Packing 2-Dimensional Bins in Harmony

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Abstract

We consider 2-Dimensional (Finite) Bin Packing (2BP), which is one of the most important generalizations of the well-known Bin Packing (BP) and calls for orthogonally packing a given set of rectangles (that cannot be rotated) into the minimum number of unit size squares.

There are many open questions concerning the approximability of 2BP, whereas the situation for the *2-stage* case, in which the items must first be packed into *shelves* that are then packed into bins, is essentially settled. For this reason, we study the asymptotic worst-case ratio between the optimal solution values of the 2-stage and general 2BP, showing that it is equal to $T_\infty = 1.691\ldots$, the well-known worst-case ratio of the *Harmonic* algorithm for BP. This ratio is achieved by packing the items into shelves by decreasing heights as in the Harmonic algorithm and then optimally packing the resulting shelves into bins. This immediately yields polynomial time approximation algorithms for 2BP whose asymptotic worst-case ratio is arbitrarily close to T_∞, i.e. substantially smaller than $2 + \varepsilon$, that was the best ratio achievable so far and constituted the first (recent) improvement over the 2.125 ratio shown in the early 80s. In particular, we manage to push the approximability threshold below 2, which is often a critical value in approximation.

The main idea in our analysis is to use the fact that the fractional and integer BP solutions have almost the same value, which is implicit in the approximation schemes for the problem, as a stand-alone structural result. This implies the existence of modified heights for the shelves whose sum yields approximately the number of bins needed to pack them. With this in mind, our proof can easily be adapted to different cases. For instance, we can derive new upper bounds on the worst-case ratio of several shelf heuristics for 2BP, among which a bound of $(\frac{17}{10})(\frac{11}{9}) = 2.077\ldots$ (rather than 2.125) on the 20-years-lasting champion mentioned above. Moreover, we can easily derive the asymptotic worst-case ratio between the 2-stage and general 2BP solution values as a function of the maximum width of the rectangles, showing that this ratio is independent of the maximum height.

1 Introduction

We consider *2-Dimensional (Finite) Bin Packing*, in which rectangles of specified size have to be packed into (larger) squares (called *bins*). The version of the problem which is most relevant for practical applications, as well as most studied in the literature, is the one in which a specified edge for each rectangle has to be packed parallel to a specified edge of a bin. Such a requirement is called *orthogonal packing without rotation.* The objective is the minimization of the number of bins used. A closely related problem is *2-Dimensional Strip Packing* (2SP), in which there is a unique bin (called also *strip*) of finite width and infinite height and the objective is to minimize the height used to pack the rectangles. In particular, we study the theoretical approximability of 2BP, focusing on the *offline* version and noting that for the *online* version there are a few results which are surveyed in [6] and will not be mentioned here.

For 2SP, a long series of results, starting in the early 80s with the work of Baker, Coffman and Rivest [1] and Coffman, Garey, Johnson and Tarjan [8] and ending in the late 90s with the elegant *Asymptotic Polynomial-Time Approximation Scheme* (APTAS) proposed by Kenyon and Rémila [17], settled the approximability of the problem. The situation for 2BP, which is certainly more complex to deal with than 2SP, is quite different. In 1982, Chung, Garey and Johnson [5] presented an approximation algorithm with asymptotic worst-case ratio at most 2.125, and then for almost 20 years nothing happened. As a result, the 2.125 value appears in a fairly large number of surveys on multidimensional packing. Although not explicitly mentioned in [17], the APTAS for 2SP in that paper can be used to achieve a ratio not larger than $(2+\varepsilon)$ for 2BP (for any $\varepsilon > 0$). This is obtained by packing all the rectangles into an infinite strip (almost minimizing the height used) and then cutting this strip into slices of height 1 so as to get bins of the required size. The rectangles that are split between two bins can be packed into additional bins. Apparently, there is no way to extend the method of [17] to 2BP so as to get a worst-case ratio strictly better than 2. For the special case in which *squares* have to be packed into squares, Seiden and van Stee [20] presented an approximation algorithm with asymptotic worst-case ratio arbitrarily close to $1.555\ldots$. We finally note that, to the best of our knowledge, there is no result that rules out the existence of an APTAS for 2BP.

Many simplified variants of 2BP and 2SP have been considered in the literature, being both easier to deal with than the original problems and important in practical applications. Among these, very famous are the *2-stage* versions of the problems, referred to as *2-Dimensional Shelf Bin Packing* (2SBP) and *2-Dimensional Shelf Strip Packing* (2SSP). 2-stage packing problems were originally introduced by Gilmore and Gomory [14] and, thinking in terms of *cutting* instead of packing, require that each item be obtained from the associated bin by at most two stages of cutting. Namely, after a first stage in which the bins are horizontally cut into *shelves*, the second stage produces *slices* which contain a single item by cutting the shelves vertically. Finally, an additional stage (called *trimming*) is allowed in order to separate an item from a waste area, as in Figure 1.

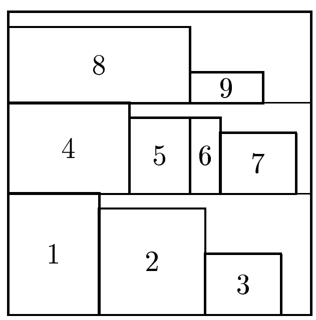

Figure 1: Packing rectangles into a bin in two stages.

