\}$$
$$C=M_1+M_3+\cdots+M_{2r-1}$$
$$n=m_1m_2\cdots m_{2r}+1$$

Let S be the subset $\{(i-j)|\ i\in C,\ j\in C,\ i\neq j\}$ of $\mathbb{Z}_n$. Let $C[m_1,\dots,m_{2r}]$ be the circulant graph $G(n,S)$. All such graphs and their variants are called CGPW_2 graphs. Every CGPW_2 graph is a circulant partitionable graph [3]. Fig. 1 shows a CGPW_2 graph.

A (k,d)-coloring of a graph G is a mapping $c:V(G)\to\mathbb{Z}_k$ such that

$$c(x)-c(y)\in\{\pm d,\dots,\pm(k-d)\}\ \forall xy\in E(G).$$

The *circular chromatic number* $\chi_c(G)$ (also called *star chromatic number*) can be thought as the best possible such coloring:

$$\chi_c(G)=\inf\left\{\frac{k}{d}: G\ has\ (k,d)-coloring\right\}.$$

Notice that, when $d=1$, we find the usual definition of proper k-coloring. This generalization of the chromatic number was introduced by Vince [9] who also proved that it takes rational values and is not too far from the ordinary chromatic number. In fact, for every graph G, $\chi(G)-1<\chi_c(G)\leq\chi(G)$.

A *fractional coloring* of a graph G is a mapping f from the set of stable sets $\mathcal{S}(G)$ of G to the interval $[0,1]$ such that, for each v of G,

$$\sum_{S\in\mathcal{S}(G),v\in S} f(S)=1$$

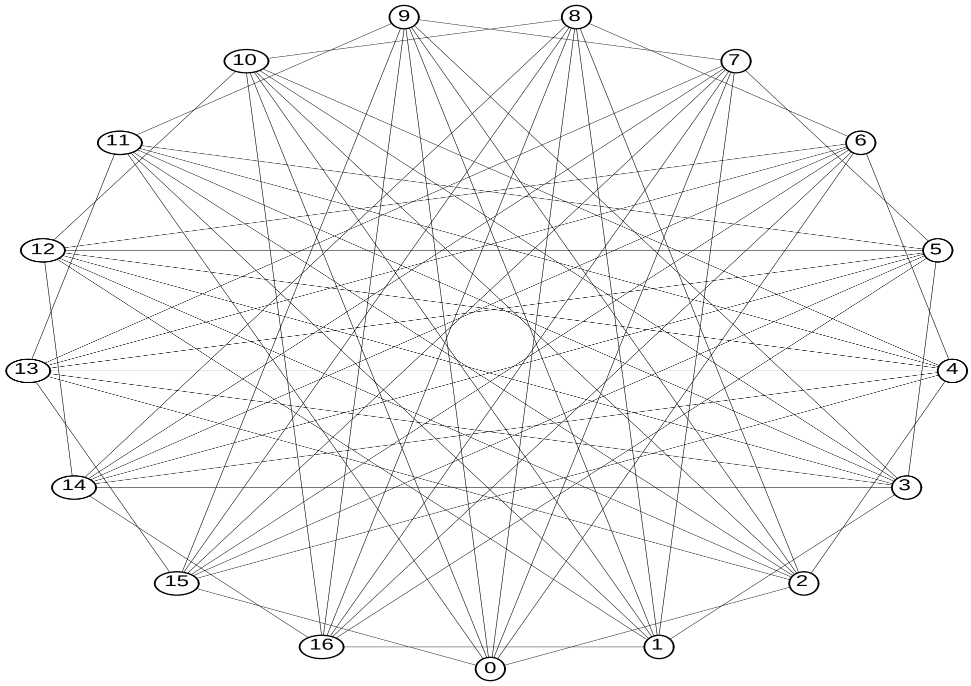

Figure 1: The CGPW$_2$ graph $C[2,2,2,2]$

The *fractional chromatic number* is

$$\chi_f(G) = min\left\{ \sum_{S\in\mathcal{S}(G)} f(S): \; f \;\; is\; frac.\; col. \right\}$$

If we replace the interval $[0,1]$ with the set $\{0,1\}$ we get the ordinary chromatic number. It is also not difficult to see that $\chi_f(G) \leq \chi_c(G)$ but the gap between the two parameters may be arbitrarily large as well as it happens for $\chi_f(G)$ and $\chi(G)$.

