Gear composition and the Stable Set Polytope

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Abstract

We present a new graph composition that produces a graph G from a given graph H and a fixed graph B called *gear* and we study its polyhedral properties. This composition yields counterexamples to a conjecture on the facial structure of $STAB(G)$ when G is claw-free.

Key words: stable set polytope, graph composition, polyhedral combinatorics, claw-free graphs.

1. Introduction

Given a graph $G = (V, E)$ and a vector $w \in \mathbb{Q}_+^V$ of node weights, the *stable set problem* is the problem of finding a set of pairwise nonadjacent nodes *(stable set)* of maximum weight.

The *stable set polytope*, denoted by $STAB(G)$, is the convex hull of the incidence vectors of the stable sets of G and it is known to be full-dimensional. A linear system $Ax \leq b$ is said to be *defining* for $STAB(G)$ if $STAB(G) = \{x : Ax \leq b\}$. The *facet defining inequalities* for $STAB(G)$ are those inequalities that constitute the unique nonredundant defining linear system of $STAB(G)$.

Clearly, finding a defining linear system for $STAB(G)$ is equivalent to transform the original optimization problem into the linear program $\max\{w^T x : Ax \leq b\}$ and, being the stable set problem NP-hard, it is unlikely to find such a system for general graphs.

Nevertheless the facial structure of the stable set polytope has been one of the most studied problems in polyhedral combinatorics. Here is a non-exhaustive list of results related with the study of facets of $STAB(G)$: facet producing graphs [17,20,15], t and h-perfectness [11], characterization of $STAB(G)$ when G is series-parallel [13], odd K_4-free [9] or quasi-line [6].

Besides the description of new classes of facets, it is of interest to find composition procedures that enable to build new families of facets for $STAB(G)$ starting from facets of a lower dimensional polytope. These compositions are usually based on graph compositions: for example, sequential lifting is based on the extension of a graph with an additional node, the Wolsey's lifting procedure [21] is based on edge subdivision, and Chvátal's compositions of polyhedra [3] are based on node substitution and clique identification.

In this paper, we present a new graph composition, named *gear composition*, which consists of 'replacing' an edge of a given graph H with a special graph

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called *gear*, to obtain the graph G. We study the polyhedral properties of this operation and derive sufficient conditions to generate facet defining inequalities of $STAB(G)$ starting from facet defining inequalities of $STAB(H)$. The gear composition can be iteratively applied to generate a very rich family of non-rank facet defining inequalities, that we name *geared inequalities*.

In the last section, we also show how to use this composition to build counterexamples to a conjecture on the facial structure of the stable set polytope of claw-free graphs.

We denote by $G = (V_G, E_G)$ any graph with node set V_G and edge set E_G. Given a vector $\beta \in \mathbb{R}^m$ and a subset $S \subseteq \{1, \ldots, m\}$, define $\beta_S \in \mathbb{R}^{|S|}$ as the subvector of β restricted on the indices of S and $\beta(S) = \sum_{i \in S} \beta_i$. Given a subset $S \subseteq \{1, \ldots, m\}$, we denote by $x^S \in \mathbb{R}^m$ the incidence vector of S.

A linear inequality $\sum_{j \in V_G} \pi_j x_j \leq \pi_0$ is said to be *valid* for $STAB(G)$ if it holds for all $x \in STAB(G)$. A valid inequality for $STAB(G)$ *defines* a facet of $STAB(G)$ if and only if it is satisfied as an equality by $|V_G|$ affinely independent incidence vectors of stable sets of G (called *roots*). It is well-known that each facet defining inequality that is not a non negative constraint has $\pi_j \geq 0$ for $j \in V_G$ and $\pi_0 > 0$. For short, we also denote a linear inequality $\pi^T x \leq \pi_0$ as (π, π_0) and the right hand side π_0 as *rhs*.

We denote by $\delta(v)$ the set of edges of G having v as endnode and by $N(v)$ the set of nodes of V_G adjacent to v. We define the *stability number* $\alpha(G)$ as the maximum cardinality of a stable set of G.

Let $\sum_{j \in V_G \setminus \{v\}} \pi_j x_j \leq \pi_0$ be a facet defining inequality of $STAB(G \setminus \{v\})$. Then $\sum_{j \in V_G \setminus \{v\}} \pi_j x_j + \pi_v x_v \leq \pi_0$ with $\pi_v = \pi_0 - \max_{x \in STAB(G \setminus N(v))} \pi^T x$ is facet defining for $STAB(G)$. This procedure, known as *sequential lifting* [17], can be iterated to generate facet defining inequalities (*lifted inequalities*) in a higher dimensional space.

