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BIPARTITE FINITE TOEPLITZ GRAPHS

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Abstract

Let $n, a_1, \ldots, a_k$ be distinct positive integers. A finite Toeplitz graph $T_n(a_1, \ldots, a_k) = (V, E)$ is a graph where $V = \{v_0, \ldots, v_{n1}\}$ and $E = \{(v_i, v_j), \text{for } |ij| \in \{a_1, \ldots, a_k\}\}$. In this paper, we characterize bipartite finite Toeplitz graphs with $k \leq 3$. As a consequence, using previous results, we get a complete characterization for the chromatic number of such graphs. In addition, we characterize some classes of bipartite Toeplitz graphs with $k \geq 4$.

Key words: Toeplitz graphs, bipartiteness, coloring, chromatic number.

1 Introduction

Let $n, a_1, a_2, \ldots, a_k$ be distinct positive integers such that $1 \leq a_1 < a_2 < \cdots < a_k < n$. By $T_n(a_1, a_2, \ldots, a_k) = (V, E)$ we denote the (simple undirected) *finite Toeplitz graph* where $V = \{v_0, v_1, \ldots, v_{n-1}\}$ and $E = \{(v_i, v_j), \text{ for } |i - j| \in \{a_1, a_2, \ldots, a_k\}\}$ (see Fig. 1). The numbers $a_1, a_2, \ldots, a_k$ are called *entries*. The name of this class of graphs is due to the fact that the adjacency matrix is a Toeplitz matrix, i.e., each of its descending diagonal from left to right is constant. Notice that in the literature also Toeplitz graphs with an infinite number of vertices are defined (*infinite Toeplitz graphs*).

In this paper we shall mainly focus on Toeplitz graps with two or three entries: we shall denote them by $T_n(a, b)$ and $T_n(a, b, c)$, respectively, where $1 \leq a < b < c < n$. Consider an arbitrary edge (v_i, v_j) of $T_n(a, b)$ or $T_n(a, b, c)$: if $|i - j| = a, b, c$ respectively, we shall say that vertices v_i v_j are *a-, b-, c-adjacent*, resp., and that $(v_i, v_j) \in E$ is an *a-, b-, c-edge*, resp. For Toeplitz graphs $T_n(a, b)$ or $T_n(a, b, c)$, by *a-path* A_p, $p = 0, 1, \ldots, a-1$, we denote the path containing vertex v_p and made of a-edges only: the vertices of A_p are v_p, $v_{p+a}, \ldots,$ v_{p+ta}, where t is such that $p + ta < n$ (that is to say, all the vertices v_x verifying $x \bmod a = p$ belong to A_p).

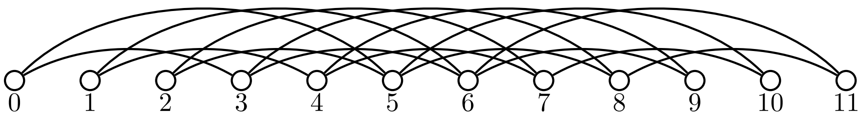


Figure 1: The Toeplitz graph $T_{12}(3, 5)$.

The problem we study is the bipartiteness of Toeplitz graphs. We remark that if $k = 1$ the problem is trivial: in fact, the Toeplitz graph $T_n(a)$ is bipartite, because it consists of a collection of $\min\{n, a\}$ vertex-disjoint paths.

In [8] the author proposes an $O(\log^2(b + 1))$ procedure to test a finite $T_n(a, b)$ for bipartiteness, and states some results for three particular subclasses of finite non-bipartite $T_{2a+1}(a, b, c)$ with $1 \leq a < b < c < n = 2a+1$. From these results one can derive the non-bipartiteness of the corresponding $T_n(a, b, c)$ with $n \geq 2a + 1$. Infinite bipartite Toeplitz graphs are characterized in [9].

In Section 2 of this paper we state a simple closed-form $\log^2 a$ condition to test a finite $T_n(a, b)$ for bipartiteness (we remark that the proof of this condition is based on easy topological properties of the graph). In Section 3 we characterize the whole family of finite bipartite $T_n(a, b, c)$'s, thereby providing an answer to the problem posed in [8]. The proved results apply to infinite bipartite Toeplitz graphs and to some integer distance graphs with two or three entries (an *integer distance graph* $G_{\mathbb{Z}}(a_1, a_2, \ldots, a_k)$ is a graph with an infinite number of vertices $\{\ldots, v_{-2}, v_{-1}, v_0, v_1, v_2, \ldots\}$, where two vertices v_x and v_y are adjacent if and only if $|x - y| \in \{a_1, a_2, \ldots, a_k\}$, see [2, 3, 6]), and are extended in Section 4 to characterize some subclasses of bipartite $T_n(a_1, \ldots, a_k)$'s with $k \geq 4$.

A consequence of our results and those in [11] is an answer to the open problem of providing a complete characterization of the chromatic number of Toeplitz graphs with three entries.

We here recall the following connectedness theorem for $T_n(a,b)$'s.

Theorem 1.1. [10, 11] *Let $T_n(a,b)$ be a Toeplitz graph with $1 \le a < b < n$.*

- *$T_n(a,b)$ is connected if and only if $\gcd(a,b) = 1$ and $n \ge a+b-1$.*

In addition

- *If $\gcd(a,b)=1$ and $n = a+b-1$, then $T_n(a,b)$ is a hamiltonian path.*
- *If $\gcd(a,b)=1$ and $n = a+b$, then $T_n(a,b)$ is a hamiltonian cycle.*
- *A $T_n(a,b)$ with $\gcd(a,b)=1$ has $\max\{1, a+b-n\}$ connected components.*

We derive the following two corollaries:

Corollary 1.2. *Vertices $v_0, \dots, v_{a+b-1}$ of a Toeplitz graph $T_n(a,b)$ with $\gcd(a,b)=1$ and $n \ge a+b$ induce a cycle.*

In Proposition 7 of [12] it is proved that $T_{a+b+1}(a,b)$ with $\gcd(a,b)=1$ has a hamiltonian path with endpoints v_0 and v_{a+b}. This result follows immediatly from the result above: delete edge (v_0, v_a) from the cycle, and add edge (v_a, v_{a+b}) to get the path.

Corollary 1.3. *Consider a Toeplitz graph $T_n(a,b)$ with $1 \le a < b < n$.*

- *If $a+b-\gcd(a,b)+1 \le n \le a+b$, then $T_n(a,b)$ is a collection of $n+\gcd(a,b)-a-b$ vertex-disjoint cycles and $a+b-n$ vertex-disjoint paths.*
- *$T_n(a,b)$ has $\max\{\gcd(a,b), a+b-n\}$ connected components, which are all paths or isolated vertices if $n \le a+b-\gcd(a,b)$.*

In the sequel of the paper we shall often make use of particular subgraphs of $T_n(a,b)$, namely the graphs T^p induced by the set V^p of vertices v_x with $x \bmod \gamma = p$, for $p = 0, \dots, \gamma-1$, where $\gamma = \gcd(a,b)$. By construction, the graph T^p induced by V^p is isomorphic to $T_{n^p}(\frac{a}{\gamma}, \frac{b}{\gamma})$, where $n^p = \left\lceil \frac{n}{\gamma} \right\rceil$ for $p = 0, \dots, n \bmod \gamma - 1$ and $n^p = \left\lfloor \frac{n}{\gamma} \right\rfloor$ for $p = n \bmod \gamma, \dots, \gamma-1$. Notice that $T_n(a,b)$ is the union of all the T^p's.

2 Bipartite $T_n(a,b)$'s

In the present section, by means of a short and combinatorial proof, we state some easy necessary and sufficient conditions for a connected $T_n(a,b)$ to be bipartite.