Note that 2-stage packing is equivalent to packing the items in the bins in shelves, where a *shelf* is a row of items having their bases on a line which is either the base of the bin or the line drawn at the top of the tallest item packed in the shelf below (see again Figure 1). Many classical heuristics for 2SP [8, 2, 9] and 2BP [5] construct solutions that are in fact feasible for the 2-stage versions. We will call such an algorithm a *shelf* algorithm. Moreover, the approximability of 2SBP and 2SSP was recently settled by Caprara, Lodi and Monaci [3], who presented APTASs for both problems. Given this situation, it is natural to ask which is the (asymptotic) worst-case ratio between the optimal 2SBP and 2BP solution values. The same question for the 2SSP and 2SP case was answered by Csirik and Woeginger [9], who proved that this ratio is equal to $T_\infty = 1.691\ldots$, the well-known worst-case ratio of the *Harmonic* on-line algorithm for BP. Note also that the 2SP algorithm in [9] that achieves this ratio is *on-line*. Actually, T_∞ is also easily seen to be a lower bound for the 2SBP versus 2BP case, although the only upper bound known so far (and fairly difficult to prove) is 2.125 [5] (the $2+\varepsilon$ ratio of [17] is not achieved by packing the items in two stages).

In this paper, we show that T_∞ is also the asymptotic worst-case ratio between the optimal 2SBP and 2BP solution values. This ratio is achieved by packing the items into shelves by decreasing heights as in the Harmonic algorithm and then optimally packing the resulting shelves into bins. This immediately yields polynomial time approximation algorithms for 2BP whose asymptotic worst-case ratio is arbitrarily close to T_∞, i.e. substantially smaller than what was known previously. In particular, we manage to push the approximability threshold below 2, which is often a critical value in approximation.

The main idea in our analysis is to use the fact that the fractional and integer BP solutions have almost the same value, which is implicit in the approximation schemes for the problem, as a stand-alone structural result. This implies the existence of modified heights for the shelves whose sum yields approximately the number of bins needed to pack them. With this in mind, our proof can easily be adapted to different cases. For instance, we can derive new upper bounds on the worst-case ratio of several shelf heuristics for 2BP, among which a bound of $(\frac{17}{10})(\frac{11}{9}) = 2.077\ldots$ (rather than 2.125) on the 20-years-lasting champion in [5] mentioned above. The correctness of this bound was mentioned as an open problem in the conclusions of [5]. Moreover, we can easily derive the asymptotic worst-case ratio between the 2-stage and general 2BP solution values as a function of the maximum width of the rectangles, showing that this ratio is independent of the maximum height.

Basic notation

In BP the input consists of a set I of m *items*, the i-th having a *size* (or *weight*) $s_i \in (0,1]$. The objective is to pack these items into the minimum number of unit capacity bins. With an abuse of notation, we will let I denote also the set of indices $\{1,\ldots,m\}$.

In 2BP, 2SP, 2SBP, and 2SSP the input consists of a set R of n *rectangles*, the j-th having a *width* $w_j \in (0,1]$ and a *height* $h_j \in (0,1]$. Again, we will let R denote also $\{1,\ldots,n\}$. For 2BP and 2SBP the objective is to pack the rectangles into the minimum number of unit size *squares*, called bins also in this case. For 2SP and 2SSP the objective is to pack the rectangles in a bin of unit width and infinite height, called *strip*, minimizing the height used. Moreover, for 2SBP and 2SSP items must be packed into shelves and the shelves into the bin(s). A *shelf* is a set $S \subseteq R$ whose *width* and *height* are defined respectively by $\sum_{j\in S} w_j$ and $\max_{j\in S} h_j$.

Consider a minimization problem P, letting $\mathcal{I}_\mathrm{P}$ denote the set of all its instances. Given an instance $I \in \mathcal{I}_\mathrm{P}$ and an approximation algorithm A for P, let $\mathrm{opt}_\mathrm{P}(I)$ denote the value of the optimal solution for I and $\mathrm{heur}_A(I)$ the value of the solution returned by A for instance I. The *asymptotic worst-case ratio* of A is defined by

$$R^\infty(A) := \lim_{z\to\infty} \sup \left\{ \frac{\mathrm{heur}_A(I)}{\mathrm{opt}_\mathrm{P}(I)} : I \in \mathcal{I}_\mathrm{P}, \mathrm{opt}_\mathrm{P}(I) \geq z \right\}.$$

An *APTAS* is an algorithm $A(\varepsilon)$ that receives on input also a required *accuracy* $\varepsilon > 0$, runs in time polynomial in the size of the instance, and satisfies $R^\infty(A(\varepsilon)) \leq (1+\varepsilon)$ for every $\varepsilon > 0$.

Given two minimization problems P_1, P_2 with the same set of instances and such that $\mathrm{opt}_{\mathrm{P}_1} \geq \mathrm{opt}_{\mathrm{P}_2}$ for all $I \in \mathcal{I}_{\mathrm{P}_1} \equiv \mathcal{I}_{\mathrm{P}_2}$, we define the *asymptotic worst-case ratio* of P_1 versus P_2 by

$$R^\infty(\mathrm{P}_1,\mathrm{P}_2) := \lim_{z\to\infty} \sup \left\{ \frac{\mathrm{opt}_{\mathrm{P}_1}(I)}{\mathrm{opt}_{\mathrm{P}_2}(I)} : I \in \mathcal{I}_{\mathrm{P}_1}, \mathrm{opt}_{\mathrm{P}_1}(I) \geq z \right\}.$$

2 Some Basic Properties of BP

In this section we present some results on (the 1-dimensional) BP which form the basis of our proof. All of the results in this section are either known or simple to derive starting from other known results. On the other hand, probably nobody ever thought of using them together since they concern different aspects of the problem (approximation schemes, practical solution methods, and on-line algorithms, respectively).

