ISTITUTO DI ANALISI DEI SISTEMI ED INFORMATICA
"Antonio Ruberti"
CONSIGLIO NAZIONALE DELLE RICERCHE

A. Galluccio, C. Gentile, P. Ventura

NEW FACET DEFINING INEQUALITIES FOR THE STABLE SET POLYTOPE

R. 641 Maggio 2006

Anna Galluccio – Istituto di Analisi dei Sistemi ed Informatica "Antonio Ruberti" del CNR, Viale Manzoni 30 - 00185 Roma, Italy. Email: galluccio@iasi.cnr.it.

Claudio Gentile – Istituto di Analisi dei Sistemi ed Informatica "Antonio Ruberti" del CNR, Viale Manzoni 30 - 00185 Roma, Italy. Email: gentile@iasi.cnr.it.

Paolo Ventura – Istituto di Analisi dei Sistemi ed Informatica "Antonio Ruberti" del CNR, Viale Manzoni 30 - 00185 Roma, Italy. Email: ventura@iasi.cnr.it.

This work has been partially supported by the EU Marie Curie Research Training Network no. 504438 ADONET

ISSN: 1128–3378

Collana dei Rapporti dell’Istituto di Analisi dei Sistemi ed Informatica “Antonio Ruberti”, CNR

viale Manzoni 30, 00185 ROMA, Italy

tel. ++39-06-77161
fax ++39-06-7716461
email: iasi@iasi.rm.cnr.it
URL: http://www.iasi.rm.cnr.it

Abstract

We present a new graph composition that produces a graph G from a given graph H and a fixed graph B called *gear* and we study its polyhedral properties. This composition yields counterexamples to a conjecture on the facial structure of $STAB(G)$ when G is claw-free.

Key words: stable set polytope, graph composition, polyhedral combinatorics.

1. Introduction

Given a graph $G = (V, E)$ and a vector $w \in \mathbb{Q}_+^V$ of node weights, the *stable set problem* is the problem of finding a set of pairwise nonadjacent nodes *(stable set)* of maximum weight.

The *stable set polytope*, denoted by $STAB(G)$, is the convex hull of the incidence vectors of the stable sets of G. A linear system $Ax \leq b$ is said to be *defining* for $STAB(G)$ if $STAB(G) = \{x : Ax \leq b\}$. The *facet defining inequalities* for $STAB(G)$ are those inequalities that constitute the unique nonredundant defining linear system of $STAB(G)$.

Clearly, finding the defining linear system for $STAB(G)$ is equivalent to transform the original optimization problem into the linear program $\max\{w^T x : Ax \leq b\}$ and, being the stable set problem NP-hard, it is unlikely to find such a system for general graphs.

Nevertheless the facial structure of the stable set polytope has been one of the most studied problems in polyhedral combinatorics. Here is a non-exhaustive list of results related with the study of facets of $STAB(G)$: facet producing graphs [22, 26, 20, 14, 4, 21], t and h-perfectness [15], lifting operations for polyhedra [22, 20], characterization of $STAB(G)$ when G is series-parallel [18], odd K_4-free [13] or quasi-line [9].

Besides the description of new classes of facets, it is of interest to find composition procedures that enable to build new families of facets for $STAB(G)$ starting from facets of a lower dimensional polytope. These compositions are usually based on graph compositions: for example, sequential lifting is based on the extension of a graph with an additional node, the Wolsey's lifting procedure [27] is based on edge subdivision, Chvátal's compositions of polyhedra [7] are based on node substitution and clique identification, and so on [1, 2, 3].

In this paper, we present a new graph composition, named *gear composition*, which consists of 'replacing' an edge of a given graph H with a special graph called *gear*, so obtaining the graph G. We study the polyhedral properties of this operation and derive sufficient conditions to generate facet defining inequalities for $STAB(G)$ starting from facet defining inequalities for $STAB(H)$. The gear composition can be iteratively applied and generates a very rich family of non-rank facet defining inequalities, that we name *geared inequalities*. For these inequalities we show that the separation problem can be solved in polynomial time in some special cases.

In the last section, we also show how to use this composition to build counterexamples to a conjecture on the facial structure of the stable set polytope of claw-free graphs.