A graph is said to be *star-extremal* if $\chi_f(G) = \chi_c(G)$.

3. Homomorphisms between partitionable graphs

Bondy and Hell proved in [2] that a graph G is (n,k)-colorable if and only if G is homomorphic to C_n^k. In this section we show that every partitionable graph is homomorphic to a circulant graph C_k^n with same maximum clique size, or its circular chromatic number is equal to its chromatic number.

Lemma 3.1. *If G is any partitionable graph with $\chi_c(G) = \frac{n}{k}$ then $\omega \leq \lfloor\frac{n}{k}\rfloor \leq \omega+1$. Furthermore $\lfloor\frac{n}{k}\rfloor = \omega+1$ if and only if $k=1$, that is $\chi_c(G) = \chi(G)$.*

Proof. We have $\frac{n}{k} = \chi_c(G) \leq \chi(G) = \omega+1$. Hence $\lfloor\frac{n}{k}\rfloor \leq \omega+1$. If $k=1$ then $\frac{n}{k} = \chi_c(G) = \chi(G) = \omega+1$. Hence $\lfloor\frac{n}{k}\rfloor = \omega+1$.

We have $\omega+1 = \chi(G) = \lceil\chi_c(G)\rceil = \lceil\frac{n}{k}\rceil$. If $\lfloor\frac{n}{k}\rfloor = \omega+1$ then $\frac{n}{k} = \omega+1$. Since n and k are relatively prime, this implies that $k=1$. Hence if $k>1$ we must have $\lfloor\frac{n}{k}\rfloor = \omega$. ∎

Thus, if G is any partitionable graph, then there are integers $k(G)$ and $q(G)$ with $1 \leq k(G) \leq \alpha(G)$ and $1 \leq q(G) \leq k(G)$ such that the circular chromatic number of G is equal to $\omega(G)+\frac{q(G)}{k(G)}$. We do not know any partitionable graph G such that $q(G) > 1$:

Problem 1. *Is every partitionable graph G such that $q(G) = 1$?*

The case $q(G) = 1$ implies that G is homomorphic to the complement of the $(\omega(G), \alpha(G))$-web, which is a variant of the $(\alpha(G), \omega(G))$-web, hence a partitionable graph with the same maximum clique size. This leads to this more general question: what does it mean for a partitionable graph to be homomorphic to another partitionable graph with same maximum clique size?

We have the following result, whose first assertion was suggested by Prop. 4 in [2].

Theorem 3.2. *Let G and H be two partitionable graphs with same maximum clique size, such that G is homomorphic to H. Then*

i) $v_G \geq v_H$*, where* v_G *(resp.* v_H*) denotes the number of vertices of* G *(resp.* H*).*

ii) the homomorphism must be surjective.

iii) G *is isomorphic to a variant of* H *if and only if* $v_G = v_H$*.*

iv) the homomorphism f *induces a surjective homomorphism* f' *from the intersection graph of* G *into the intersection graph of* H*.*

Proof.

Let f be a homomorphism from G into H. For every vertex i of H, we denote by H_i the induced subgraph $H \setminus \{i\}$ and let $G_i := f^{-1}(H_i)$. Let $\omega = \omega(G) = \omega(H)$.