A *k-hole* $C_k = (v_1, v_2, \ldots, v_k)$ is a chordless cycle of length k. A 5-*wheel* $W = (h : v_1, \ldots, v_5)$ consists of a 5-hole $C = (v_1, \ldots, v_5)$, called *rim* of W, and a node h (*hub* of W) adjacent to every node of C. The inequality $\sum_{i=1}^{5} x_{v_i} + 2x_h \leq 2$ is facet defining for $STAB(W)$ and it is called *5-wheel inequality*. A *gear* B is a graph of eight nodes $\{h_1, h_2, a, b_1, b_2, c, d_1, d_2\}$ such that $(h_1 : a, d_1, b_1, c, h_2)$ and $(h_2 : a, d_2, b_2, c, h_1)$ are 5-wheels (see Fig. 1); moreover, the edges of these wheels are the only edges of B.

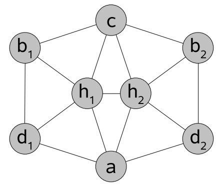


Figure 1. The gear with nodes $h_1, h_2, a, b_1, b_2, c, d_1, d_2$.

2. Gear composition

In this section we introduce the gear composition. An edge v_1v_2 of a graph H is said to be *simplicial* if $K_1 = N(v_1) \setminus \{v_2\}$ and $K_2 = N(v_2) \setminus \{v_1\}$ are two nonempty cliques of H. Notice that the two cliques K_1 and K_2 might intersect.

Definition 1 *Let $H = (V_H, E_H)$ be a graph with a simplicial edge v_1v_2 and let $B = (V_B, E_B)$ be a gear. The* gear composition *of H and B produces a new graph G such that:*

$$V_G = V_H \setminus \{v_1, v_2\} \cup V_B$$
$$E_G = E_H \setminus (\delta(v_1) \cup \delta(v_2)) \cup E_B \cup F_1 \cup F_2,$$
$$\textit{where } F_i = \{d_i u, b_i u | u \in K_i\} \textit{ for } i = 1, 2.$$

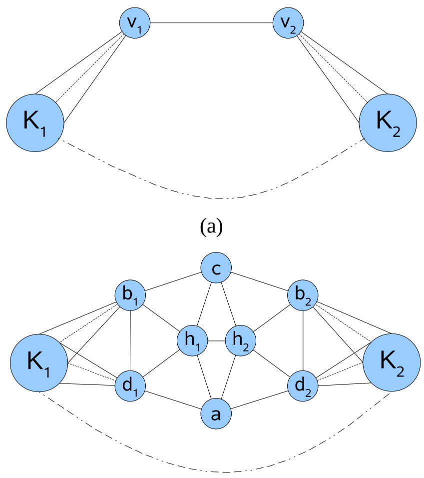
A graph G resulting from the gear composition of two graphs H and B along the simplicial edge v_1v_2 will be denoted by (H, B, v_1v_2). A sketch of how the gear composition works is shown in Fig. 2.

Definition 2 *Let $H = (V_H, E_H)$ be a graph containing the simplicial edge v_1v_2. The inequality (π, π_0) is said to be* g-extendable *with respect to v_1v_2 if it is valid for $STAB(H)$, it has $\pi_{v_1} = \pi_{v_2} = \lambda > 0$ and it is not the clique inequality $x_{v_1} + x_{v_2} \leq 1$. If $B = (V_B, E_B)$ is a gear, the following inequality*

$$\sum_{i \in V'} \pi_i x_i + \lambda \sum_{i \in V_B \setminus \{h_1, h_2\}} x_i + 2\lambda(x_{h_1} + x_{h_2}) \leq \pi_0 + 2\lambda \quad (1)$$

where $V' = V_H \setminus \{v_1, v_2\}$, is called the geared inequality *associated with (π, π_0) and will be denoted as $(\bar{\pi}, \bar{\pi}_0)$.*

In the following we show that geared inequalities are essential in the linear description of the stable set polytope of geared graphs. We first prove that they are valid inequalities for $STAB(G)$.



(b)

Figure 2. (a) A graph H with a simplicial edge v_1v_2; (b) The geared graph $G = (H, B, v_1v_2)$.