From Corollary 1.3 it immediately follows that:

Theorem 2.1. *A Toeplitz graph $T_n(a,b)$ with $n \le a+b-\gcd(a,b)$ is bipartite.*

As for the remaining cases, we have that:

Theorem 2.2. *A Toeplitz graph $T_n(a,b)$ with $n \geq a+b-\gcd(a,b)+1$ is bipartite if and only if $\frac{a}{\gcd(a,b)}$ and $\frac{b}{\gcd(a,b)}$ are odd.*

Proof. If part. If all the entries are odd, every edge connects an even-indexed vertex to an odd-indexed one.
Only if part. We have to prove that if $T_n(a,b)$ with $n \geq a+b-\gcd(a,b)+1$ is bipartite, then $\frac{a}{\gcd(a,b)}$ and $\frac{b}{\gcd(a,b)}$ are odd. Let $\gamma = \gcd(a,b)$, and consider the subgraph T^0. Recalling that, by definition, T^0 is isomorphic to $T_{\lceil \frac{n}{\gamma} \rceil}(\frac{a}{\gamma}, \frac{b}{\gamma})$, since $\left\lceil \frac{n}{\gamma} \right\rceil \geq \frac{a}{\gamma} + \frac{b}{\gamma}$, by Corollary 1.2 it contains a cycle on the first $\frac{a}{\gamma} + \frac{b}{\gamma}$ vertices. The assumption $T_n(a,b)$ bipartite implies T^0 bipartite, thus $\frac{a}{\gamma} + \frac{b}{\gamma}$ is even, that is to say, $\frac{a}{\gamma}$ and $\frac{b}{\gamma}$ have the same parity. By definition of $\gamma = \gcd(a,b)$, they are both odd. □

3 Bipartite $T_n(a,b,c)$'s

The present section is devoted to characterize the bipartite $T_n(a,b,c)$'s. The following preliminary results hold:

Theorem 3.1. *A Toeplitz graph $T_n(a,b,c)$ with $\gcd(a,b,c) = 1$ and a,b,c odd is bipartite.*

Proof. If all the entries are odd, every edge connects an even-indexed vertex to an odd-indexed one. □

Theorem 3.2. *A Toeplitz graph $T_n(a,b,c)$ is bipartite if and only if $T_{\lceil \frac{n}{\gcd(a,b,c)} \rceil}\left(\frac{a}{\gcd(a,b,c)}, \frac{b}{\gcd(a,b,c)}, \frac{c}{\gcd(a,b,c)}\right)$ is.*

Proof. Let $\delta = \gcd(a,b,c)$. Consider the set $\overline{V}^q$ of vertices v_t with $t \bmod \delta = q$, $q = 0, \ldots, \delta-1$. By construction, the graph $\overline{T}^q$ induced by $\overline{V}^q$ is isomorphic to $T_{n^q}(\frac{a}{\delta}, \frac{b}{\delta}, \frac{c}{\delta})$, where $n^q = \lceil \frac{n}{\delta} \rceil$ for $q = 0, \ldots, n \bmod \delta - 1$ and $n^q = \lfloor \frac{n}{\delta} \rfloor$ for $q = n \bmod \delta, \ldots, \delta-1$. By definition of $\gcd(a,b,c)$ and since $\overline{T}^1$, $\overline{T}^2$, …, $\overline{T}^{\delta-1}$ are all isomorphic to subgraphs of $\overline{T}^0$, the claim follows. □

As a consequence of the theorem above, in the sequel we shall limit ourselves to consider $T_n(a,b,c)$'s with $\gcd(a,b,c) = 1$: each of the following subsections proves the bipartiteness of $T_n(a,b,c)$'s according to three different value-ranges for n. In addition, as a consequence of Theorem 3.1, it is convenient to apply the results proved in the next three subsections to Toeplitz graphs with at least one even entry. A preliminary result is the following.

Theorem 3.3. *If b or c is an even multiple of a, then $T_n(a,b,c)$ is non-bipartite.*

Proof. If $b = ha$, where h is an even integer, then $\{(v_0, v_a), (v_a, v_{2a}), \ldots, (v_{(h-1)a}, v_{ha}), (v_{ha}, v_0)\}$ is an odd cycle. Similarly if $c = ha$. □

3.1 $T_n(a,b,c)$'s with $\gcd(a,b,c) = 1$ and $n \leq c + \gcd(a,b) - 1$

We shall first prove that the bipartiteness of the Toeplitz graphs $T_n(a,b,c)$ we consider in the present section, depends on the bipartiteness of $T_n(a,b)$. From this result and the results of Section 2, we derive Theorems 3.5 and 3.6.

Lemma 3.4. *A Toeplitz graph $T_n(a,b,c)$ with $n \leq c + \gcd(a,b) - 1$ and $\gcd(a,b,c) = 1$ is bipartite if and only if $T_n(a,b)$ is.*

Proof. Let $\gamma = \gcd(a,b)$. $T_n(a,b,c)$ has $n - c$ c-edges. The assumption $n \leq c + \gamma - 1$ implies that the graph has at most $\gamma - 1$ c-edges. Since $\gcd(c,\gamma) = 1$ (as implied by $\gcd(a,b,c) = 1$), no cycle containing a c-edge exists when $T_n(a,b,c)$ contains less than γ c-edges (notice that when $\gamma = 1$ the graph has no c-edge at all). Hence $T_n(a,b,c)$ is bipartite if and only if $T_n(a,b)$ is. □

An example is shown in Fig. 2.

Combining the lemma above with Theorems 2.1 and 2.2 we get the following two theorems.

Theorem 3.5. *A Toeplitz graph $T_n(a,b,c)$ with $\gcd(a,b,c) = 1$ and $n \leq \min\{a + b - \gcd(a,b); c + \gcd(a,b) - 1\}$ is bipartite.*

Theorem 3.6. *A Toeplitz graph $T_n(a,b,c)$ with $\gcd(a,b,c) = 1$ and $a+b-\gcd(a,b)+1 \leq n \leq c + \gcd(a,b) - 1$ is bipartite if and only if $\frac{a}{\gcd(a,b)}$ and $\frac{b}{\gcd(a,b)}$ are odd.*

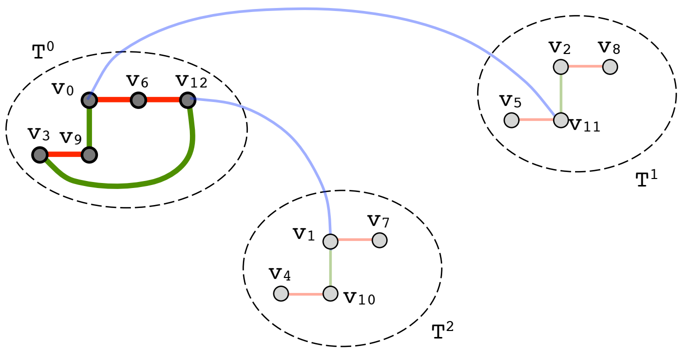


Figure 2: The Toeplitz graph $T_{13}(6,9,11)$.

3.2 $T_n(a,b,c)$'s with $\gcd(a,b,c) = 1$ and $n \geq \max\{a + b - \gcd(a,b) + 1; c + \gcd(a,b)\}$

In this section we prove that:

Theorem 3.7. *A Toeplitz graph $T_n(a,b,c)$ with $\gcd(a,b,c) = 1$ and $n \geq \max\{a + b - \gcd(a,b) + 1; c + \gcd(a,b)\}$ is bipartite if and only if a,b,c are odd.*

Proof. The *If part* follows from Theorem 3.1, so let us turn to prove the *Only if part.* We have to show that if $T_n(a,b,c)$ is bipartite, then a,b,c are odd. The assumption that $T_n(a,b,c)$ is bipartite implies that $T_n(a,b)$ is bipartite too. Let $\gamma = \gcd(a,b)$. By Theorem 2.2 it follows that $\frac{a}{\gamma}, \frac{b}{\gamma}$ are odd. We distinguish two cases: $\gamma = 1$ and $\gamma > 1$.

If $\gamma = 1$, by Theorem 1.1 $T_n(a,b)$ is connected, and the (unique) bipartition of the set of vertices of $T_n(a,b)$ is the one which separates the odd-indexed vertices from the even-indexed ones. Since we are assuming that $T_n(a,b,c)$ is bipartite, it must be the case that c is odd, too.

Now consider the case $\gamma > 1$, and consider the subgraphs T^p's of $T_n(a,b)$, for $p = 0,\ldots,\gamma-1$. The assumption $n \geq a+b-\gamma+1$ implies that each T^p is connected, by Theorem 1.1. In addition, each c-edge connects distinct T^p's, because $\gcd(c,\gamma) = 1$ (as implied by $\gcd(a,b,c) = 1$). Since $n \geq c + \gcd(a,b)$, $T_n(a,b,c)$ has $n - c \geq \gamma$ c-edges. The first γ of them define the cycle $\mathcal{C}$ which alternates a (suitable) c-edge with a path (made of a- and b-edges, only) within a (suitable) T^p. Precisely: $\mathcal{C}$ contains c-edge (v_0, v_c), then a path within $T^{c \bmod \gamma}$ from v_c to $v_{c \bmod \gamma}$, then c-edge $(v_{c \bmod \gamma}, v_{c \bmod \gamma + c})$, then a path within $T^{((c \bmod \gamma + c) \bmod \gamma)}$ and so on, until a (suitable) c-edge brings $\mathcal{C}$ back to a vertex $v_y \in T^0$, and a path from v_y to v_0 within T^0 closes it.