Obviously, f induces a homomorphism from G_i into H_i. Since $\chi(H_i) = \chi_c(H_i) = \omega$, there is a homomorphism from H_i into the clique C_1^{ω} of size ω. Hence there is a homomorphism from G_i into C_1^{ω}. Thus G_i is of size at most $v_G - 1$ for every vertex i of H.

Let $\overrightarrow{\chi_{G_i}}$ denote the incidence vectors of the set G_i. Let x be any vertex of G. Notice that $x \in G_i$ if and only if $f(x) \neq i$. Hence x belongs to exactly $v_H - 1$ sets G_i. Thus we get

$$\sum_{i \in V(H)} \overrightarrow{\chi_{G_i}} = (v_H - 1)\overrightarrow{1}$$

Therefore we have

$$\begin{aligned} \sum_{i \in V(H)} |G_i| &= \sum_{i \in V(H)} \overrightarrow{\chi_{G_i}} . \overrightarrow{1} \\ &= v_H(v_H - 1) \leq v_H(v_G - 1) \end{aligned}$$

Thus $v_G \geq v_H$.

We now prove that f must be surjective. Let $H[\mathrm{Im}(f)]$ be the subgraph of H induced by the image of the homomorphism f. The graph H is a partitionable graph with chromatic number $\omega + 1$. Hence for every induced subgraph H' of H distinct of H, the chromatic number of H' is at most ω.

Thus, if f is not surjective then $H[\mathrm{Im}(f)]$ is a proper induced subgraph of H and therefore $\chi_c(H[\mathrm{Im}(f)]) \leq \chi(H[\mathrm{Im}(f)] \leq \omega$. If $\chi_c(G) = \omega$ then $\chi(G) = \omega$, a contradiction. Hence $\chi_c(G) > \omega$.

Then we get, $\omega < \chi_c(G) \leq \chi_c(H[\mathrm{Im}(f)]) \leq \omega$, which is impossible. Hence f is surjective.

If G is isomorphic to a variant of H then obviously $v_G = v_H$.

If $v_G = v_H$ then since f is surjective, f is bijective. Hence H contains a partial subgraph H' which is isomorphic to G. Thus H' has exactly n maximum cliques of size ω. This implies that H' is a variant of H. Therefore, G is a variant of H.

Since $\omega(G) = \omega(H)$, the homomorphism f maps a maximum clique of G onto a maximum clique of H. Let f' be the map from the set of maximum cliques of G into the set of maximum cliques of H induced by f. If Q and Q' are two maximum cliques of G such that $Q \cap Q' \neq \emptyset$, then $f'(Q) \cap f'(Q') = f(Q) \cap f(Q') \neq \emptyset$. Hence f' is a homomorphism from the intersection graph of G into the intersection graph of H. Hence f' is surjective. ∎

Notice that the first assertion of Theorem 3.2 is not implied by Prop. 4 in [2], as H is not assumed to be vertex transitive.

Now, let G be a partitionable graph. Since $\chi(G) = \omega + 1$, there is a homomorphism from G into the clique $C_1^{\omega+1}$ of size $\omega + 1$.

By Theorem 3.2, for every $r > \alpha(G)$ there is no homomorphism from G into $C_r^{r\omega(G)+1}$.

Let us denote by rank(G) the greatest integer r such that there is a homomorphism from G into $C_r^{r\omega(G)+1}$. We have that

$$1 \leq \text{rank}(G) \leq \alpha(G).$$

Notice that the rank gives the following upper bound of χ_c :

$$\chi_c(G) \leq \omega + \frac{1}{\text{rank}(G)}.$$

Furthermore, if G is a partitionable graph homomorphic to a partitionable graph H with the same maximum clique size then rank$(G) \geq$ rank(H).

There are partitionable graphs of rank 1 (Fig. 1 gives an example).

The next theorem concerns the rank of the intersection graph of a partitionable graph. The following lemma is useful for its proof.