The fractional BP

In order to define the *Fractional* BP (FBP), we slightly change the notation to represent a BP instance. In particular, consider such an instance I and let $s_1 > s_2 > \ldots > s_p$ be the *distinct* sizes of the items in I and $n_1, \ldots, n_p$ be the numbers of items of each size. A *feasible pattern* is a vector $v = (v_1, \ldots, v_p)$ such that $\sum_{j=1}^{p} v_j s_j \leq 1$, i.e. v_j items of size s_j $(j = 1, \ldots, p)$ would fit in a bin. (Note that we *do not* impose $v_j \leq n_j$, i.e. we consider also patterns with more items of size s_j than there are in the instance – this is somehow crucial, as we will point out later.) Let $\mathcal{V}$ donote the collection of all feasible patterns for I. FBP corresponds to the *Linear Program* (LP)

$$\begin{array}{rcl} \min \sum_{v \in \mathcal{V}} x_v, & & \\ \sum_{v \in \mathcal{V}} v_j x_v & \geq & n_j, \quad j = 1, \ldots, p, \\ x_v & \geq & 0, \quad v \in \mathcal{V}, \end{array} \tag{1}$$

noting that the corresponding *Integer LP* (ILP) (with x_v constrained to be integer for each $v \in \mathcal{V}$) is a formulation of BP. The *LP dual* of (1) reads

$$\begin{array}{rcl} \max \sum_{j=1}^{p} n_j \pi_j, & & \\ \sum_{j=1}^{p} v_j \pi_j & \leq & 1, \quad v \in \mathcal{V}, \\ \pi_j & \geq & 0, \quad j = 1, \ldots, p. \end{array} \tag{2}$$

By LP duality, the optimal solution values of (1) and (2) coincide.

The following is a very easy property of the optimal solutions of (2) that we need in our analysis and whose proof is in the appendix.

Lemma 1 *There exists an optimal solution π^* of (2) such that $\pi_1^* \geq \pi_2^* \geq \ldots \geq \pi_p^*$ (recalling $s_1 > s_2 > \ldots > s_p$).*

In the following we will let $\text{opt}_{\text{FBP}}(I)$ denote the optimal solution value of LPs (1) and (2). The following relevant result is an implicit byproduct of the APTAS for BP presented in the landmark paper of Fernandez de la Vega and Lueker [12]. Actually the result does not appear to be widely known (for instance, it is posed as an open problem in [4] and [11]). Therefore, we report a detailed proof in the appendix.

Lemma 2 *For each* BP *instance I and $\varepsilon > 0$, there exists a constant $\delta(\varepsilon)$ such that*

$$\text{opt}_{\text{BP}}(I) \leq (1 + \varepsilon)\ \text{opt}_{\text{FBP}}(I) + \delta(\varepsilon). \tag{3}$$

Using the techniques by Karmarkar and Karp [16] it is possible to show a substantially stronger result, namely

$$\text{opt}_{\text{BP}}(I) \leq \text{opt}_{\text{FBP}}(I) + O(\log^2(\text{opt}_{\text{FBP}}(I)),$$

but (3) is sufficient for our purposes and its proof is slightly simpler.

Dual feasible functions

A function $f : [0,1] \to [0,1]$ is called a *dual feasible function* if, for each sequence $x_1, \ldots, x_m$ with $x_1 \in [0,1]$ for $i = 1, \ldots, m$ and $\sum_{i=1}^m x_i \leq 1$, we have $\sum_{i=1}^m f(x_i) \leq 1$. It is easy to check that, giving the distinct sizes $s_1, \ldots, s_p$ in a BP instance and an arbitrary dual feasible function $f(\cdot)$, the vector $(f(s_1), \ldots, f(s_p))$ is a feasible solution of (2). Conversely, the next simple lemma, proved in the appendix, shows how to construct a dual feasible function from any feasible solution of (2).

Lemma 3 *Let $\overline{\pi}$ be a feasible solution of (2) meeting the requirements of Lemma 1, and define $s_0 := 1, s_{p+1} := 0, \overline{\pi}_{p+1} := 0$. Then the function defined by $g(0) := 0$ and*

$$g(x) := \overline{\pi}_j, \qquad \text{for } x \in [s_j, s_{j-1}) \text{ and } j = 1, \ldots, p+1, \tag{4}$$

is a dual feasible function.

The above Lemma would not hold if $\mathcal{V}$ contained only the feasible patterns v for which $v_j \leq n_j$ for $j = 1, \ldots, p$.

Dual feasible functions are essentially the same as the *weighting functions* used in the analysis of approximation algorithms for BP. The name was introduced by Fekete ad Schepers [10, 11], who used these functions in the practical solution of BP and its multidimensional generalizations. One of the main advantages of dual feasible functions is that they can be used to derive lower bounds on the 2BP solution value by modifying the widths and heights accordingly. In particular, the following key lemma is proved in [10].

Lemma 4 *Let R be a set of rectangles that fit within a unit square and $f(\cdot)$, $g(\cdot)$ be dual feasible functions. Then,*

$$\sum_{j \in R} f(w_j) \cdot g(h_j) \leq 1. \tag{5}$$

Lemma 4 is actually a corollary of a much stronger result in [10], stating that if the conditions of the lemma apply then the rectangles with modified widths and heights *fit* within a unit square (and this holds not only for dimension 2 but for any dimension).