Let us determine the number of edges of $\mathcal{C}$. $\mathcal{C}$ contains γ c-edges and γ paths within T^p's. The bipartiteness of $T_n(a,b)$ implies that all the paths connecting two given vertices v_z and v_p in T^p do have the same parity. For example consider a path, say P_z with i_z a-edges and j_z b-edges, where i_z and j_z are such that $p + i_z a + j_z b = z$. We get $i_z \frac{a}{\gamma} + j_z \frac{b}{\gamma} = \frac{z-p}{\gamma}$ (notice that the definition of T^p implies that $\frac{z-p}{\gamma}$ is integer). Recalling that $\gamma = \gcd(a,b)$, we know that $\frac{a}{\gamma}$ and $\frac{b}{\gamma}$ are coprime, and odd, in particular. Thus the length $i_z + j_z$ of P_z and $i_z \frac{a}{\gamma} + j_z \frac{b}{\gamma} = \frac{z-p}{\gamma}$ have the same parity. In particular: for the $(\gamma - c \bmod \gamma)$ vertices $v_c \in T^{c \bmod \gamma}$, $\ldots$, $v_{c+\gamma-1-c \bmod \gamma} \in T^{\gamma-1}$ the parity of the corresponding paths P_c, $\ldots$, $P_{c+\gamma-1-c \bmod \gamma}$ is the same as the parity of $\frac{c - c \bmod \gamma}{\gamma}$; and for the $(c \bmod \gamma)$ vertices $v_{c+\gamma-c \bmod \gamma} \in T^0$, $\ldots$, $v_{c+\gamma-1} \in T^{c \bmod \gamma - 1}$ the parity of the corresponding paths $P_{c+\gamma-1-c \bmod \gamma}$, $\ldots$, $P_{c+\gamma-1}$ is the same as the parity of $\frac{c+\gamma-c \bmod \gamma}{\gamma}$. Thus, the parity of the number of edges in $\mathcal{C}$ is the same as the parity of $\gamma + (\gamma - c \bmod \gamma)\frac{c - c \bmod \gamma}{\gamma} + (c \bmod \gamma)\frac{c+\gamma-c \bmod \gamma}{\gamma} = \gamma + c$.

Recalling that $T_n(a,b,c)$ is bipartite, the quantity $\gamma + c$ is even, that is to say, γ and c have the same parity. Recalling that $\gamma = \gcd(a,b)$ and $\gcd(a,b,c) = 1$, γ and c are both odd. The conditions γ odd and $\frac{a}{\gamma}, \frac{b}{\gamma}$ odd imply a,b odd, and prove the theorem. □

An example is drawn in Fig. 3

3.3 $T_n(a,b,c)$'s with $\gcd(a,b,c) = 1$ and $c + \gcd(a,b) \leq n \leq a + b - \gcd(a,b)$

All the remaining cases are the Toeplitz graphs $T_n(a,b,c)$ with $\gcd(a,b,c) = 1$ verifying $c + \gcd(a,b) \leq n \leq a + b - \gcd(a,b)$, which we now focus on.

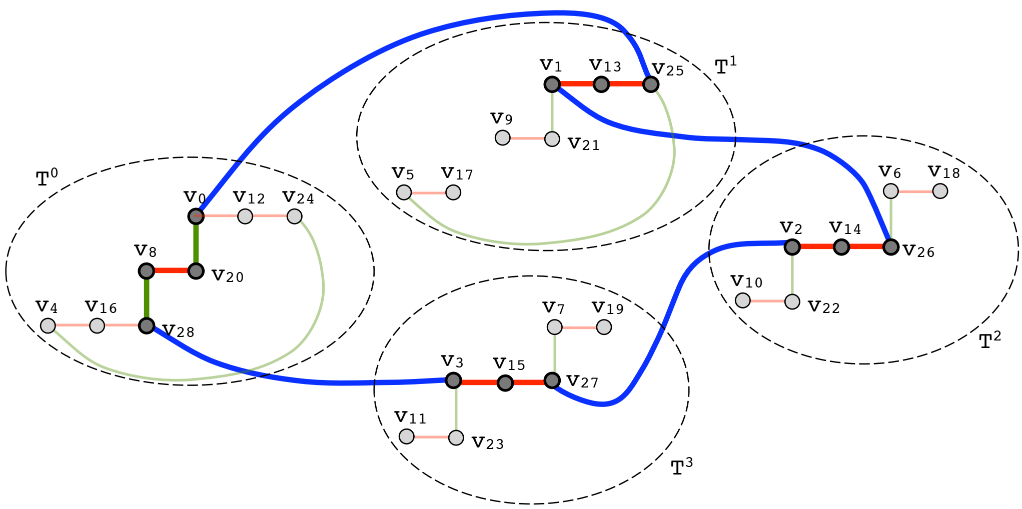


Figure 3: The Toeplitz graph $T_{29}(12, 20, 25)$; in bold, the edges belonging to the odd cycle C_{13}.

We define $E_n(a, b, c) = [e_{i,j}]$ (see Fig. 4) as the matrix whose rows and columns are indexed from 0 to $a - 1$, and from b to $n - 1$, respectively, where $e_{i,j} = q\text{A}$ if $|i - j| = qa$, for some integer q, and $e_{i,j} = \text{B}$ (C, resp.) if $|i - j| = b$ ($|i - j| = c$, resp.). If b (c, resp.) is a multiple of a, that is $b = pa$ ($c = pa$, resp.), we set $e_{i,j} =$ "pA,B" ($e_{i,j} =$ "pA,C", resp.). In the sequel, w.l.o.g., we shall assume that p is odd: in fact, by Theorem 3.3 we can rule out all the cases where b or c is an even multiple of a. The *multiplicity* of $e_{i,j}$ is the number of edges it represents (when $e_{i,j} =$ "pA,B" or $e_{i,j} =$ "pA,C", fix the multiplicity either to p or 1, which have the same parity, by definition). In what follows, by *non-empty* we mean "containing a pA, a B, a C, or a $*$".

$E_n(a, b, c)$ is a diagonal matrix (see Fig. 4). Precisely, in E we find (not necessarily distinct): the leftmost A-diagonal of elements qA, (the A^{left}-*diagonal*); a diagonal of B-elements (the B-*diagonal*); a diagonal of C-elements (the C-*diagonal*); and the rightmost diagonal of elements $(q + 1)$A (the A^{right}-*diagonal*), if and only if $n \geq a(1 + \lfloor \frac{b}{a} \rfloor) + 1$; all the remaining diagonals are empty. By construction, it is the case that $q = \lfloor \frac{b}{a} \rfloor$. Depending on the parity of q, the leftmost A-diagonal is an *even-diagonal* or an *odd-diagonal*; if both A-diagonals are present, either one is an even-diagonal and the other one is an odd-diagonal.

We distinguish two types of matrices: if $\lfloor \frac{b}{a} \rfloor = \lfloor \frac{c}{a} \rfloor$ then $E_n(a, b, c)$ and $T_n(a, b, c)$ are of *type 1*; if $\lfloor \frac{b}{a} \rfloor < \lfloor \frac{c}{a} \rfloor$ then $E_n(a, b, c)$ and $T_n(a, b, c)$ are of *type 2*. In a type 1 matrix, from left to right, we find the A^{left}-diagonal, the B-diagonal, the C-diagonal, and, if $n \geq a(1 + \lfloor \frac{b}{a} \rfloor) + 1$, the A^{right}-diagonal (see $E_{17}(7, 11, 12)$ in Fig. 4(a)); in a type 2 matrix, from left to right, we find the A^{left}-diagonal, the B-diagonal, the A^{right}-diagonal, and the C-diagonal (see $E_{25}(7, 19, 24)$ in the Fig. 4 (b)).

Since the subgraph induced by all the a-edges is a collection of disjoint a-paths, a cycle in the graph must necessarily contain some b- or c-edge (this explains why matrix $E_n(a, b, c)$ is defined on columns b to $n - 1$). The following lemma holds:

	11	12	13	14	15	16
0	B	C		2A		
1		B	C		2A	
2			B	C		2A
3				B	C	
4	1A				B	C
5		1A				B
6			1A			

(a)

	19	20	21	22	23	24
0	B		3A			C
1		B		3A		
2			B		3A	
3				B		3A
4					B	
5	2A					B
6		2A				

(b)

Figure 4: (a) The type 1 matrix $E_{17}(7, 11, 12)$ and (b) the type 2 matrix $E_{25}(7, 19, 24)$.