Lemma 3.3. *If H is the normalized graph of a circulant partitionable graph G then the intersection graph of G is isomorphic to H.*

Proof. Let H be the normalized graph of a circulant partitionable graph $G(n, S)$. Let Q be any maximum clique of $G(n, S)$. For every $x \in \mathbb{Z}_n$, $Q + x$ is obviously a maximum clique of $G(n, S)$. If there are two distinct elements x and y of $\mathbb{Z}_n$ such that $Q + x = Q + y$ then Q must be a disjoint union of cosets of the subgroup generated by $y - x$ of $\mathbb{Z}_n$. By Lagrange's theorem, the size of this subgroup is a divisor of n and therefore $|Q|$ and n are not relatively prime, in contradiction with the relation $n = \alpha(G(n, S))|Q| + 1$.

Hence the n maximum cliques of $G(n, S)$ (and also of H), are $\{Q + x | \ x \in Z_n\}$.

Let ij be any edge of H: there exists a maximum clique $Q + x$ such that $i \in Q + x$ and $j \in Q + x$. Hence we get that $Q + i$ intersects $Q + j$ (with $i \neq j$). Thus $(Q + i)(Q + j)$ is an edge of the intersection graph of $G(n, S)$.

Conversely, let $(Q + i)(Q + j)$ be any edge of the intersection graph of $G(n, S)$. We have $(i - j) \in (Q - Q) \setminus \{0\}$. Hence there are q and q' in Q such that $i - j = q - q'$. Thus we get that i is an element of $Q + j - q'$. Obviously, j is also an element of $Q + j - q'$, as $q' \in Q$. Hence ij is a determined edge of $G(n, S)$ and therefore an edge of H.

Thus H and the intersection graph of $G(n, S)$ are isomorphic. ∎

Theorem 3.4. *If $I(G)$ denotes the intersection graph of a normalized partitionable graph G then*

$$\mathrm{rank}(G) = \mathrm{rank}(I(G))$$

Proof. By (iv) of Theorem 3.2, we have that $\mathrm{rank}(I(G)) \geq \mathrm{rank}(I(C_{\mathrm{rank}(G)}^{\mathrm{rank}(G)\times\omega+1}))$.

Since $C_{\mathrm{rank}(G)}^{\mathrm{rank}(G)\omega+1}$ is a normalized circulant partitionable graph, we have, by Lemma 3.3, that $I(C_{\mathrm{rank}(G)}^{\mathrm{rank}(G)\omega+1}) = C_{\mathrm{rank}(G)}^{\mathrm{rank}(G)\omega+1}$. Hence $\mathrm{rank}(I(G)) \geq \mathrm{rank}(G)$.

If $\mathrm{rank}(I(G)) > \mathrm{rank}(G)$ then $\mathrm{rank}(I(I(G))) > \mathrm{rank}(G)$. But it is well-known [7] that $I(I(G))$ is the normalized graph of G, hence we get a contradiction. ∎

4. Circulant partitionable graphs

A partitionable graph G with $\mathrm{rank}(G) = \alpha(G)$ is said to have maximal rank. Notice that every odd hole or antihole G has maximal rank. Due to Theorem 3.2, we get the following characterization of the class of partitionable graphs having maximal rank:

Corollary 4.1. *The partitionable graphs having maximal rank are the variants of partitionable webs.*

Lemma 4.2. *A star-extremal circulant partitionable graph is a variant of a web.*

Proof. Let G be any star-extremal circulant partitionable graph. Since G is vertex transitive, we have $\chi_f(G) = \frac{n}{\alpha}$. Hence $\chi_c(G) = \omega + \frac{1}{\alpha}$, that is G is of maximal rank. Hence G must be a variant of a web. ∎

We are now ready to state the main theorem that proves the existence of an infinite family of circulant graphs which are not star-extremal.

Theorem 4.3. *There is an infinite number of circulant partitionable graphs not star-extremal with a prime number of vertices.*

Proof. By Lemma 4.2, it suffices to prove that there are infinitely many circulant partitionable graphs with a prime number of vertices which are not variants of webs.