We now introduce some notation and basic definitions. We denote by $G = (V_G, E_G)$ any graph with node set V_G and edge set E_G. Given a vector $\beta \in \mathbb{R}^m$ and a subset $S \subseteq \{1, \ldots, m\}$, define $\beta_S \in \mathbb{R}^{|S|}$ as the subvector of β restricted on the indices of S and $\beta(S) = \sum_{i \in S} \beta_i$. Given a subset $S \subseteq \{1, \ldots, m\}$, we denote by $x^S \in \mathbb{R}^m$ the incidence vector of S.

A linear inequality $\sum_{j \in V_G} \pi_j x_j \leq \pi_0$ is said to be *valid* for $STAB(G)$ if it holds for all $x \in STAB(G)$. A valid inequality for $STAB(G)$ *defines* a facet of $STAB(G)$ if and only if it is satisfied as an equality by $|V_G|$ affinely independent incidence vectors of stable sets of G (called *roots*). If the support of a facet defining inequality coincides with V_G, we say that the graph G *produces* the corresponding facet. For short, we also denote a linear inequality $\pi^T x \leq \pi_0$ as (π, π_0) and the right hand side π_0 as *rhs*.

We denote by $\delta(v)$ the set of edges of G having v as endnode and by $N(v)$ the set of nodes of V_G adjacent to v. If $w : V_G \rightarrow \mathbb{Q}_+$ is any weighting of the nodes of G, then $\alpha(G, w)$ denotes the maximum weight of a stable set of G. We refer to $\alpha(G) = \alpha(G, \mathbb{1})$ ($\mathbb{1}$ being the vector of all ones) as the *stability number* of G.

A *k-hole* $C_k = (v_1, v_2, \ldots, v_k)$ is a chordless cycle of length k. A 5-*wheel* $W = (h : v_1, \ldots, v_5)$ is a graph consisting of a 5-hole $C = (v_1, \ldots, v_5)$, called *rim* of W, and a node h (*hub* of W) adjacent to every node of C.

A *gear* B is a graph of eight nodes $\{h_1, h_2, a, b_1, b_2, c, d_1, d_2\}$ such that $(h_1 : a, d_1, b_1, c, h_2)$ and

$(h_2 : a, d_2, b_2, c, h_1)$ are 5-wheels (see Fig. 1); moreover, the edges of these wheels are the only edges of B.

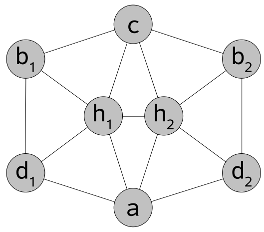

Figure 1: The gear with nodes $h_1, h_2, a, b_1, b_2, c, d_1, d_2$.

2. Gear composition

In this section, we introduce a graph operation, named *gear composition*, and we show some of its polyhedral properties. In particular we show under which conditions the gear composition preserves the property of a graph of being facet producing.

An edge v_1v_2 of a graph H is said to be *simplicial* if $K_1 = N(v_1) \setminus \{v_2\}$ and $K_2 = N(v_2) \setminus \{v_1\}$ are two nonempty cliques of H.

Definition 2.1. *Let $H = (V_H, E_H)$ be a graph with a simplicial edge v_1v_2 and let $B = (V_B, E_B)$ be a gear. The* gear composition *of H and B produces a new graph $G = (H, B, v_1v_2)$, called* geared graph *such that:*

$$V_G = V_H \setminus \{v_1, v_2\} \cup V_B$$
$$E_G = E_H \setminus (\delta(v_1) \cup \delta(v_2)) \cup E_B \cup F_1 \cup F_2, \text{ where } F_i = \{d_iu, b_iu | u \in K_i\} \text{ for } i = 1, 2$$

A sketch of how the gear composition works is shown in Fig. 2.

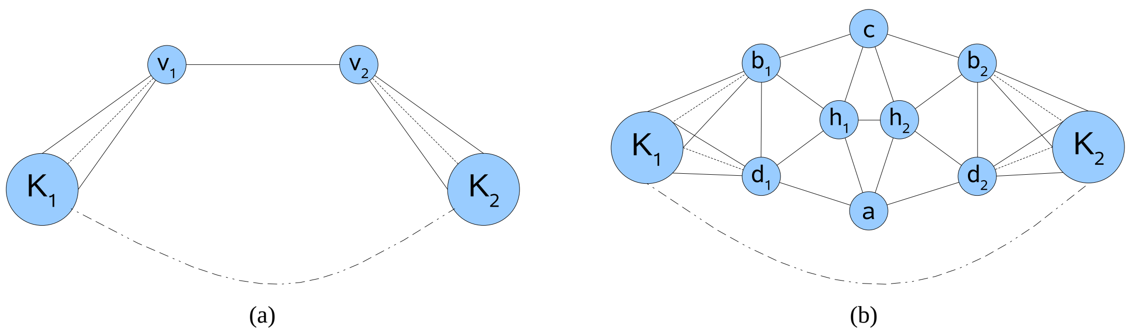

Figure 2: (a) A graph H with a simplicial edge v_1v_2; (b) The geared graph $G = (H, B, v_1v_2)$.