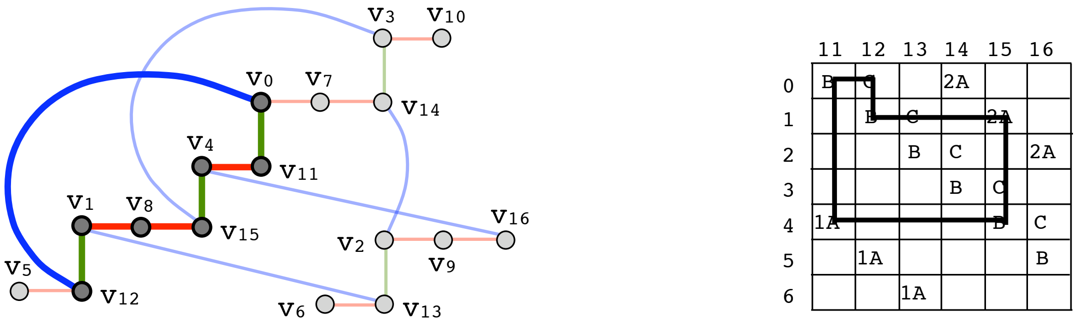
Lemma 3.8. *Consider a $T_n(a, b, c)$. If $n \leq a + b - \gcd(a, b)$, then every b-edge and every c-edge connects the smallest-indexed vertex of an a-path with the largest-indexed vertex of a different a-path, and every vertex has at most one b-edge and at most one c-edge incident to it.*

Proof. As for the first claim, focus on the b-edges. Let $\gamma = \gcd(a, b)$. We distinguish two cases: $\gamma = 1$ and $\gamma > 1$. If $\gamma = 1$, consider the graph $T_n(a, b)$ and an arbitrary b-edge (v_x, v_{x+b}). One has $0 \leq x \leq n - 1 - b = a - 2$, otherwise v_x, v_{x+b} are not in $T_n(a, b)$. The inequality $x \leq a - 2$ shows that $x \bmod a = x$, thus vertex v_x is the smallest-indexed vertex of a-path A_x. On the other hand, no a-edge (v_{x+b}, v_{x+b+a}) exists in $T_n(a, b)$, as $x + b + a > n - 1$, for all $x \geq 0$, showing that v_{x+b} is the largest indexed vertex of $A_{(x+b) \bmod a}$. Notice that A_x and $A_{(x+b) \bmod a}$ are not the same path, since $\gcd(a, b) = 1$. If $\gamma > 1$, consider graphs T^p, for $p = 0, \ldots, \gamma - 1$. The assumption $n \leq a + b - \gamma$ implies $n^0 = \lceil \frac{n}{\gamma} \rceil \leq \frac{a}{\gamma} + \frac{b}{\gamma} - 1$. This proves that each T^p is isomorphic to a subgraph of $T_{\frac{a}{\gamma}+\frac{b}{\gamma}-1}(\frac{a}{\gamma}, \frac{b}{\gamma})$, which the proof above applies to, and the first claim follows for the b-edges. The same result holds for c-edges, as $n \leq a + b - \gamma \leq a + b - 1 \leq a + c - 1$.
As for the second claim, consider the b-edges (*a fortiori* the same result holds for c-edges). An arbitrary vertex v_x has two b-edges incident to it if and only if vertices v_{x-b} and v_{x+b} are both in the graph. That is to say, if both $x \geq b$ and $x + b \leq n - 1$ hold. By definition $b \geq a + 1$, and we get $n \geq a + b + 2$, which contradicts the assumption. □

Two are the consequences of the lemma above. One is that every cycle $\mathcal{C}$ consists of maximal sequences of consecutive a-edges, separated by non-empty paths alternating b- and c-edges; the other one is that every maximal sequence of consecutive a-edges is an a-path. Hence $\mathcal{C}$ can be written as $A_{i_1}, F_{i_2}, A_{i_3}, F_{i_4}, \ldots, A_{i_p}$, where each A_{i_q} is an a-path, and the F_{i_q}'s are the non-empty paths alternating a b- and a c-edge. These two facts show that on matrix $E_n(a, b, c)$ all the cycles of $T_n(a, b, c)$ are easily represented, as we now explain.

A cycle $\mathcal{C}$ of $T_n(a, b, c)$ can be mapped onto a $\mathcal{C}$losed $\mathcal{M}$anhattan $\mathcal{C}$urve ($\mathcal{CMC}$, for short) with corners in the non-empty elements of $E_n(a, b, c)$, and viceversa. $\mathcal{CMC}$ has a corner in an element $e_{i,h}$ if and only if all the edges the element represents, belong to $\mathcal{C}$

(as an example, see Fig. 5. The number of edges in $\mathcal{C}$ can be easily computed by summing up the multiplicity of the elements which are corners of the correponding $\mathcal{CMC}$. In the example of Fig. 5, $\mathcal{C}$ is defined on $2+1+1+1+1+1 = 7$ edges.



(a) (b)

Figure 5: (a) The Toeplitz graph $T_{17}(7, 11, 12)$ (in bold, the edges belonging to C_7) and (b) the matrix $E_{17}(7, 11, 12)$ with the $\mathcal{CMC}$ corresponding to the odd cycle.

In order to prove the bipartiteness of $T_n(a, b, c)$, we are interested in the parity of the number of edges in a cycle, and not in the number itself. An easy way to compute it, is to consider the vertical segments of a $\mathcal{CMC}$. We notice that the lengths of the vertical segments in a $\mathcal{CMC}$ are limited to few quantities, namely the distances among two consecutive non-empty diagonals and the sum of suitable pairs of them.

In a type 1 matrix we have the following types of vertical segments:

- s_1, of length $k_1 = b \bmod a$, which connects an element of the A^{left}-diagonal with an element of the B-diagonal;
- s_2, of length $k_2 = c - b$, which connects an element of the B- with an element of the C-diagonal;
- if $n \geq a(1 + \lfloor \frac{b}{a} \rfloor) + 1$, we also have s_3, of length $k_3 = a - c \bmod a$, which connects an element of the C-diagonal with an element of the A^{right}-diagonal;

where, clearly, $k_1 + k_2 + k_3 = a$ (in $T_{17}(7, 11, 12)$ one has $(k_1, k_2, k_3) = (4, 1, 2)$).

In a type 2 matrix we identify these types of vertical segments:

- s_1, of length $k_1 = b \bmod a$, which connects an element of the A^{left}-diagonal with an element of the B-diagonal;
- s_2, of length $k_2 = a - b \bmod a = a \lceil \frac{b}{a} \rceil - b$, which connects an element of the B- with an element of the A^{right}-diagonal;
- s_3, of length $k_3 = c \bmod a$, which connects an element of the C-diagonal with an element of the A^{right}-diagonal;

where, clearly, $k_1 + k_2 = a$ (in $T_{25}(7, 19, 24)$ one has $(k_1, k_2, k_3) = (5, 2, 3)$).

In both matrices, the sum of the multiplicity of the endpoints of each vertical segment is an odd quantity if and only if one endpoint is in the even A-diagonal. For this reason, we observe that

Observation 3.9. *An odd cycle in $T_n(a,b,c)$ is represented by a $\mathcal{CMC}$ of $E_n(a,b,c)$ with an odd number of vertical segments incident to elements of the even* A*-diagonal.*

In some cases, $\mathcal{CMC}$'s are easily identified in the matrix, and the observation above allows for immediately recognizing the odd ones. However, we first discuss a general method which allows to recognize an arbitrary bipartite/non-bipartite Toeplitz graph. Later on we shall present some easy cases. We remark that the method we are going to present applies to *all* the graphs studied in the paper, that is, to all $T_n(a,b)$'s, and to all $T_n(a,b,c)$'s.

Since the $\mathcal{CMC}$ can only make use of the vertical segments s_1, s_2, s_3 of length k_1, k_2, k_3, respectively, or of a sum of them, the following diophantine equation can be written:

$$k_1y_1 + k_2y_2 + k_3y_3 = 0 \tag{1}$$

where $|y_i|$, $i = 1,\dots,3$ represents the number of vertical segments of type s_i in the $\mathcal{CMC}$ ($y_i \geq 0$ if the segments are taken downwards, and $y_i \leq 0$ if taken upwards).
A solution (y_1, y_2, y_3) to this equation corresponds to a $\mathcal{CMC}$ in $E_n(a,b,c)$ if it verifies some constraints. First of all we have to ensure that the number of vertical segments available in $E_n(a,b,c)$ is not exceeded. In addition, since we are looking for odd cycles of $T_n(a,b,c)$, we require that a solution contains an *odd* number of vertical segments incident to the *even* A-diagonal (see Observation 3.9).

In order to compute the maximum number of available vertical segments, we have to identify those elements of $E_n(a,b,c)$, if any, which will never be corners of a $\mathcal{CMC}$. Consider a row or a column with one non-empty element only, say $e_{i,j}$. No *closed* curve can exist with a corner in $e_{i,j}$ (unless we allow the curve to go back where it was coming from, obtaining an even subcycle, which is clearly useless both from the point of view of a $\mathcal{CMC}$ and of its parity) (see $e_{4,23}$ in $E_{25}(7,19,24)$). Each of these elements, in fact, represents a *dangling* path, that is, an a-path, a b- or a c-edge with an endpoint of degree 1. For this reason, we can delete it from the matrix. Such deletion may cause other elements to be deleted for the same reason (in $E_{25}(7,19,24)$, after $e_{4,23}$, we delete $e_{2,23}$, $e_{2,21}$, and $e_{0,21}$). The process can be repeated until no row or column of $E_n(a,b,c)$ contains a unique element (as a consequence, in $E_{25}(7,19,24)$ we also delete $e_{6,20}$, $e_{1,20}$, $e_{1,22}$, $e_{3,22}$, and $e_{3,24}$). The resulting matrix will be denoted by $\widehat{E}_n(a,b,c)$ (see Fig. 3.3).