Lemma 1 *If G is a geared graph, then the geared inequality (1) is valid for $STAB(G)$.*

Proof: Let S be a stable set of G. Since each non trivial facet defining inequality of $STAB(G)$ has non negative coefficients, we can assume that S is maximal. To prove the lemma we distinguish two cases depending on the intersection of S with the subset $\{b_1, b_2, d_1, d_2\}$ of V_B.

If $|S \cap \{b_1, b_2, d_1, d_2\}| \geq 1$, then we can suppose without loss of generality that $b_1 \in S$. Then $(S \setminus V_B) \cup \{v_1\}$ is a stable set of H and therefore $\pi(S \setminus V_B) = \bar{\pi}(S \setminus V_B) \leq \pi_0 - \lambda$. Moreover, it is not difficult to check that $\bar{\pi}(S \cap V_B) \leq 3\lambda$ and thus, $\bar{\pi}(S \setminus V_B) + \bar{\pi}(S \cap V_B) \leq \pi_0 - \lambda + 3\lambda = \pi_0 + 2\lambda$.

If $|S \cap \{b_1, b_2, d_1, d_2\}| = 0$ then $S \setminus V_B$ is a stable set in H. By the maximality of S, exactly one among the sets $\{h_1\}$, $\{h_2\}$, and $\{a, c\}$, is contained in S, thus implying that $\bar{\pi}(S \cap V_B) = 2\lambda$. Hence, $\bar{\pi}(S \setminus V_B) + \bar{\pi}(S \cap V_B) \leq \pi_0 + 2\lambda$ and the thesis follows. □

Theorem 1 *Let (π, π_0) be a g-extendable inequality. If (π, π_0) is facet defining for $STAB(H)$, then the associated geared inequality (1) is facet defining for $STAB(G)$.*

Proof: Suppose $\beta^T x \leq \beta_0$ is facet defining for $STAB(G)$ and contains all the roots of (1): we show below that such inequality is equivalent to (1).

We start with the following three observations.

i) Let x^{S_1} be a root of (π, π_0) such that $v_2 \in S_1$. Consider the sets:

$$S_1^1 = S_1 \setminus \{v_2\} \cup \{h_1, d_2\}$$
$$S_1^2 = S_1 \setminus \{v_2\} \cup \{h_1, b_2\}.$$

They are stable sets of G and their incidence vectors $x^{S_1^1}$ and $x^{S_1^2}$ are roots of (1); consequently, they are roots of (β, β_0). As $\beta(S_1^1) = \beta(S_1^2) = \beta_0$, we have that $\beta_{b_2} = \beta_{d_2}$. Symmetrically, we prove that $\beta_{b_1} = \beta_{d_1}$.

ii) Let x^{S_2} be a root of (π, π_0) such that $v_1, v_2 \notin S_2$. The existence of such a root is guaranteed by the fact that (π, π_0) is not the clique inequality $x_{v_1} + x_{v_2} \leq 1$. Consider now the sets

$$S_2^1 = S_2 \cup \{h_1\}$$
$$S_2^2 = S_2 \cup \{a, c\}.$$

They are stable sets of G and their incidence vectors $x^{S_2^1}$ and $x^{S_2^2}$ are roots of (1), and hence, of (β, β_0). This implies that $\beta_a + \beta_c = \beta_{h_1}$. Replacing S_2^1 with $S_2 \cup \{h_2\}$, we get $\beta_a + \beta_c = \beta_{h_2}$ and then $\beta_{h_1} = \beta_{h_2}$.

iii) Let $x^{S'}$ be a root of (π, π_0) such that $(K_2 \cup \{v_2\}) \cap S' = \emptyset$. This root always exists because (π, π_0) is not the clique inequality defined by $K_2 \cup \{v_2\}$ (since by hypothesis $\pi_{v_1} = \pi_{v_2} = \lambda > 0$). Then $v_1 \in S'$, since otherwise $S' \cup \{v_2\}$ would be a stable set violating (π, π_0). Let $S_3 = S' \setminus \{v_1\}$: we have that $\pi(S_3) = \pi_0 - \lambda$, as (π, π_0) is g-extendable. Finally, consider the following stable sets whose incidence vectors are roots of (1):

$$S_3^1 = S_3 \cup \{d_1, d_2, c\}$$
$$S_3^2 = S_3 \cup \{b_1, b_2, a\}$$
$$S_3^3 = S_3 \cup \{b_2, h_1\}.$$

From $\beta(S_3^1) = \beta(S_3^2)$ and (i) it follows that $\beta_a = \beta_c$, and so, by (ii), $\beta_{h_1} = 2\beta_a$. From $\beta(S_3^2) = \beta(S_3^3)$ it follows that $\beta_{b_1} + \beta_a = \beta_{h_1}$, that is $\beta_{b_1} = \beta_a$. Replacing S_3^3 with $S_3 \cup \{b_1, h_2\}$, we get $\beta_{b_2} = \beta_a$.