Definition 2.2. *Let $H = (V_H, E_H)$ be a graph containing the simplicial edge v_1v_2. The inequality (π, π_0) is said to be* g-extendable *with respect to v_1v_2 if it is valid for $STAB(H)$, it has $\pi_{v_1} = \pi_{v_2} = \lambda > 0$ and it is not the clique inequality $x_{v_1} + x_{v_2} \leq 1$. If $B = (V_B, E_B)$ is a gear, the following inequality produced by $G = (H, B, v_1v_2)$*

$$\sum_{i \in V_H \setminus \{v_1, v_2\}} \pi_i x_i + \lambda \sum_{i \in V_B \setminus \{h_1, h_2\}} x_i + 2\lambda(x_{h_1} + x_{h_2}) \leq \pi_0 + 2\lambda \tag{1}$$

is called the geared inequality *associated with* (π, π_0) *and will be denoted as* $(\bar{\pi}, \bar{\pi}_0)$.

In the following we show that geared inequalities are essential in the linear description of the stable set polytope of geared graphs. We first prove that they are valid inequalities for $STAB(G)$.

Lemma 2.3. *If G is a geared graph, then the geared inequality (1) is valid for $STAB(G)$.*

Proof. Let S be a maximal stable set of G. To prove the lemma we distinguish three cases depending on the intersection of S with the subset $\{b_1, b_2, d_1, d_2\}$ of V_B.

If $|S \cap \{b_1, b_2, d_1, d_2\}| = 2$, then $K_1 \cap S = K_2 \cap S = \emptyset$ and the set $S \setminus V_B$ is a stable set of H. It follows that $\pi(S \setminus V_B) = \bar{\pi}(S \setminus V_B) \leq \pi_0 - \lambda$, since otherwise the stable set $S \setminus V_B \cup \{v_1\}$ of H would violate (π, π_0). Moreover, it is not difficult to check that $\bar{\pi}(S \cap V_B) \leq 3\lambda$ and thus, $\bar{\pi}(S \setminus V_B) + \bar{\pi}(S \cap V_B) \leq \pi_0 - \lambda + 3\lambda = \pi_0 + 2\lambda$.

If $|S \cap \{b_1, b_2, d_1, d_2\}| = 1$, we first suppose that $b_1 \in S$; then, $b_2, h_1, c, d_1, d_2 \notin S$ and $S \cap V_B$ contains exactly one node in $\{h_2, a\}$. Since $S \cap K_1 = \emptyset$, $(S \setminus V_B) \cup \{v_1\}$ is a stable set of H; hence, as in the previous case, $\pi(S \setminus V_B) = \bar{\pi}(S \setminus V_B) \leq \pi_0 - \lambda$ and the result follows. The cases with $b_2 \in S$, $d_1 \in S$, or $d_2 \in S$ are analogous.

In the last case, $|S \cap \{b_1, b_2, d_1, d_2\}| = 0$ and $S \setminus V_B$ is a stable set in H. By the maximality of S, exactly one among the sets $\{h_1\}$, $\{h_2\}$, and $\{a, c\}$, is contained in S, thus implying that $\bar{\pi}(S \cap V_B) = 2\lambda$. Hence, $\bar{\pi}(S \setminus V_B) + \bar{\pi}(S \cap V_B) \leq \pi_0 + 2\lambda$ and the thesis follows. ∎

Theorem 2.4. *Let (π, π_0) be a g-extendable inequality. If (π, π_0) is facet defining for $STAB(H)$, then the associated geared inequality (1) is facet defining for $STAB(G)$.*

Proof. Suppose $\beta^T x \leq \beta_0$ is facet defining for $STAB(G)$ and contains all the roots of (1): we show below that such inequality is equivalent to (1).