The harmonic algorithm

The results in this section are due to Lee and Lee [18]. Consider a BP instance I and let k be a positive integer. We say that an item i is of *type* q if $s_i \in (\frac{1}{q+1}, \frac{1}{q}]$ for $q = 1, \ldots, k-1$, and of *type* k if $s_i \in (0, \frac{1}{k}]$. The *Harmonic*$_k$ (H_k) algorithm of [18] packs items of different types into different bins. For each type q, the current bin is *closed* and a new one is opened when the current item of type q does not fit in the bin (i.e., for $q < k$, when the bin contains q items).

Let

$$t_1 := 1, \quad t_{q+1} := t_q(t_q + 1) \text{ for } i \geq 1.$$

(We have $t_2 = 2, t_3 = 6, t_4 = 42, \ldots$.) For a positive integer k, let $m(k)$ be the integer such that $t_{m(k)} < k \leq t_{m(k)+1}$. Moreover, let

$$T_k := \sum_{q=1}^{m(k)} \frac{1}{t_q} + \frac{1}{t_{m(k)+1}} \cdot \frac{k}{k-1}$$

and

$$T_\infty := \lim_{k\to\infty} T_k = \sum_{q=1}^{\infty} \frac{1}{t_q} = 1.691\ldots,$$

noting that $T_k \le T_\infty + \frac{1}{k-1}$.

The asymptotic worst-case ratio of H_k is $R^\infty(H_k) = T_k$. To prove this, one can define the following function

$$f_k(x) := \begin{cases} \dfrac{1}{q}, & \text{for } x \in (\frac{1}{q+1}, \frac{1}{q}] \text{ and } q = 1, \ldots, k-1, \\ \dfrac{k}{k-1} \cdot x, & \text{for } x \in (0, \frac{1}{k}], \end{cases} \tag{6}$$

and then apply the following two lemmas

Lemma 5 *For each bin that is closed by H_k with items $\{i_1, \ldots, i_b\}$,*

$$\sum_{j=1}^{b} f_k(s_{i_j}) \ge 1.$$

and

Lemma 6 *For each sequence $x_1, \ldots, x_m$ with $x_i \in (0, 1]$ for $i = 1, \ldots, m$ and $\sum_{i=1}^m x_i \le 1$,*

$$\sum_{i=1}^{m} f_k(x_i) \le T_k.$$

Corollary 1 *The function $f'_k(\cdot)$ defined by $f'_k(0) := 0$ and $f'_k(x) := \frac{f_k(x)}{T_k}$ for $x \in (0, 1]$ is a dual feasible function.*

3 Shelf versus General 2BP

In this section we prove our main result, namely $R^\infty(\text{2SBP}, \text{2BP}) = T_\infty$. We recall that $R^\infty(\text{2SSP}, \text{2SP}) = T_\infty$ was already shown in the paper by Csirik and Woeginger [9], from which we will borrow some ideas.

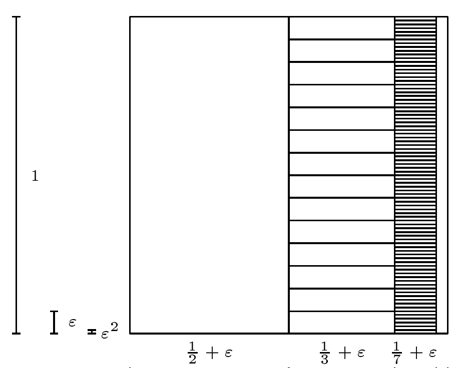

Figure 2: Bad example for 2-stage packing.

The bad examples in [9] can be adapted in a straightforward way to show the following

Lemma 7 $R^\infty(\text{2SBP}, \text{2BP}) \geq T_\infty$.

The detailed proof, deferred to the appendix, is naturally suggested by Figure 2, that shows the structure the bins in the optimal 2BP solution for the bad examples.

The interesting part of our result amounts to showing $R^\infty(\text{2SBP}, \text{2BP}) \leq T_\infty$. To this aim, we next present a shelf algorithm for 2BP that produces a solution whose value can be made arbitrarily close to T_∞. In this algorithm, shelves are formed by the following *Harmonic Decreasing Height*$_k$ (HDH_k) heuristic, which is a natural generalization of the H_k heuristic for BP. Here the harmonic partitioning concerns the *widths* of the rectangles, namely a rectangle is of *type* q if $w_j \in (\frac{1}{q+1}, \frac{1}{q}]$ for $q = 1, \ldots, k-1$, and of type k if $w_j \in (0, \frac{1}{k}]$. Shelves are formed with rectangles of the same type. Namely, for each type we pack all the corresponding rectangles into shelves by *next fit decreasing height* [8] considering these rectangles in *decreasing order of height* and packing them into new shelves by a next fit policy, closing the current shelf and starting a new one when the current rectangle does not fit in the shelf. The shelves formed by HDH_k are then packed into bins optimally (note that this is NP-hard, but in practice we may use an APTAS for BP to get arbitrarily close to the optimum). We remark that HDH_k is a simplification of the method to form shelves in [9]. This simplification, which is essential to achieve our result, makes it impossible to form the shelves on-line.

We next show that the optimal solution of the BP instance whose items correspond to the shelves formed by HDH_k (the size of each item being given by the height of the corresponding shelf) yields a good 2BP solution for the original instance.