The number of available vertical segments can now be computed. The number of available vertical segments of type s_1 is the number U_1 of columns of $\widehat{E}_n(a,b,c)$ containing both the endpoints of a segment s_1, namely a B and an A^{left} element; similarly for segments of type s_2 and s_3 in either type of matrix (these upper bounds will be called U_2 and U_3, respectively).

Actually, in a type 2 matrix things are slightly more complicated. Consider a column, say j, of a type 2 matrix $\widehat{E}_n(a,b,c)$ having exactly a B-element and a C-element (hence $j \geq c$). This column represents vertex v_j which is the endpoint of a b-edge, of a c-edge, and is the largest-indexed vertex of a dangling a-path: this a-path can not belong to any cycle, but the b- and c-edges incident to v_j may. Precisely, column j tells us that the three vertices v_{j-c}, v_j, v_{j-b} might belong to a cycle C if and only if they are consecutive in C, as well as the corresponding two edges (v_{j-c}, v_j) and (v_j, v_{j-b}). By definition of s_2 and s_3, any such column j allows for increasing U_2, U_3 by one unit, and the fact that the two edges (v_{j-c}, v_j) and (v_j, v_{j-b}) be consecutive corresponds to constrain y_2, y_3 to have the same sign, that is to say $y_2 y_3 \geq 0$ (in fact, the B and C elements of a column with only these two elements can be corners of a $\mathcal{CMC}$ if and only if the segments s_2 and s_3 are taken either both upwards or both downwards). We shall denote by w the number of columns of $\widehat{E}_n(a,b,c)$ having exactly a B and a C elements. It is convenient to keep this information in the matrix: we do that by placing a "$*$" in correspondence of the deleted A^{right} element in every column of $\widehat{E}_n(a,b,c)$ having exactly a B and a C elements (see the"$*$" in $e_{3,24}$ of $\widehat{E}_{25}(7,19,24)$). Clearly, w is exactly the number of "$*$" in $\widehat{E}_n(a,b,c)$.

	11	12	13	14	15	16
0	B	C		2A		
1		B	C		2A	
2			B	C		2A
3				B	C	
4	1A				B	C
5		1A				B
6						

(a)

	19	20	21	22	23	24
0	B					C
1						
2						
3						*
4						
5	2A					B
6						

(b)

Figure 6: (a) The matrix $\widehat{E}_{17}(7,11,12)$ and (b) the matrix $\widehat{E}_{25}(7,19,24)$.

As a result, for a type 1 matrix we get the set of constraints in Fig. 7 (a); for a type 2 matrix we have to use the set of constraints in Fig. 7 (b) or, if $w > 0$, the set of constraints in Fig. 7 (c) (where the bounds on $|y_2|$ and $|y_3|$ are looser, but y_2 and y_3 are requested to have the same sign).

$$\begin{array}{c} |y_1| \leq U_1 \\ |y_2| \leq U_2 \\ |y_3| \leq U_3 \\ y_1 \text{ odd if } \left\lfloor \frac{b}{a} \right\rfloor \text{ even} \\ y_3 \text{ odd if } \left\lfloor \frac{b}{a} \right\rfloor \text{ odd} \\ \\ (a) \end{array} \qquad \begin{array}{c} |y_1| \leq U_1 \\ |y_2| \leq U_2 \\ |y_3| \leq U_3 \\ y_1 \text{ odd if } \left\lfloor \frac{b}{a} \right\rfloor \text{ even} \\ y_2 + y_3 \text{ odd if } \left\lfloor \frac{b}{a} \right\rfloor \text{ odd} \\ \\ (b) \end{array} \qquad \begin{array}{c} |y_1| \leq U_1 \\ |y_2| \leq U_2 + w \\ |y_3| \leq U_3 + w \\ y_1 \text{ odd if } \left\lfloor \frac{b}{a} \right\rfloor \text{ even} \\ y_2 + y_3 \text{ odd if } \left\lfloor \frac{b}{a} \right\rfloor \text{ odd} \\ y_2 y_3 \geq 0 \\ (c) \end{array}$$

Figure 7: The sets of constraints for (a) a type 1 matrix, (b) a type 2 matrix with $w = 0$, and (c) a type 2 matrix with $w > 0$.

Let us define *constrained* a solution (y_1, y_2, y_3) of the diophantine equation (1) which verifies the constraints above. We can prove that:

Lemma 3.10. *Consider a* $T_n(a,b,c)$ *with* $\gcd(a,b,c) = 1$ *and* $c + \gcd(a,b) \leq n \leq a+b-\gcd(a,b)$*. Every constrained solution* (y_1, y_2, y_3) *of the diophantine equation* (1) *corresponds to an odd cycle of* $T_n(a,b,c)$*, and viceversa.*

Proof. First, we show how to derive a constrained solution from an odd cycle $\mathcal{C}$ of a $T_n(a,b,c)$ undergoing the hypothesis. Map $\mathcal{C}$ onto the corresponding $\mathcal{CMC}$, as described above. Arbitrarily define a direction on it, and count the number of vertical segments of type s_1, s_2 and s_3 which $\mathcal{CMC}$ is made of. Precisely, set the counters c_1, c_2, c_3 to 0 and start following $\mathcal{CMC}$: if the current segment is vertical and is of type j, increase (decrease, resp.) c_j by one unit if the segment is taken downward (upwards, resp.). By construction (c_1, c_2, c_3) is a solution to (1), and verifies all the required constraints.
We now show how to derive an odd cycle from an arbitrary constrained solution (y_1, y_2, y_3) of the diophantine equation (1).

Consider a type 1 matrix (similar reasonings hold for a type 2 matrix). Let $\underline{s}_j$, $\overline{s}_j$, for $j = 1,2,3$, represent a vertical segment of type j directed downwards and upwards, respectively. Call *head* (*tail*, resp.) the element of the matrix where the segment is directed to (originates from). Define Σ_M as a multiset of $|y_1| + |y_2| + |y_3|$ segments. Precisely, Σ_M contains $|y_1|$ occurences of $\underline{s}_1$ if $y_1 \geq 0$ or of $\overline{s}_1$ if $y_1 \leq 0$, $|y_2|$ occurences of $\underline{s}_2$ if $y_2 \geq 0$ or of $\overline{s}_2$ if $y_2 \leq 0$, and $|y_3|$ occurences of $\underline{s}_3$ if $y_3 \geq 0$ or of $\overline{s}_3$ if $y_3 \leq 0$. We also define the set Σ_F of the segments one might need to construct a $\mathcal{CMC}$ in addition to the mandatory segments in Σ_M. Precisely, in Σ_F we find $\lfloor \frac{U_1 - |y_1|}{2} \rfloor$ copies of the pair $\underline{s}_1, \overline{s}_1$, $\lfloor \frac{U_2 - |y_2|}{2} \rfloor$ copies of the pair $\underline{s}_2, \overline{s}_2$, and $\lfloor \frac{U_3 - |y_3|}{2} \rfloor$ copies of the pair $\underline{s}_3, \overline{s}_3$.

For each non-empty row r of $\widehat{E}_n(a,b,c)$, define the *segment set* S_r as follows. For $p = 1,2,3$, segment $\underline{s}_p$ ($\overline{s}_p$, resp.) belongs to S_r iff $e_{r,j}$ and $e_{r+k_p,j}$ ($e_{r-k_p,j}$, resp.) are both non-empty (in $E_{25}(7,19,24)$, for example, the only segment sets are $S_1 = \{\underline{s}_1, \underline{s}_3\}$, $S_3 = \{\underline{s}_2, \overline{s}_3\}$, and $S_5 = \{\overline{s}_1, \overline{s}_2\}$). Notice that, by definition, given the row index r and a segment $s \in S_r$, the row and column indices of the tail and head of s are fixed: precisely, for a type 1 matrix, if $s = \underline{s}_1$, then its tail and head are found in $e_{r,b+r}$ and $e_{r+k_1,b+r+k_1}$, resp.; if $s = \overline{s}_1$, then its tail and head are found in $e_{r,b+r}$ and $e_{r-k_1,b+r-k_1}$, resp.; if $s = \underline{s}_2$, then its tail and head are found in $e_{r,c+r}$ and $e_{r+k_2,c+r+k_2}$, resp.; if $s = \overline{s}_2$, then its tail and head are found in $e_{r,c+r}$ and $e_{r-k_2,c+r-k_2}$, resp.; if $s = \underline{s}_3$, then its tail and head are found in $e_{r,\lceil \frac{c}{a} \rceil a+r}$ and $e_{r+k_3,\lceil \frac{c}{a} \rceil a+r+k_3}$, resp.; and if $s = \overline{s}_3$, then its tail and head are found in $e_{r,\lceil \frac{c}{a} \rceil a+r}$ and $e_{r-k_3,\lceil \frac{c}{a} \rceil a+r-k_3}$, resp..