We start with the following three observations.

i) Let x^{S_1} be a root of (π, π_0) such that $v_2 \in S_1$. Consider the sets:

$$S_1^1 = S_1 \setminus \{v_2\} \cup \{h_1, d_2\}$$
$$S_1^2 = S_1 \setminus \{v_2\} \cup \{h_1, b_2\}.$$

They are stable sets of G and their incidence vectors $x^{S_1^1}$ and $x^{S_1^2}$ are roots of (1); consequently, they are roots of (β, β_0). As $\beta(S_1^1) = \beta(S_1^2) = \beta_0$, we have that $\beta_{b_2} = \beta_{d_2}$. Symmetrically, we prove that $\beta_{b_1} = \beta_{d_1}$.

ii) Let x^{S_2} be a root of (π, π_0) such that $v_1, v_2 \notin S_2$. The existence of such a root is guaranteed by the fact that (π, π_0) is not the clique inequality $x_{v_1} + x_{v_2} \leq 1$. Consider now the sets

$$S_2^1 = S_2 \cup \{h_1\}$$
$$S_2^2 = S_2 \cup \{a, c\}.$$

They are stable sets of G and their incidence vectors $x^{S_2^1}$ and $x^{S_2^2}$ are roots of (1), and hence, of (β, β_0). This implies that $\beta_a + \beta_c = \beta_{h_1}$. Replacing S_2^1 with $S_2 \cup \{h_2\}$, we get $\beta_a + \beta_c = \beta_{h_2}$ and then $\beta_{h_1} = \beta_{h_2}$.

iii) Let $x^{S'}$ be a root of (π, π_0) such that $(K_2 \cup \{v_2\}) \cap S' = \emptyset$. This root always exists because (π, π_0) is not the clique inequality defined by $K_2 \cup \{v_2\}$ (since by hypothesis $\pi_{v_1} = \pi_{v_2} = \lambda > 0$). Then $v_1 \in S'$, since otherwise $S' \cup \{v_2\}$ would be a stable set violating (π, π_0). Let $S_3 = S' \setminus \{v_1\}$: we have that $\pi(S_3) = \pi_0 - \lambda$, as (π, π_0) is g-extendable. Finally, consider the following stable sets whose incidence vectors are roots of (1):

$$\begin{array}{l} S_3^1 = S_3 \cup \{d_1, d_2, c\} \\ S_3^2 = S_3 \cup \{b_1, b_2, a\} \\ S_3^3 = S_3 \cup \{b_2, h_1\}. \end{array}$$

From $\beta(S_3^1) = \beta(S_3^2)$, and (i) it follows that $\beta_a = \beta_c$, and so, by (ii), $\beta_{h_1} = 2\beta_a$. From $\beta(S_3^2) = \beta(S_3^3)$ it follows that $\beta_{b_1} + \beta_a = \beta_{h_1}$, that is $\beta_{b_1} = \beta_a$. Replacing S_3^3 with $S_3 \cup \{b_1, h_2\}$, we get $\beta_{b_2} = \beta_a$.

Without loss of generality, we can fix $\beta_{d_1} = \pi_{v_1} = \lambda$, and so, by (i)-(iii), we have that $\beta_v = \lambda$ for each $v \in V_B \setminus \{h_1, h_2\}$ and $\beta_{h_1} = \beta_{h_2} = 2\lambda$.

Let M be a matrix whose rows are $|V_H|$ incidence vectors of stable sets of H which are linearly independent roots of (π, π_0), i.e.,

$$M\pi = \pi_0 \mathbb{1}. \tag{2}$$

Any stable set $\tilde{S}$ of H can be transformed into a stable set S of G as follows: set $S = \tilde{S} \setminus \{v_1, v_2\} \cup S_B$, where S_B is a stable set of B such that $d_i \in S_B$ if and only if $v_i \in \tilde{S}$ for $i = 1, 2$. It is not difficult to verify that if $x^{\tilde{S}}$ defines a root of (π, π_0) then S_B can be chosen so that x^S defines a root of (1) such that $\beta(S \cap \{h_1, h_2, a, b_1, b_2, c\}) = 2\lambda$. By replacing V_H with $V' = V_H \setminus \{v_1, v_2\} \cup \{d_1, d_2\}$, we have $M\beta_{V'} = (\beta_0 - 2\lambda)\mathbb{1}$ and by (2),

$$\beta_{V'} = (\beta_0 - 2\lambda)M^{-1}\mathbb{1} = \frac{\beta_0 - 2\lambda}{\pi_0}\pi.$$