So, by (i)-(iii), we have that $\beta_v = \beta_{d_1}$ for each $v \in V_B \setminus \{h_1, h_2\}$ and $\beta_{h_1} = \beta_{h_2} = 2\beta_{d_1}$.

Let M be a matrix whose rows are $|V_H|$ incidence vectors of stable sets of H which are linearly independent roots of (π, π_0), i.e.,

$$M\pi = \pi_0 \mathbb{1}. \tag{2}$$

Any stable set $\tilde{S}$ of H can be transformed into a stable set S of G as follows: set $S = \tilde{S} \setminus \{v_1, v_2\} \cup S_B$, where S_B is a stable set of B such that $d_i \in S_B$ if and only if $v_i \in \tilde{S}$ for $i = 1, 2$. It is not difficult to verify that if $x^{\tilde{S}}$ defines a root of (π, π_0) then S_B can be chosen so that x^S defines a root of (1) such that $\beta(S \cap \{h_1, h_2, a, c\}) = 2\beta_{d_1}$. By replacing V_H with $V' = V_H \setminus \{v_1, v_2\} \cup \{d_1, d_2\}$, we have $M\beta_{V'} = (\beta_0 - 2\beta_{d_1})\mathbb{1}$ and by (2),

$$\beta_{V'} = (\beta_0 - 2\beta_{d_1})M^{-1}\mathbb{1} = \frac{\beta_0 - 2\beta_{d_1}}{\pi_0}\pi.$$

In particular, we have

$$\beta_{d_1} = \frac{\beta_0 - 2\beta_{d_1}}{\pi_0}\pi_{v_1} = \frac{\beta_0 - 2\beta_{d_1}}{\pi_0}\lambda. \qquad (3)$$

Then $\beta_{d_1} > 0$ and, without loss of generality, we can fix $\beta_{d_1} = \pi_{v_1} = \lambda$; consequently, we have that

$$\begin{aligned}
&\beta_0 = \pi_0 + 2\lambda, \\
&\beta_v = \pi_v \qquad \text{for each } v \in V_H \setminus \{v_1, v_2\}, \\
&\beta_v = \lambda \qquad \text{for each } v \in V_B \setminus \{h_1, h_2\}, \\
&\beta_{h_1} = \beta_{h_2} = 2\lambda,
\end{aligned}$$

and the theorem follows. □

The following example shows a geared graph obtained by two applications of the gear composition to a 5-hole and the corresponding geared inequalities.

Example 1 Consider the 5-hole $C_5 = (v_1^1, v_2^1, u, v_2^2, v_1^2)$ and the geared 5-hole $H_1 = (C_5, B^1, v_1^1 v_2^1)$ in Fig. 3. Two simplicial edges are emphasized as thick lines.

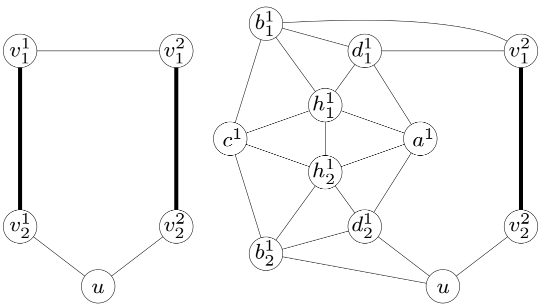


Figure 3. A 5-hole and a geared 5-hole

As the 5-hole inequality $x(V_{C_5}) \le 2$ is facet defining for $STAB(C_5)$ and g-extendable, we have, by Theorem 1, that

$$x(V_{H_1} \setminus \{h_1^1, h_2^1\}) + 2x_{h_1^1} + 2x_{h_2^1} \le 4 \qquad (4)$$

is facet defining for $STAB(H_1)$.

Observe that the gear composition can be applied iteratively provided that the graph involved in the operation at the i-th step has a simplicial edge. For instance, the graph H_1 in the Example 1 contains $v_1^2 v_2^2$ and thus it can be composed with another gear B^2 to obtain the graph $G = (H_1, B^2, v_1^2 v_2^2)$ shown in Fig. 4. A further application of Theorem 1 yields the following "double" geared facet defining inequality

$$x(V_G \setminus T) + 2x(T) \le 6, \qquad (5)$$

where $T = \{h_1^1, h_2^1, h_1^2, h_2^2\}$.