Lemma 8 *For a* 2BP *instance* R*, let* $I(R)$ *denote the be* BP *instance associated with the shelves formed by* HDH_k *for* R*. For each* $\varepsilon > 0$*, there exists a constant* $\gamma(\varepsilon, k)$ *such that*

$$\text{opt}_{\text{BP}}(I(R)) \leq (1+\varepsilon) T_k \ \text{opt}_{\text{2BP}}(R) + \gamma(\varepsilon, k). \tag{7}$$

Proof Consider BP instance $I(R)$ and let $\overline{\pi}$ be a corresponding optimal solution of (2) satisfying the conditions of Lemma 1. Define the *modified width* and *height* for each rectangle $j \in R$ by $f_k(w_j)$ and $g(h_j)$, respectively, where $f_k(\cdot)$ is defined by (6) and $g(\cdot)$ by (4) (using $\overline{\pi}$). The *modified area* of rectangle j is given by $f_k(w_j) \cdot g(h_j)$. By Lemmas 3 and 4 and Corollary 1, we have that the overall modified area of the rectangles in R is

$$\sum_{j \in R} f_k(w_j) \cdot g(h_j) \leq T_k \ \text{opt}_{\text{2BP}}(R)$$

(it would be at most $\text{opt}_{\text{2BP}}(R)$ if the heights were modified according to $f'_k(\cdot)$ defined in Corollary 1).

Considering now the shelves formed by HDH_k, let the *type* of a shelf be the type of the rectangles that it contains. For $q = 1, \ldots, k$, we let m^q be the number of shelves of type q formed by HDH_k and $s_1^q \geq s_2^q \geq \ldots \geq s_{m^q}^q$ be the corresponding heights in decreasing order. For convenience we also define $s_{m^q+1}^q := 0$ for all q.

Claim 1 *The total modified area of the rectangles in the* i*-th shelf of type* q *in the* HDH_k *solution is at least* $g(s_{i+1}^q)$ *(noting that* $g(s_{m^q+1}^q) = g(0) = 0$*).*

Proof of Claim 1 For each rectangle j in the i-th shelf of type q, we have $s_{i+1}^q \leq h_j \leq s_i^q$, which implies $g(h_j) \geq g(s_{i+1}^q)$ by the definition of $g(\cdot)$. Moreover, the overall modified width of the rectangles in the shelf is at least 1 by Lemma 5, with the possible exception of the last shelf for which the claim is trivially true. □

The following computation is based on the "collapsing sum" principle of [8]. Letting $G^q := \sum_{i=1}^{m^q} g(s_i^q)$ denote the overall modified height of the shelves of type q, we have that the total modified area of the associated rectangles is at least

$$\sum_{i=2}^{m^q+1} g(s_i^q) = G^q - g(s_1^q) \geq G^q - 1.$$

This implies, summing over all types,

$$\sum_{q=1}^{k}(G^q - 1) = \sum_{q=1}^{k} G^q - k \leq \sum_{j \in R} f_k(w_j) \cdot g(h_j) \leq T_k \ \mathrm{opt}_{\mathrm{2BP}}(R)$$

i.e., noting that the optimal FBP solution value for instance $I(R)$ is given by the sum of the modified heights of the shelves, we have

$$\mathrm{opt}_{\mathrm{FBP}}(I(R)) = \sum_{q=1}^{k} G^q \leq T_k \ \mathrm{opt}_{\mathrm{2BP}}(R) + k,$$

which yields, along with (3),

$$\mathrm{opt}_{\mathrm{BP}}(I(R)) \leq (1+\varepsilon) \ \mathrm{opt}_{\mathrm{FBP}}(I(R)) + \delta(\varepsilon) \leq (1+\varepsilon) T_k \ \mathrm{opt}_{\mathrm{2BP}}(R) + k + \delta(\varepsilon),$$

completing the proof. □

Lemma 9 $R^\infty(\mathrm{2SBP}, \mathrm{2BP}) \leq T_\infty$.

Proof Noting that $\mathrm{opt}_{\mathrm{2SBP}}(R) \leq \mathrm{opt}_{\mathrm{BP}}(I(R))$, by (7), we have

$$\mathrm{opt}_{\mathrm{2SBP}}(R) \leq (1+\varepsilon) T_k \ \mathrm{opt}_{\mathrm{2BP}}(R) + \gamma(\varepsilon, k).$$

Choosing $k := \frac{1+\varepsilon}{\varepsilon}$ so that $T_k \leq T_\infty + \varepsilon$ and taking the limit for $\varepsilon \to 0$ yields the claim. □

The following main result is the combination of Lemmas 7 and 9.

Theorem 1 $R^\infty(\mathrm{2SBP}, \mathrm{2BP}) = T_\infty$.

To conclude the section, we point out that our analysis immediately extends to the case in which the rectangles have bounded width. Consider the value $T_\infty(\alpha)$, which is defined by

$$T_\infty(\alpha) := \lim_{k \to \infty} \max \left\{ \sum_{i=1}^{m} f_k(x_i) : x \in (0, \alpha]^m, \sum_{i=1}^{m} x_i \leq 1 \right\},$$

and is easily seen to be constant for $\alpha \in (\frac{1}{q+1}, \frac{1}{q}]$ for each positive integer q. Letting $R^\infty(\mathrm{2SBP}, \mathrm{2BP}, \alpha)$ denote the asymptotic worst-case ratio for instances where all rectangles have width bounded by α, we have the following

Corollary 2 *For each* $\alpha \in (0, 1]$, $R^\infty(\mathrm{2SBP}, \mathrm{2BP}, \alpha) = T_\infty(\alpha)$.