Claim 1. *If $r \geq 2$ and $m_1, \ldots, m_{2r}$ are integers with $m_i \geq 2$ for every $1 \leq i \leq 2r$, then the partitionable graph $C[m_1, \ldots, m_{2r}]$ with $m_1 m_2 \ldots m_{2r} + 1$ vertices is not a variant of a web.*

In a web, there are always two mawimum cliques sharing $\omega - 1$ vertices: we are going to show that this does not hold in $C[m_1, \ldots, m_{2r}]$. The maximum cliques of $C[m_1, \ldots, m_{2r}]$ are the sets $C + x,\ x \in Z_n$. Assume that there is x such that $|C \cap (C + x)| = \omega - 1$. We want to show that this leads to a contradiction.

It is easy to check that

$$C \cap (C + m_1) \quad = \quad \emptyset. \tag{1}$$

Hence there is a subset T of $\mathbb{Z}_n$ such that the sets $M_1 + t,\ t \in T$ are pairwise disjoints and $C = \cup_{t \in T}(M_1 + t)$. Due to Eq. 1 and since $|C \cap (C + x)| = \omega - 1$, for every t in T, there exists an unique element $g(t)$ of T such that $(M_1 + t) \cap (M_1 + x + g(t)) \neq \emptyset$.

Hence we get, $C \cap (C + x) = \cup_{t \in T}((M_1 + t) \cap (M_1 + x + g(t)))$ and this union is disjoint.

Since $|C \cap (C+x)| = \omega - 1$, there is an unique t_0 in T such that

$$|(M_1 + t_0) \cap (M_1 + x + g(t_0))| \quad = \quad m_1 - 1 \tag{2}$$

and

$$\forall t \neq t_0,\ t \in T,\ M_1 + t \quad = \quad M_1 + x + g(t) \tag{3}$$

Notice that $t_0 + m_1 m_2$ or $t_0 - m_1 m_2$ is an element of T.

If $t_0 + m_1 m_2$ is an element of T: assume that $g(t_0) + m_1 m_2$ is not an element of T. Then $(M_1 + t_0 + m_1 m_2) \cap (C + x) = \emptyset$: a contradiction. Hence $g(t_0) + m_1 m_2$ is an element of T. Since $|(M_1 + g(t_0) + m_1 m_2) \cap (M_1 + t_0 + m_1 m_2)| = m_1 - 1 \neq 0$ by Eq. 2, we get $g(t_0 + m_1 m_2) = g(t_0) + m_1 m_2$. Then Eq. 3 gives $M_1 + t_0 = M_1 + x + g(t_0)$ in contradiction with Eq. 2.

The case $(t_0 - m_1 m_2) \in T$ is similar. Thus $|C \cap (C+x)| \neq \omega - 1$, which ends the proof of Claim 1.

Claim 1 says that most of CGPW_2 graphs are not variants of webs. In particular, for every integer $k \geq 2$, the graph $C[2,2,2,k]$ with $8k+1$ vertices is not a variant of a web. Since 8 and 1 are relatively prime, this implies that there are infinitely many circulant partitionable graphs with a prime number of vertices which are not variants of webs. ∎

In Zhu's survey [10], it was asked for a characterization of graphs G such that

$$\chi_c(G - v) = \chi_c(G) - 1 \quad \text{for each vertex } v \text{ of } G \qquad (*)$$

It is well known that every partitionable graph G is such that for every vertex v, we have $\omega(G) = \chi(G - v) = \chi_c(G - v)$ ([1],[6]). It follows that a partitionable graph G has property $(*)$ if and only if $\chi_c(G) = \chi(G) = \omega(G) + 1$.

A computer check revealed that the graph $C[2,2,2,2]$ is such a graph. Hence it should be interesting to look for other such graphs in the class of CGPW_2 graphs.

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