In particular, we have

$$\beta_{d_1} = \frac{\beta_0 - 2\lambda}{\pi_0}\pi_{v_1} = \frac{\beta_0 - 2\lambda}{\pi_0}\lambda. \tag{3}$$

As, by assumption, $\beta_{d_1} = \lambda$, we have that

$$\begin{array}{ll} \beta_0 = \pi_0 + 2\lambda, & \\ \beta_v = \pi_v & \text{for each } v \in V_H \setminus \{v_1, v_2\}, \\ \beta_v = \lambda & \text{for each } v \in V_B \setminus \{h_1, h_2\}, \\ \beta_{h_1} = \beta_{h_2} = 2\lambda, & \end{array}$$

and the theorem follows. ∎

The following example shows a geared graph obtained by a single application of the gear composition to a 5-hole and the relative geared inequality.

Example 2.1. Consider the 5-hole $C_5 = (v_1^1, v_2^1, u, v_1^2 v_2^2)$ and the geared 5-hole $H_1 = (C_5, B^1, v_1^1 v_2^1)$ depicted in Fig. 3. Two simplicial edges are emphasized as thick lines.

As the 5-hole inequality $x(V_{C_5}) \leq 2$ is facet defining for $STAB(C_5)$, by Theorem 2.4

$$x(V_{H_1} \setminus \{h_1^1, h_2^1\}) + 2x_{h_1^1} + 2x_{h_2^1} \leq 4 \tag{4}$$

is facet defining for $STAB(H_1)$. □

Observe that the gear composition can be applied iteratively provided that the graph involved in the operation at the i-th step has a simplicial edge. For instance, the graph H_1 in the Example 1 contains $v_1^2 v_2^2$ and thus it can be composed with another gear B^2 to obtain the graph $G = (H_1, B^2, v_1^2 v_2^2)$ shown

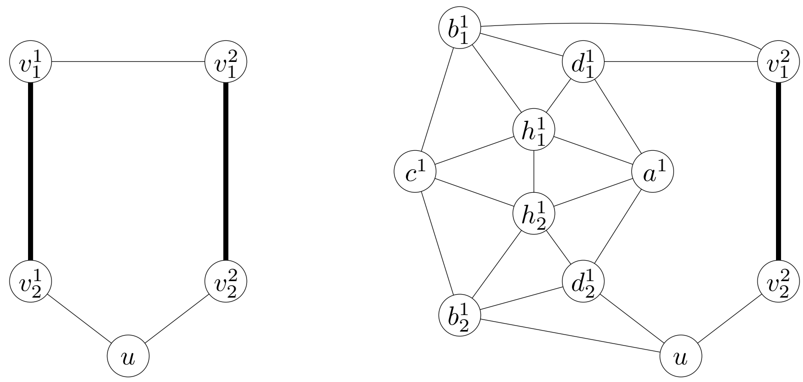

Figure 3: A 5-hole and a geared 5-hole

in Fig. 4. A further application of Theorem 2.4 yields the following "double" geared facet defining inequality

$$x(V_G \setminus T) + 2x(T) \leq 6, \tag{5}$$

where $T = \{h_1^1, h_2^1, h_1^2, h_2^2\}$.

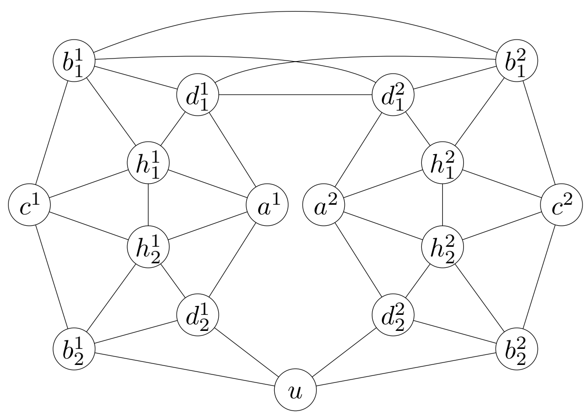

Figure 4: A double geared graph

As expected, in the description of the stable set polytope of H_1, there are facet defining inequalities which are not geared inequalities and whose supporting graphs contain nodes of both C_5 and B^1. Therefore, when a new gear composition is performed on H_1 using another simplicial edge and another gear B^2, beyond inequality (5) several geared inequalities appear in the linear description of the stable set polytope of the resulting graph G.