In order to define a (directed) $\mathcal{CMC}$ corresponding to the given solution (y_1, y_2, y_3), apply the following algorithm. Let S_ρ be a segment set having non-empty intersection with Σ_M, and choose a segment $\sigma \in \Sigma_M \cap S_\rho$ (if such a segment set does not exist, consider solution $(-y_1, -y_2, -y_3)$ and start again): start constructing $\mathcal{CMC}$ (vertically) connecting the tail of σ to its head, fix the tail of $\mathcal{CMC}$ in the tail of σ, fix the current head of $\mathcal{CMC}$ in the head of σ, and let ρ denote its row index. Remove σ from Σ_M. While $\Sigma_M \neq \emptyset$ do: choose a segment $\sigma \in \Sigma_M \cap S_\rho$, if any; if such a segment does not exist, take a segment

$\sigma \in \Sigma_F \cap S_\rho$, insert into Σ_M the segment σ and the segment σ' paired with σ, and remove the pair σ, σ' from Σ_F; horizontally connect the current head of $\mathcal{CMC}$ to the tail of σ, vertically connect the tail of σ to its head, update the current head of $\mathcal{CMC}$ in the head of σ, and update ρ to its row index; remove σ from Σ. Finally, connect the current head of $\mathcal{CMC}$ to its tail, and derive from $\mathcal{CMC}$ the corresponding odd cycle. Since 0 is an integral linear combination of the constrained solution (y_1, y_2, y_3), the thesis follows. □

While deriving a $\mathcal{CMC}$ from a given constrained solution, as in the proof above, it may happen that the row index of the current head of $\mathcal{CMC}$ equals the row index of its tail. In this case, we have found a sub-cycle $\mathcal{C}'$ of the odd cycle corresponding to the computed $\mathcal{CMC}$. With a backwards reasoning, the solution (y'_1, y'_2, y'_3) corresponding to $\mathcal{C}'$ can be constructed by counting the number of segments, and their direction. If (y'_1, y'_2, y'_3) happens to be constrained, then $\mathcal{C}'$ is an odd cycle, otherwise it is even.

A consequence of Lemma 3.10 is the following:

Theorem 3.11. *$T_n(a, b, c)$ with $\gcd(a, b, c) = 1$ and $c + \gcd(a, b) \leq n \leq a + b - \gcd(a, b)$ is bipartite if and only if the diophantine equation (1) has no constrained solution.*

We now briefly discuss a method to solve the diophantine equation (1), which always admits a solution as $\gcd(k_1, k_2, k_3)$ divides 0 [4]. If equation (1) admits a solution, then it admits an infinite number of them [4]. However, since we are interested in constrained solutions, only, we suggest to proceed as follows [1].

Consider a type 1 matrix, first, and let $\lfloor \frac{b}{a} \rfloor$ be odd. In this case, a constrained solution has to verify the first three constraints and the fifth one of the set in Fig. 7 (a). Consider the diophantine equation $k_1 y_1 + k_2 y_2 = -k_3 i$ for $i \in \{r \in \mathbb{Z}, 0 \leq r \leq U_3, r \text{ odd}\}$: it has a solution if and only if $-k_3 i$ is a multiple of $\gcd(k_1, k_2)$, say $-k_3 i = \delta^i \gcd(k_1, k_2)$ (notice that $\delta^i < 0$). Use Euclid's algorithm to find a solution (p_0, q_0) to the diophantine equation $k_1 p + k_2 q = \gcd(k_1, k_2)$. Let $\tilde{y}_1^i = \delta^i p_0$ and $\tilde{y}_2^i = \delta^i q_0$: one has that $(\tilde{y}_1^i, \tilde{y}_2^i)$ is a particular solution to the diophantine equation $k_1 y_1 + k_2 y_2 = -k_3 i$. Let $y_1^i = \tilde{y}_1^i + \frac{k_2}{\gcd(k_1,k_2)}\gamma^i$ and $y_2^i = \tilde{y}_2^i - \frac{k_1}{\gcd(k_1,k_2)}\gamma^i$, with $\gamma^i \in \mathbb{Z}$: by varying γ^i in all possible ways we generate all the solutions (y_1^i, y_2^i) to $k_1 y_1 + k_2 y_2 = -k_3 i$, that is to say (y_1^i, y_2^i, i) is a solution to $k_1 y_1 + k_2 y_2 + k_3 y_3 = 0$ and verifies the third and fifth constraints of Fig. 7 (a). If among these solutions there are some which verify also the first and the second constraint of Fig. 7 (a), then the given graph $T_n(a, b, c)$ contains odd cyles, that is to say, it is non-bipartite. If no solution among these is constrained, or if no $i \in \{r \in \mathbb{Z}, 0 \leq r \leq U_3, r \text{ odd}\}$ exists such that $-k_3 i$ is a multiple of $\gcd(k_1, k_2)$, then the given graph is bipartite. We remark that if $\gcd(k_1, k_2) > 1$ and $\gcd(k_1, k_2)$ does not divide k_3, then it must necessarily divide i: thus, one can limit the search to those i which are multiples of $\gcd(k_1, k_2)$, and conclude stating that if, in addition, $\gcd(k_1, k_2) > U_3$, then the given graph is bipartite.

When the matrix is of type 1 and $\lfloor \frac{b}{a} \rfloor$ is even, we exchange the role of y_3 and y_1 and solve equation $k_2y_2 + k_3y_3 = -k_1i$ for $i \in \{r \in \mathbb{Z}, 0 \leq r \leq U_1, r \text{ odd}\}$, as required by the corresponding constraint.

When the matrix is of type 2 and $\lfloor \frac{b}{a} \rfloor$ is odd, we have several possibilities. One of them is to start solving the equation $k_1y_1 + k_2y_2 = -k_3j$ for $j \in \{0, \ldots, U_3\}$ (the constrained solution in this case must verify $|y_2| \leq U_2$ with y_2 and y_3 of opposite parity, see Fig. 7 (b) or (c), depending on w); if no constrained solution is found, and if $w \geq 1$, we can try solving the equation $k_1y_1+k_2y_2 = -k_3j$ for $j \in \{U_3+1, \ldots, U_3+w\}$ (in this case, recalling that y_2 and y_3 must have the same sign, the range for y_2 becomes $0 \leq y_2 \leq U_2 + w$, see Fig. 7 (c)).

Finally, when the matrix is of type 2 and $\lfloor \frac{b}{a} \rfloor$ is even, we solve equation $k_2y_2 + k_3y_3 = -k_1i$ for $i \in \{r \in \mathbb{Z}, 0 \leq r \leq U_1, r \text{ odd}\}$, taking care y_2 and y_3 to satisfy either one of the two sets of constraints in Fig. 7 (b), (c).

As for the computational complexity of the above algorithm, we recall that Euclid's algorithm requires $O(\log^2(n-b))$ elementary operations [5] (in fact, $U_i \leq n-b$ for $i \in \{1, 2, 3\}$ and, if $k_i > n-b$ for some $i \in \{1, 2, 3\}$, then $U_i = 0$), and that if our goal is that of deciding the bipartiteness of $T_n(a, b, c)$, we can stop searching for a solution to (1) as soon as we find a constrained one. For a type 1 matrix with $\lfloor \frac{b}{a} \rfloor$ odd, let us solve w.r.t. y_3. Recalling that

$$y_1^i = \tilde{y}_1^i + \tfrac{k_2}{\gcd(k_1,k_2)}\gamma^i, \qquad |y_1^i| \leq U_1, \qquad y_2^i = \tilde{y}_2^i - \tfrac{k_1}{\gcd(k_1,k_2)}\gamma^i, \qquad |y_2^i| \leq U_2,$$

we get that the feasible γ^i's, if any, are those satisfying both the following constraints

$$\left\lceil -(U_1 + \tilde{y}_1^i)\tfrac{\gcd(k_1,k_2)}{k_2} \right\rceil \leq \gamma^i \leq \left\lfloor (U_1 - \tilde{y}_1^i)\tfrac{\gcd(k_1,k_2)}{k_2} \right\rfloor$$

and

$$\left\lceil (\tilde{y}_2^i - U_2)\tfrac{\gcd(k_1,k_2)}{k_1} \right\rceil \leq \gamma^i \leq \left\lfloor (\tilde{y}_2^i + U_2)\tfrac{\gcd(k_1,k_2)}{k_1} \right\rfloor.$$

If such a γ^i does not exist, the equation $k_1y_1 + k_2y_2 = -k_3i$ admits no solution. If such a γ^i does exist, then we have found a solution to $k_1y_1 + k_2y_2 = -k_3i$: if it is constrained, then Theorem 3.11 applies. In the last case we have to check at most $O((n-b)^2)$ solutions to possibly find a constrained one, giving an overall computational complexity of $O((n-b)^2 + \log^2(n-b))$.