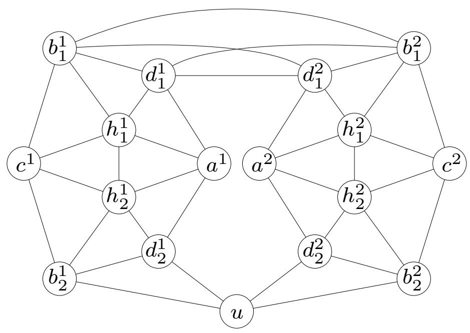


Figure 4. A double geared graph

Notice that other geared inequalities appear in the linear description of $STAB(G)$. In fact, the following inequalities:

$$x(V_{H_1} \setminus A) \le 3 \qquad (6)$$

where $A \in \{\{d_2^1, a^1\}, \{d_1^1, a^1\}, \{b_2^1, c^1\}, \{b_1^1, c^1\}, \{a^1, c^1\}\}$, are rank facet defining for $STAB(H_1)$ and they are also g-extendable with respect to $v_1^2 v_2^2$. Hence, by Theorem 1, each of the inequalities (6) generates a geared inequality which is facet defining for $STAB(G)$ and different from (5).

Simmetrically, other geared inequalities are generated by performing gear compositions in a different order: first build $H_2 = (C_5, B^2, v_1^2 v_2^2)$, and then $G = (H_2, B^1, v_1^1 v_2^1)$ as the gear composition of H_2 and B^1. As above, the first gear composition generates five rank inequalities (similar to (6)) which are facet defining for $STAB(H_2)$ and g-extendable while the second gear composition generates their associated geared inequalities. All the inequalities mentioned so far are different and, by Theorem 1, they are all facet defining for $STAB(G)$. It follows that two applications of the gear composition to a 5-hole have produced 11 geared inequalities which are facet defining for the stable set polytope of G. □

The situation illustrated above may be generalized to the case when G contains k gears: in this case, any subset of gears may be possibly involved in a facet defining inequality, thus producing an exponential number of geared inequalities. To see this we need a preliminary result:

Theorem 2 *Let (π, π_0) be a g-extendable inequality. If (π, π_0) is facet defining for $STAB(H)$, then the inequality*

$$\sum_{i \in V_G \setminus \{v_1, v_2\}} \pi_i x_i + \pi_{v_1} \sum_{i \in V_B \setminus \{a, c\}} x_i \leq \pi_0 + \pi_{v_1} \qquad (7)$$

is facet defining for $STAB(G)$.

Proof: Consider the graph G' obtained from H by subdividing the edge $e = v_1 v_2$ with two nodes h_1 and h_2 and renaming v_i as d_i, $i = 1, 2$. Clearly G' is a subgraph of G and, by a result of Wolsey [21] on edge subdivisions, the following inequality

$$\sum_{i \in V_G \setminus \{v_1, v_2\}} \pi_i x_i + \pi_{v_1} \sum_{i \in \{d_1, h_1, h_2, d_2\}} x_i \leq \pi_0 + \pi_{v_1}$$

is facet defining for $STAB(G')$. This inequality can be lifted to yield a facet defining inequality of $STAB(G)$ by observing that b_1 and b_2 can be lifted with coefficient π_{v_1}, and then a and c can be lifted with coefficient zero. This completes the proof. □

We now show an example where a linear number of gear compositions yields an exponential number of facet defining geared inequalities. Consider the graph H as a $(2k+1)$-hole $(v_1, v_2, \ldots, v_{2k+1})$ and the following set $F = \{e_i = v_{2i} v_{2i+1} : i = 1, \ldots, k\}$ of disjoint simplicial edges of H. Let $F' = \{e_{i_1}, e_{i_2}, \ldots, e_{i_h}\} \subseteq F$ such that $i_1 < i_2 < \cdots < i_h \leq k$ and let $G_{F'}$ denote the graph obtained from H by iteratively applying the gear composition on the edge e_{i_j} for $j = 1, 2, \ldots, h$ (notice that, since the edges in F are disjoint, the edges in $F \setminus F'$ remain simplicial in $G_{F'}$). Denote by T the set of hubs' pairs belonging to the h gears of $G_{F'}$. As $x(V_H) \leq k$ is facet defining for $STAB(H)$ and g-extendable with respect to each edge of F, by iteratively applying Theorem 1, we have that the geared inequality

$$\sum_{v \in V_{G_{F'}} \setminus T} x_v + 2 \sum_{v \in T} x_v \leq k + 2h$$

is facet defining for $STAB(G_{F'})$. Moreover, this inequality may be extended to a facet defining inequality for $STAB(G_F)$ by applying Theorem 2 to the $k - h$ edges of $F \setminus F'$. Since this procedure can be applied to any subset F' of F, we have that an exponential number of geared inequalities appear in the linear description of $STAB(G_F)$.