We explain what happens with an example. Consider the graph H_1 of Fig. 3: the following rank inequalities

$$\begin{array}{c} x(V_{H_1} \setminus \{d_2^1, a^1\}) \leq 3 \\ x(V_{H_1} \setminus \{d_1^1, a^1\}) \leq 3 \\ x(V_{H_1} \setminus \{b_2^1, c^1\}) \leq 3 \\ x(V_{H_1} \setminus \{b_1^1, c^1\}) \leq 3 \\ x(V_{H_1} \setminus \{a^1, c^1\}) \leq 3 \end{array} \tag{6}$$

are facet defining for $STAB(H_1)$ and they are also g-extendable with respect to $v_1^2 v_2^2$. Hence, by Theorem 2.4, each of the inequalities (6) generates a geared inequality which is facet defining for $STAB(G)$

and it is different from (5).

But the situation turns out to be even more complex! In fact, the graph G of Fig. 4 may also be constructed by applying the gear compositions in a different order: first construct $H_2 = (C_5, B^2, v_1^2 v_2^2)$, and then $G = (H_2, B^1, v_1^1 v_2^1)$ as the gear composition of H_2 and B^1. As noticed before, the first gear composition generates five rank inequalities (similar to (6)) which are facet defining for $STAB(H_2)$ while the second gear composition generates their associated geared inequalities. All the inequalities mentioned so far are different and, by Theorem 2.4, they are all facet defining for $STAB(G)$. It follows that two applications of the gear composition to a 5-hole have produced 11 geared inequalities which are facet producing for the stable set polytope of G.

It is not difficult to see that this situation may be generalized to the case when G contains k gears: in this case, any subset of gears may be possibly involved in a facet defining inequality. Therefore, an exponential number of geared inequalities might appear in the facial description of $STAB(G)$. One has still to be careful since it is not true that any possible subset of gear's hubs appears with a 2λ coefficient in a facet defining inequality. In the following example we show how a graph obtained by two gear compositions produces facet defining inequalities with each gear, but not with both.

Example 2.2. Let G be the graph depicted in Fig. 5. It is easy to see that G may be obtained by the

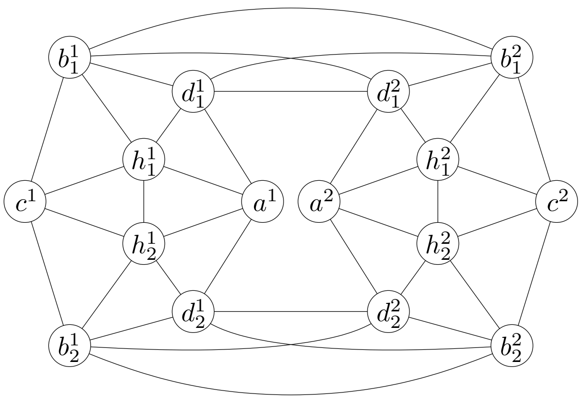

Figure 5: A graph with two alternative gears

composition of two gears B^1 and B^2 starting from a cycle $C_4 = (v_1^1, v_2^1, v_1^2, v_2^2)$ with simplicial edges $v_1^1 v_2^1$ and $v_1^2 v_2^2$. As C_4 does not produce a facet defining inequality, Theorem 2.4 cannot be applied right after the first gear composition is performed. Nevertheless, three geared inequalities which are facet defining for $STAB(G)$ will be generated by the second gear composition. In fact, let $L_1 = (C_4, B^1, v_1^1 v_2^1)$ be the graph obtained from C_4 and B^1 and let $G = (L_1, B^2, v_1^2 v_2^2)$. Theorem 2.4 can be applied to each of the following facet defining (rank) inequalities of $STAB(L_1)$:

$$
\begin{aligned}
&x(\{v_1^2, v_2^2, b_2^1, c^1, b_1^1\}) \leq 2 \\
&x(\{v_1^2, v_2^2, d_2^1, a^1, d_1^1\}) \leq 2 \\
&x(V_{L_1}) \leq 3,
\end{aligned}
$$

thus producing three geared inequalities that are facet defining for $STAB(G)$. Symmetrically, there are three geared facet defining inequalities for $STAB(G)$ generated by considering the construction of G as $G = (L_2, B^1, v_1^1, v_2^1)$, where $L_2 = (C_4, B^1, v_1^1 v_2^1)$.