As already mentioned, there are some easy cases where $\mathcal{CMC}$ are easily identified in the matrix $E_n(a, b, c)$, and Observation 3.9 allows for immediately recognizing the odd ones (a sketch of the proof of some of them, only, is given). In other words, we now show that in some cases we know *a-priori* that equation (1) has a constrained solution.

For matrices of either type, we can prove that:

Lemma 3.12. *Consider a* $T_n(a, b, c)$ *verifying* $\gcd(a, b, c) = 1$ *and* $c + \gcd(a, b) \leq n \leq a + b - \gcd(a, b)$*. If* $k_1 > n - b$ *and* $\lfloor \frac{b}{a} \rfloor$ *is even, then* $T_n(a, b, c)$ *is bipartite.*

Proof. The assumption $k_1 > n-b$ implies that the elements of the A^{left}-diagonal are alone in a row (see Figure 8 (a)), thus no $\mathcal{CMC}$ exists having a corner in one of these elements. The assumption $\lfloor \frac{b}{a} \rfloor$ even tells us that the only even diagonal is exactly the A^{left}-diagonal. The thesis follows from Observation 3.9. □

The following lemma applies to type 1 matrices with no A^{right}-diagonal:

Lemma 3.13. *Consider a* $T_n(a,b,c)$ *verifying* $\gcd(a,b,c)=1$, $\lfloor \frac{b}{a} \rfloor = \lfloor \frac{c}{a} \rfloor$, *and* $c+\gcd(a,b) \leq n \leq \min\left\{a+b-\gcd(a,b); a(1+\lfloor \frac{b}{a} \rfloor)\right\}$.

- *If* $\lfloor \frac{b}{a} \rfloor$ *is odd, then* $T_n(a,b,c)$ *is bipartite.*
- *If* k_2 *is an odd multiple of* k_1 *and* $\lfloor \frac{b}{a} \rfloor$ *is even, then* $T_n(a,b,c)$ *is non-bipartite.*
- *If* $k_1 \leq n-b-1$ *is a multiple of* k_2 *and* $\lfloor \frac{b}{a} \rfloor$ *is even, then* $T_n(a,b,c)$ *is non-bipartite.*

Proof. We shall give a sketch of the proof for the third case only. A possible schema for this case is depicted in Fig. 8 (b), where $k_1 = 3k_2$. Consider the $\mathcal{CMC}$ drawn in the figure and represented by elements $e_{0,b}$, $e_{0,c}$, $e_{k_2,c}$, $e_{k_2,c+k_2}$, $e_{2k_2,c+k_2}$, $e_{2k_2,c+2k_2}$, $e_{3k_2,c+2k_2}$, $e_{k_1,b}$. We claim that it corresponds to an odd cycle. Since $e_{k_1,b}$ is the only element of the $\mathcal{CMC}$ belonging to the (unique) even diagonal of $E_n(a,b,c)$, the thesis follows. □

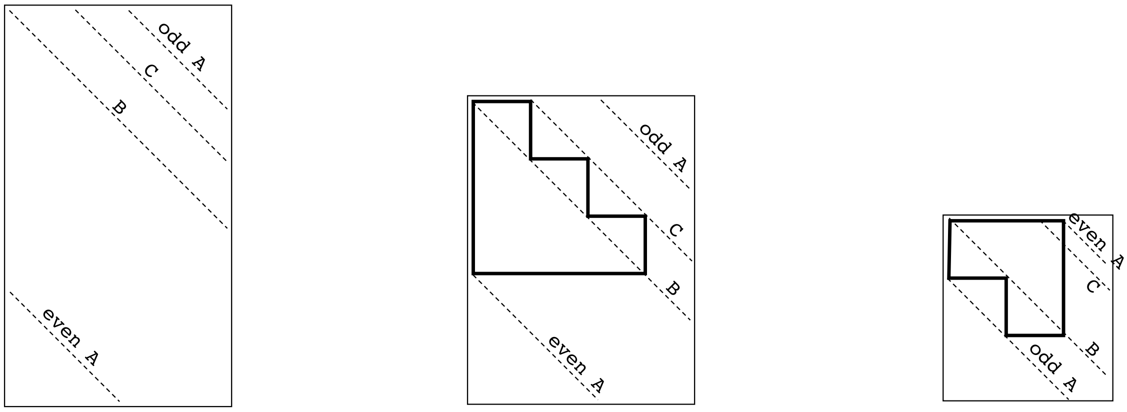
As an example of the third case of the lemma above, as required by Theorem 3.11, a constrained solution to the corresponding diophantine equation $(3k_2)y_1 + k_2y_2 + k_3y_3 = 0$ is (y_1, y_2, y_3) with y_1 odd. It is trivial to see that $(y_1, y_2, y_3) = (1, -3, 0)$ is a solution corresponding to the $\mathcal{CMC}$ drawn in Fig. 8 (b). $T_{58}(11, 50, 52)$ and $T_{19}(6, 15, 16)$ are examples of Toeplitz graphs fitting into the structure of Fig. 8 (b).

Corollary 3.14. *Consider a* $T_n(a,b,c)$ *verifying* $\gcd(a,b,c)=1$, $\lfloor \frac{b}{a} \rfloor = \lfloor \frac{c}{a} \rfloor$, *and* $c+\gcd(a,b) \leq n \leq \min\left\{a+b-\gcd(a,b); a(1+\lfloor \frac{b}{a} \rfloor)\right\}$. *If* $k_1 = k_2$ *and* $\lfloor \frac{b}{a} \rfloor$ *is even, then* $T_n(a,b,c)$ *is non-bipartite.*

For type 1 matrices having the A^{right}-diagonal, we can prove that:

Lemma 3.15. *Consider a* $T_n(a,b,c)$ *verifying* $\gcd(a,b,c)=1$, $\lfloor \frac{b}{a} \rfloor = \lfloor \frac{c}{a} \rfloor$, *and* $\max\left\{c+\gcd(a,b); a(1+\lfloor \frac{b}{a} \rfloor)+1\right\} \leq n \leq a+b-\gcd(a,b)$.

- *If* $k_1 \leq n-b-1$ *is a multiple of* k_2+k_3 *and* $\lfloor \frac{b}{a} \rfloor$ *is even, then* $T_n(a,b,c)$ *is non-bipartite.*
- *If* $k_1 \leq n-b-1$ *is an odd multiple of* k_2+k_3 *and* $\lfloor \frac{b}{a} \rfloor$ *is odd, then* $T_n(a,b,c)$ *is non-bipartite.*
- *If* k_2+k_3 *is a multiple of* k_1 *and* $\lfloor \frac{b}{a} \rfloor$ *is odd, then* $T_n(a,b,c)$ *is non-bipartite.*
- *If* k_2+k_3 *is an odd multiple of* k_1 *and* $\lfloor \frac{b}{a} \rfloor$ *is even, then* $T_n(a,b,c)$ *is non-bipartite.*



(a) (b) (c)

Figure 8: The schema (a) for Lemma 3.12, and for the third case (b) of Lemma 3.13 and (c) of Lemma 3.15.

Proof. We shall give a sketch of the proof for the third case only. A possible schema for this case is depicted in Fig. 8 (c), where $k_2 + k_3 = 2k_1$. Consider the $\mathcal{CMC}$ drawn in the figure and represented by elements $e_{0,b}$, $e_{0,b+k_2+k_3}$, $e_{k_2+k_3,b+k_2+k_3}$, $e_{k_2+k_3,b+k_2+k_3-k_1}$, $e_{k_2+k_3-k_1,b+k_2+k_3-k_1}$, $e_{k_2+k_3-k_1,b+k_2+k_3-2k_1}$. We claim that it corresponds to an odd cycle. Since $e_{0,b+k_2+k_3}$ is the only element of the $\mathcal{CMC}$ belonging to the (unique) even diagonal of $E_n(a,b,c)$, the thesis follows. □

As an example of the third case of the theorem above, as required by Theorem 3.11, a constrained solution to the corresponding diophantine equation (1) is (y_1, y_2, y_3) with y_3 odd. It is trivial to see that $(y_1, y_2, y_3) = (2, -1, -1)$ is a solution corresponding to the $\mathcal{CMC}$ drawn in Fig. 8 (c). $T_{58}(11, 50, 52)$ and $T_{19}(6, 15, 16)$ are examples of Toeplitz graphs fitting into the structure of Fig. 8 (c).