Table 1 contains the values of $R^\infty(\mathrm{2SBP}, \mathrm{2BP}, \alpha)$ for different ranges of α.

$\alpha \in$	$(\frac{1}{2}, 1]$	$(\frac{1}{3}, \frac{1}{2}]$	$(\frac{1}{4}, \frac{1}{3}]$	$(\frac{1}{5}, \frac{1}{4}]$
$R^\infty(\text{2SBP}, \text{2BP}, \alpha)$	$1.691\ldots$	$1.423\ldots$	$1.302\ldots$	$1.234\ldots$

Table 1: The value of $R^\infty(\text{2SBP}, \text{2BP}, \alpha)$ for $\alpha > \frac{1}{5}$.

4 Implications on Approximation

In this section, we discuss the implications of the result (and the proof methods) of the previous section on the approximability of 2BP.

Given a method T to pack rectangles into shelves and an approximation algorithm A for BP, let $S(T, A)$ be the shelf algorithm for 2BP that constructs the shelves according to T and then packs these shelves into bins according to A. A corollary of the results of the previous section is

Corollary 3 *For each approximation algorithm A for* BP*,*

$$\lim_{k\to\infty} R^\infty(S(HDH_k, A)) = T_\infty \cdot R^\infty(A).$$

Of course, considering bounded widths, the above result applies with $T_\infty(\alpha)$ instead of T_∞.

Taking for A the linear time APTAS of [12] we get an $O(n \log n)$-time approximation algorithm for 2BP with asymptotic worst-case ratio arbitrarily close to T_∞. On the other hand, there are also algorithms A for BP such that $R^\infty(A) = 1$ whose running time is polynomial (although much worse than $O(n \log n)$). This was first observed by Johnson [15], and the best such algorithm is by Karmarkar and Karp [16].

In fact, the idea of using in the analysis the heights modified according to (4) appears to be quite fruitful. For instance, it is immediate to check that the analyses performed in [8] for the *Next Fit Decreasing Height* ($NFDH$) and the *First Fit Decreasing Height* ($FFDH$) methods to form shelves (in the 2SP context) carry over also to the 2BP case. More specifically, we have

Lemma 10 *For a* 2BP *instance R, let $I(R)$ denote the be* BP *instance associated with the shelves formed by $NFDH$ for R. For each $\varepsilon > 0$, there exists a constant $\gamma(\varepsilon)$ such that*

$$\text{opt}_{\text{BP}}(I(R)) \le (1+\varepsilon)2\ \text{opt}_{\text{2SP}}(R) + \gamma(\varepsilon). \tag{8}$$

Lemma 11 *For a* 2BP *instance R, let $I(R)$ denote the be* BP *instance associated with the shelves formed by $FFDH$ for R. For each $\varepsilon > 0$, there exists a constant $\gamma(\varepsilon)$ such that*

$$\text{opt}_{\text{BP}}(I(R)) \le (1+\varepsilon)1.7\ \text{opt}_{\text{2SP}}(R) + \gamma(\varepsilon). \tag{9}$$

Lemma 12 *For a* 2BP *instance R such that all rectangles in R have width at most $\frac{1}{m}$ for some fixed $m \ge 2$, let $I(R)$ denote the be* BP *instance associated with the shelves formed by $FFDH$ for R. For each $\varepsilon > 0$, there exists a constant $\gamma(\varepsilon)$ such that*

$$\text{opt}_{\text{BP}}(I(R)) \le (1+\varepsilon)(1+\tfrac{1}{m})\ \text{opt}_{\text{2SP}}(R) + \gamma(\varepsilon). \tag{10}$$

The proofs of the above lemmas are obtained by reproducing the original ones in [8] (Theorems 1, 2, and 3, respectively), which are all based on considerations on the area of the items, using the modified heights defined by (4) instead of the original heights. (Actually, for the counterpart of Lemma 11 a modified width, along with the corresponding modified area, is used in [8], but the height is always unchanged, as is natural for 2SP.)

Corollary 4 *For each approximation algorithm A for* BP,

$$R^\infty(S(NFDH, A)) \le 2 \cdot R^\infty(A).$$

Corollary 5 *For each approximation algorithm A for* BP,

$$R^\infty(S(FFDH, A)) \le 1.7 \cdot R^\infty(A).$$

A special case of the last corollary is $R^\infty(S(FFDH, FFD)) \le (\frac{17}{10})(\frac{11}{9}) = 2.077\ldots$, where FFD is the well-known *First Fit Decreasing* heuristic for BP. Algorithm $S(FFDH, FFD)$ is the 20-years-lasting champion for 2BP proposed in [5] (and therein called *Hybrid First Fit*), where a (fairly complicated) proof that $R^\infty(S(FFDH, FFD)) \le 2.125$ is presented, and the "amusing"possibility that $R^\infty(S(FFDH, FFD)) = (\frac{17}{10})(\frac{11}{9})$ is mentioned in the conclusions. Although we do not know if equality holds (the best lower bound that we are aware of is 2.022... from [5]), our upper bound is stronger and very easy to prove starting from the results of the previous section and those in [8].