3. Geared rank inequalities

In this section we show how to use the gear composition to build a new class of inequalities that *naturally* extend the inequalities supported by the line graph of *hypomatchable graphs* [5].

It is well-known that the stable set polytope $STAB(L(G))$ of a line graph $L(G)$ is equivalent to the matching polytope $\mathcal{M}(G)$ of G. Since the only nontrivial inequalities describing $\mathcal{M}(G)$ are rank inequalities [4], we have that the same holds for $STAB(L(G))$.

However these inequalities are not sufficient to describe $STAB(G)$ as long as G is not a line graph and the structure of $STAB(G)$ becomes quite complex even for those graphs that are natural generalizations of line graphs as the *claw-free graphs*, i.e., graphs such that the neighborhood of each node has no stable set of size three. For claw-free graphs, as for the line graphs, the optimization problem over the stable set polytope is polynomial time solvable [14] and, by a well-known result of Grötschel, Lovász and Schrijver (see [11]), the same holds for the separation problem. Thus, it is expected that $STAB(G)$ has a *nice* linear description when G is claw-free. But up to now no explicit set of facet defining inequalities is known despite many research efforts [8,10,12,16,18] and several disproved conjectures [10].

A complete linear description of $STAB(G)$ was given by Eisenbrand et al. [6] for a subclass of claw-free graphs: the quasi-line graphs (a graph is quasi-line if the neighborhood of each node can be partitioned into two cliques). These graphs generalize the line graphs and their stable set polytope is completeley described by the *clique family inequalities* [16] which are a generalization of the *Edmonds' inequalities* [2].

It remains open the problem of finding the linear description of $STAB(G)$ when G is claw-free and not quasi-line. It is well-known [7] that any claw-free graph G which is not quasi-line and has $\alpha(G) \geq 4$, contains at least one 5-wheel and no odd antihole $\bar{C}_{2p+1}$ with $p \geq 3$. Recently, Stauffer [19] conjectured that:

Conjecture 1 *The stable set polytope of a claw-free graph G which is not quasi-line and has $\alpha(G) \geq 4$ is described by: non-negativity inequalities, rank inequalities and (lifted) 5-wheel inequalities.*

To build counterexamples to the above conjecture it suffices to observe that each geared inequality is

- *supported by a graph G that is not quasi-line (since it contains a 5-wheel)* and moreover, is a
- *non-rank valid inequality for $STAB(G)$ with rhs greater than 2.*

Thus, any geared inequality that is facet defining for $STAB(G)$, when G is claw-free with $\alpha(G) \geq 4$, is a counterexample to Conjecture 1 because G is not quasi-line and the rhs of the geared inequality is greater than the rhs of a (lifted) 5-wheel inequality which is 2. Instances of such inequalities are provided in Example 1.

We define recursively the family $\mathcal{G}_{\mathcal{R}}$ of *geared rank inequalities* as the family of geared inequalities associated with inequalities that: either are rank inequalities or belong to $\mathcal{G}_{\mathcal{R}}$. By repeated applications of Definition 2, we have that the coefficients of geared rank inequalities are all 1's apart from some pairs of gears' hubs which have coefficient 2; moreover, their rhs is greater than 2.

Geared rank inequalities play a role in the study of $STAB(G)$ when G is claw-free. The results of this paper imply that the geared rank inequalities have to be necessarily added to the defining linear system of $STAB(G)$. Moreover, the recent decomposition theorem for claw-free graphs of Chudnovsky and Seymour [1] made us quite confident that geared rank inequalities are also sufficient to give a linear description of $STAB(G)$ when G is claw-free, not quasi-line and has stability number greater than 3. This led us to conjecture that

Conjecture 2 *The stable set polytope of a claw-free graph G which is not quasi-line and has $\alpha(G) \geq 4$ is described by:*

non-negativity inequalities
rank inequalities
(lifted) 5-wheel inequalities
(lifted) geared rank inequalities.

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