Corollary 3.16. *Consider a* $T_n(a,b,c)$ *verifying* $\gcd(a,b,c) = 1$, $\lfloor \frac{b}{a} \rfloor = \lfloor \frac{c}{a} \rfloor$, *and* $\max\left\{c + \gcd(a,b); a(1 + \lfloor \frac{b}{a} \rfloor) + 1\right\} \le n \le a + b - \gcd(a,b)$. *If* $k_1 = k_2 + k_3$, *then* $T_n(a,b,c)$ *is non-bipartite.*

Finally, for type 2 matrices we have the following result:

Lemma 3.17. *Consider a* $T_n(a,b,c)$ *verifying* $\gcd(a,b,c) = 1$, $\lfloor \frac{b}{a} \rfloor < \lfloor \frac{c}{a} \rfloor$ *and* $c + \gcd(a,b) \le n \le a + b - \gcd(a,b)$.

- *If* $k_1 = k_2 + k_3$ *and* $\lfloor \frac{b}{a} \rfloor$ *is even, then* $T_n(a,b,c)$ *is non-bipartite.*
- *If* k_2 *is a multiple of* k_1 *and* $\lfloor \frac{b}{a} \rfloor$ *is odd, then* $T_n(a,b,c)$ *is non-bipartite.*
- *If* k_2 *is an odd multiple of* k_1 *and* $\lfloor \frac{b}{a} \rfloor$ *is even, then* $T_n(a,b,c)$ *is non-bipartite;*
- *If* $k_1 \le n - b - 1$ *is a multiple of* k_2 *and* $\lfloor \frac{b}{a} \rfloor$ *is even, then* $T_n(a,b,c)$ *is non-bipartite.*

- *If* $k_1 \leq n-b-1$ *is an odd multiple of* k_2 *and* $\lfloor \frac{b}{a} \rfloor$ *is odd, then* $T_n(a,b,c)$ *is non-bipartite.*
- *If* $k_2 \leq \frac{n-b-1}{2}$ *is an even multiple of* k_3 *and* $\lfloor \frac{c}{a} \rfloor$ *is even, then* $T_n(a,b,c)$ *is non-bipartite.*
- *If* k_3 *is an even multiple of* k_2 *and* $\lfloor \frac{c}{a} \rfloor$ *is even, then* $T_n(a,b,c)$ *is non-bipartite.*

Corollary 3.18. *Consider a* $T_n(a,b,c)$ *verifying* $\gcd(a,b,c)=1$, $\lfloor \frac{b}{a} \rfloor < \lfloor \frac{c}{a} \rfloor$, *and* $c+\gcd(a,b) \leq n \leq a+b-\gcd(a,b)$, *and. If* $k_1 = k_2$, *then* $T_n(a,b,c)$ *is non-bipartite.*

4 Generalizations

In this section we show that some of the previous results can be extended to Toeplitz graphs with $k \geq 4$ entries, to infinite Toeplitz graphs, or to integer distance graphs. The following is an extension of Lemma 3.4 to $T_n(a_1,\dots,a_k)$'s:

Lemma 4.1. *A Toeplitz graph* $T_n(a_1,\dots,a_k)$ *with* $\gcd(a_1,a_2,a_i)=1$ *for each* $i \in \{3,\dots,k\}$ *and* $n \leq a_3+\gcd(a_1,a_2)-1$ *is bipartite if and only if* $T_n(a_1,a_2)$ *is.*

Combining the lemma above with Theorems 2.1 and 2.2 we get the following two theorems.

Theorem 4.2. *A Toeplitz graph* $T_n(a_1,\dots,a_k)$ *with* $\gcd(a_1,a_2,a_i) = 1$ *for each* $i \in \{3,\dots,k\}$ *and* $n \leq \min\{a+b-\gcd(a_1,a_2); a_3+\gcd(a_1,a_2)-1\}$ *is bipartite.*

Theorem 4.3. *A Toeplitz graph* $T_n(a_1,\dots,a_k)$ *with* $\gcd(a_1,a_2,a_i) = 1$ *for each* $i \in \{3,\dots,k\}$ *and* $a_1+a_2-\gcd(a_1,a_2)+1 \leq n \leq a_3+\gcd(a_1,a_2)-1$ *is bipartite if and only if* $\frac{a_1}{\gcd(a_1,a_2)}$ *and* $\frac{a_2}{\gcd(a_1,a_2)}$ *are odd.*

Notice that the upper bound on n in the Lemma/Theorems 4.1 to 4.3 above bounds the number and the value of entries larger than a_3.

As for Theorem 3.7 we remark that it applies almost literally to connected integer distance graphs $G_{\mathbb{Z}}(a,b,c)$.

In addition, since $\infty > \max\{a+b-\gcd(a,b)+1; c+\gcd(a,b)\}$, from Theorems 3.2 and 3.7, we immediately get that

Corollary 4.4. *An infinite Toeplitz graph with three entries* a,b,c *is bipartite if and only if* $\frac{a}{\gcd(a,b,c)}, \frac{b}{\gcd(a,b,c)}, \frac{c}{\gcd(a,b,c)}$ *are odd.*

Finally, Theorem 3.7 can be generalized to Toeplitz graphs with $k \geq 4$ entries, as follows.

Theorem 4.5. *Consider a Toeplitz graph* $T_n(a_1,\dots,a_k)$ *and let* $\alpha_1,\alpha_2,\alpha_3 \in \{a_1,\dots,a_k\}$ *be three distinct entries verifying* $\gcd(\alpha_1,\alpha_2,\alpha_3)=1$. *Let* $n \geq \max\{\alpha_i+\alpha_j-\gcd(\alpha_i,\alpha_j)+1; \alpha_h+\gcd(\alpha_i,\alpha_j)\}$ *for* $\{i,j,h\} = \{1,2,3\}$. $T_n(a_1,\dots a_k)$ *is bipartite if and only if* $a_1,\dots,a_k$ *are odd.*

Clearly, if more than one triple $\alpha_1, \alpha_2, \alpha_3$ exists verifying the conditions in the theorem above, the strongest lower bound for n is the one given by that triple which minimizes $\max\{\alpha_i + \alpha_j - \gcd(\alpha_i, \alpha_j) + 1; \alpha_h + \gcd(\alpha_i, \alpha_j)\}$.

The theorem above applies to integer distance graphs $G_{\mathbb{Z}}(a_1, a_2, \dots, a_k)$, as already proved in the unpublished paper [2], and, partially, in the earlier papers [3] and [6].

5 Conclusions

In this paper we first easily characterize the bipartite Toeplitz graphs with two entries. This closed-form characterization, based on easy topological properties, is important because it also helps in the characterization of bipartite Toeplitz graphs with three entries.

We also completely characterize the bipartite Toeplitz graphs with three entries. The considered graphs are subdivided into three different classes. In the first two classes, the characterization consists in verifying a simple condition on n, a, b, c; the bipartiteness of the graphs in the third class is proved to depend on the existence of a suitable solution to a linear diophantine equation in three variables (some cases also allow for a closed-form characterization, as discussed). These results and those in [11] completely solve the open problem of determining the chromatic number of Toeplitz graphs with three entries.

We remark that all the Toeplitz graphs with two or three entries can be analyzed in the context of constrained diophantine equations too. Nevertheless, the specific conditions that we proposed in Sections 2, 3.1, and 3.2 are much easier to verify.

Finally we remark that in Example 1 at the end of [8] the Toeplitz graphs $T_{2a+1}(a, b, c)$ verifying $a < b < c < n = 2a + 1$, $a = (c - b) + a \bmod (c - b)$, and $a < 3(b - a)$ (thus, $c - b \leq a \leq \min\{2(c - b) - 1, 3(b - a) - 1\}$) are said to be bipartite. However, there are *non-bipartite* Toeplitz fitting into these assumptions (for example: $T_{17}(8, 11, 16)$, where vertices v_0, v_8, v_{16} induce an odd cycle). This fact has been confirmed in [7].

As for Example 2 at the end of [8], it recognizes bipartite Toeplitz graphs $T_{2a+1}(a, b, c)$ fitting into given assumptions, but from this result we can not derive that $T_n(a, b, c)$ for $n \geq 2a + 1$ is bipartite: as an example, $T_{37}(18, 23, 35)$ is a bipartite graph fitting into the assumptions of Example 2 of [8], but from Theorem 3.7 we know that there exists a threshold value μ such that $T_n(18, 23, 35)$ with $n \geq \mu$ is non-bipartite (in this case, by Theorem 3.7 we get that $T_n(18, 23, 35)$ is non-bipartite for $n \geq 41$; actually, in application of Theorem 3.11, if we search for a constrained solution to the diophantine equation (1) we find that $T_n(18, 23, 35)$ is bipartite for $n \leq 38$ and non-bipartite for $n \geq 39$).

Some of our results have been extended to Toeplitz graphs with $k \geq 4$ entries, to infinite Toeplitz graphs, or to integer distance graphs.

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