Table 2 presents the upper bounds on the asymptotic worst-case ratios of $S(T, A)$ for different combinations of T (rows) and A (columns). HDH_∞ indicates that we are taking the limit for $k \to \infty$, A^* is a generic heuristic for BP with $R^\infty(A^*) = 1$, and $MFFD$ is the *Modified First Fit Decreasing* heuristic of Garey and Johnson [13], the algorithm with the best asymptotic worst-case ratio among the "practical" ones, which runs in $O(n \log n)$ time without huge constants hidden inside. Note that, in order to have a worst-case ratio better than 2 for 2BP we apparently need an APTAS to pack the shelves into bins at the moment (at least as long as we stick to *deterministic* algorithms).

	A^*	$MFFD$	FFD
HDH_∞	1.691...	2.001...	2.066...
$NFDH$	2	2.366...	2.444...
$FFDH$	1.7	2.011...	2.077...

Table 2: Upper bounds on $R^\infty(T, A)$ for various combinations of T (rows) and A (columns).

5 Final Remarks

The main open problem is of course to find out more about the approximability of BP. At first glance it may seem that, in order to improve on the T_∞ ratio, one has to forget about shelf algorithms. However, this may not be the case: consider the optimal 2SBP solution in which shelves are *horizontal* (as was the case throughout the paper) as well as the optimal 2SBP solution in which they are *vertical*. (Recall that near-optimal 2SBP solutions can be found in polynomial – although astronomical – time [3].) There is no evidence that the worst-case ratio between the best of these two solutions and the optimal 2BP one can be as bad as T_∞. On the other hand, it is easy to construct examples showing that we cannot do better than T_∞ if both horizontal and vertical shelves are formed by HDH_k.

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Appendix

Proof of Lemma 1 Suppose we have $s_i > s_j$ and $\pi_i^* < \pi_j^*$. We will show how to define another optimal solution $\overline{\pi}$ of (2) such that $\overline{\pi}_i = \overline{\pi}_j$ and $\overline{\pi}_h = \pi_h^*$ for $h \neq i, j$, which clearly implies the claim. Consider the solution $\overline{\pi}$ defined by

$$\overline{\pi}_i := \overline{\pi}_j := \frac{n_i \pi_i^* + n_j \pi_j^*}{n_i + n_j}, \quad \overline{\pi}_h := \pi_h \text{ for } h \neq i, j.$$

The value of $\overline{\pi}$ is clearly equal to that of π. Moreover, the feasibility of $\overline{\pi}$ follows from $\overline{\pi}_i = \overline{\pi}_j < \pi_j^*$ and the fact that, for each pattern $v \in V$, also the pattern v' defined by $v'_i := 0$, $v'_j := v_i + v_j$, $v'_h := v_h$ for $h \neq i, j$ is feasible. Therefore,

$$\sum_{j=1}^{p} v_j \overline{\pi}_j = \sum_{j=1}^{p} v'_j \overline{\pi}_j \leq \sum_{j=1}^{p} v'_j \pi_j^* \leq 1,$$

since π^* is feasible. □

Proof of Lemma 2 (In this proof we use the notation s_i to denote both the size of a specific item and one of the distinct sizes in a BP instance; this should generate no confusion since it will always be clear which one of the two things we mean.)

Let $\sigma := \frac{\varepsilon}{1+\varepsilon}$, $S := \{i \in I : s_i < \sigma\}$ be the set of *small items* and $L := \{i \in I : s_i \geq \sigma\}$ the set of *large items*, letting $s_1 \geq s_2 \geq \ldots \geq s_\ell$ be the sizes of the items in L in decreasing order.

Define the following set $\overline{L}$ of ℓ items by changing the sizes of the large items according to the well-known *linear grouping* technique of [12]. If $\ell < \frac{2}{\sigma^2}$, then $\overline{L}$ coincides with L, and we let $p := \ell$. Otherwise, we define $q := \lfloor \ell \sigma^2 \rfloor$ and form $p := \lceil \frac{\ell}{q} \rceil$ *groups* $L_1, \ldots, L_p$ of consecutive items in L, starting from the first item and letting L_p contain $\ell - (p-1)q$ items. The sizes of the items in $\overline{L}$ are obtained from those in L by replacing the size of each item in group L_k ($k = 1, \ldots, p$) by $\overline{s}_k := \min_{i \in L_k} s_k$.

A feasible solution of BP for I is obtained as follows. If grouping was performed (i.e. if $\ell \geq \frac{2}{\sigma^2}$), pack each of the q items in L_1 in a separate bin. For the remaining items in L, solve FBP associated with the items in $\overline{L}$ obtaining an optimal basic solution and round up this solution. This yields a feasible packing for the items in $\overline{L}$ as well as a feasible packing for the items in $L \setminus L_1$, which is obtained by replacing each of the q items of size $\overline{s}_k$ in the packing for $\overline{L}$

by an item of L_{k+1} (the space reserved in the solution for the items of size $\overline{s}_q$ is not used, as well as part of the space for the items of size $\overline{s}_{q-1}$). If grouping was not performed, solve FBP for $L \equiv \overline{L}$ obtaining an optimal basic solution and round up this solution. Let heur(L) denote the number of bins used to pack the items in L. Finally, the small items are packed in an arbitrary order, starting from the bins already containing some large items and considering a new bin only when the current small item does not fit in the current bin. Let heur$(I) \geq$ heur(L) denote the number of bins used by the solution.

If heur$(I) >$ heur(L), then all the bins with the possible exception of the last one contain items for a total size of at least $(1 - \sigma)$. This implies

$$\text{heur}(I) \leq \frac{\sum_{j=1}^{p} n_j s_j}{1-\sigma} + 1,$$

i.e., considering the trivial dual solution of (2) defined by $\pi'_i := s_i$ $(j = 1, \dots, p)$ and recalling the definition of σ, we have

$$\text{opt}_{\text{BP}}(I) \leq \text{heur}(I) \leq (1+\varepsilon) \sum_{j=1}^{p} n_j \pi'_j + 1 \leq (1+\varepsilon)\ \text{opt}_{\text{FBP}}(I) + 1. \tag{11}$$

If heur(I) = heur(L), note that the number of bins to pack the items in $\overline{L}$ is at most $\text{opt}_{\text{FBP}}(\overline{L}) + p$, since a basic solution of (1) for $\overline{L}$ has at most p nonzero components, p being the number of constraints in the LP. Since the items in L_1 are packed into q additional bins if grouping was performed, and are already packed in the rounded LP solution otherwise, we have

$$\text{heur}(I) \leq \text{opt}_{\text{FBP}}(\overline{L}) + p + q. \tag{12}$$

Note that $\text{opt}_{\text{FBP}}(I) \geq \text{opt}_{\text{FBP}}(L) \geq \text{opt}_{\text{FBP}}(\overline{L})$, since L is defined from I by removing some items and $\overline{L}$ is defined form L by decreasing the size of some other items. Moreover, it is easy to check that $p \leq \frac{3}{\sigma^2} + 1$ and $q \leq \sigma\ \text{opt}_{\text{FBP}}(I)$. The first relation is implied by

$$p \leq \frac{\ell}{\lfloor \ell\sigma^2 \rfloor} + 1 \leq \frac{\ell}{\ell\sigma^2 - 1} + 1 \leq \frac{3}{\sigma^2} + 1,$$

where the last inequality holds since $\ell\sigma^2 \geq 2$. As to the second relation, it follows from $q \leq \ell\sigma^2$ and $\text{opt}_{\text{FBP}}(I) \geq \text{opt}_{\text{FBP}}(L) \geq \ell\sigma$, recalling the trivial dual solution π' mentioned above, in which $\pi'_j \geq \sigma$ for $s_j \geq \sigma$. Therefore (12) along with the definition of $\sigma (< \varepsilon)$ imply

$$\text{opt}_{\text{BP}}(I) \leq \text{heur}(I) \leq (1+\varepsilon)\ \text{opt}_{\text{FBP}}(I) + \frac{3(1+\varepsilon)^2}{\varepsilon^2} + 1. \tag{13}$$

The combination of (11) and (13) yields the desired result. □

Proof of Lemma 3 Consider a sequence $x_1, \dots, x_m$ with $x_1 \in [0,1]$ for $i = 1, \dots, m$ and $\sum_{i=1}^{m} x_i \leq 1$. For $i = 1, \dots, m$, let j_i be such that $x_i \in [s_{j_i}, s_{j_i - 1})$, noting that we can assume $x_i > 0$. We have $\sum_{i=1}^{m} s_{j_i} \leq 1$, i.e. the sizes $s_{j_1}, \dots, s_{j_m}$ are associated with a feasible pattern in $\mathcal{V}$ (defined by counting the number of occurrences of each size). Then the constraints in (2) guarantee

$$\sum_{i=1}^{m} g(x_i) = \sum_{i=1}^{m} \overline{\pi}_{j_i} \leq 1.$$

□

Proof of Lemma 7 Consider positive integers b, k and a real value $\varepsilon \in (0, \frac{1}{2}]$ such that b is a multiple of t_q for $q = 1, \ldots, k$ and $\frac{1}{\varepsilon}$ is integer. Define the the instance R with $b \cdot \frac{1}{\varepsilon^{q-1}}$ rectangles of width $\frac{1}{t_q+1} + \varepsilon$ and height ε^{q-1} for $q = 1, \ldots, k$ (this means that there are b rectangles of width $\frac{1}{2} + \varepsilon$ and height 1, $\frac{b}{\varepsilon}$ rectangles of width $\frac{1}{3} + \varepsilon$ and height ε, $\ldots$; see Figure 2). It is easy to check that these rectangles fit into b unit squares, as illustrated in Figure 2. On the other hand, if the rectangles have to be packed into shelves (and then the shelves into bins), an almost optimal solution packs the items of different heights into different shelves. More precisely, it is proved in [3] that the best value, say z, of the 2SBP solution for R under the additional requirement that only items of the same height be packed into a shelf satisfies

$$z \leq (1 + 4\varepsilon)\mathrm{opt}_{2\mathrm{SBP}}(R) + 1. \tag{14}$$

Packing items of the same height in the same shelf optimally means that, for $q = 1, \ldots, k$, t_q items fit in each shelf, i.e. there are $\frac{b}{t_q \varepsilon^{q-1}}$ shelves of height ε^{q-1}, whose overall height is $\frac{b}{t_q}$. Since the total height of the shelves is a lower bound on z, we get

$$z \geq b\left(\frac{1}{t_1} + \ldots + \frac{1}{t_k}\right) = \mathrm{opt}_{\mathrm{BP}}(R)\left(T_\infty - \sum_{q=k+1}^{\infty} \frac{1}{t_q}\right). \tag{15}$$

The proof follows from (14) and (15) by taking the limit for $\varepsilon \to 0$, $k \to \infty$, $b \to \infty$. $\square$