□

3. Separation of geared inequalities

Although the stable set problem is one of the most studied problems in polyhedral combinatorics and many facet defining inequalities are known, only few classes of valid inequalities are known to be polynomially separable. Among them, there are the odd cycle inequalities ([12], [8]) and the blossom inequalities for line graphs [23]. Odd wheel inequalities can be separated by modifying the Padberg and Rao's procedure for the odd cycle inequalities [15]. Other separation procedures have been proposed for generalization of wheels [2, 4]. Here, we propose polynomial time separation algorithms for geared odd cycle inequalities and geared blossom inequalities, i.e., geared inequalities associated with odd cycle inequalities and blossom inequalities, respectively.

3.1. Geared odd cycle inequalities

Given a cycle $C = (V_C, E_C)$ with an odd number of nodes, the odd cycle inequality is

$$\sum_{i \in V_C} x_i \leq \frac{|V_C| - 1}{2}.$$

In the following we denote by $G \setminus uw$ the graph $(V_G, E_G \setminus \{uw\})$.

Theorem 3.1. *Given a graph G, geared odd cycle inequalities can be separated in polynomial time over the set of points satisfying the edge formulation for STAB(G).*

Proof. First observe that the number of gears in G is $\mathrm{O}(n)$, since two gears cannot intersect. Moreover, it is not difficult to prove that all the gears of G can be identified in time $\mathrm{O}(n^2)$, since one gear is defined as an induced subgraph of G with a fixed number of vertices. Once a gear B is fixed, consider the graph $H = (V_H, E_H)$ obtained from G by substituting the gear B with the edge v_1v_2, i.e., $V_H = V_G \setminus V_B \cup \{v_1, v_2\}$ and $E_H = E_G \setminus (E_B \cup F_1 \cup F_2) \cup L_1 \cup L_2 \cup \{v_1v_2\}$, where $F_i = \{d_iu, b_iu | u \in K_i\}$ and $L_i = \{v_iu | u \in K_i\}$, for $i = 1, 2$. Then, any geared odd cycle inequality valid for $STAB(G)$ is of the form:

$$\sum_{i \in V_C \setminus \{v_1, v_2\}} x_i + \gamma_B(x) \leq \frac{|V_C| - 1}{2} + 2, \tag{7}$$

where $\gamma_B(x) = \sum_{i \in V_B \setminus \{h_1, h_2\}} x_i + 2x_{h_1} + 2x_{h_2}$, and $C = (V_C, E_C)$ is an odd cycle of H containing the simplicial edge v_1v_2. Define the edge weights y_e for $e \in E_H$ as follows:

- $y_{uw} = 1 - x_u - x_w$ for each $uw \in E_G \setminus (\delta(v_1) \cup \delta(v_2))$,
- $y_{v_iu} = 1 - x_u$ for each $v_iu \in \delta(v_i) \setminus \{v_1v_2\}$, $i = 1, 2$.

Inequality (7) is equivalent to

$$\sum_{e \in E_C \setminus \{v_1v_2\}} y_e \geq 2\gamma_B(x) - 4.$$

Therefore, a fractional point x^* violates an inequality of type (7) if and only if there exists an odd cycle C in H, containing the edge v_1v_2, and such that

$$\sum_{e \in E_C \setminus \{v_1v_2\}} y^*_e < 2\gamma_B(x^*) - 4. \tag{8}$$

Observe now that, for a fixed gear B, the *rhs* of (8) is a constant and, if x^* satisfies the edge formulation, we have that $y^*_e \geq 0$, for all $e \in E_C$. Therefore, the problem of finding the odd cycle C containing the edge v_1v_2 such that the related geared inequality is violated by x^* reduces to find an even length path from v_1 to v_2 of minimum weight, in a graph $H \setminus v_1v_2$ with nonnegative costs on the edges. Thus it is polynomially solvable by using the algorithm proposed in [16]. ∎

3.2. Geared blossom inequalities

Consider a geared graph $G = (H, B, v_1v_2)$ such that H is the line graph of some root graph $R(H)$ and let w_1z and w_2z be the two adjacent edges of $R(H)$ corresponding to the endpoints v_1 and v_2 of the simplicial edge of H. Since any non clique inequality (π,π_0) valid for $STAB(H)$ corresponds to an Edmonds' inequality [8] for the matching polytope of $R(H)$, the associated geared inequality is called a *geared blossom* inequality. Proceeding as in the case of geared odd cycles inequalities, we can rewrite such an inequality in the form

$$\sum_{i \in V_T \setminus \{v_1,v_2\}} x_i + \gamma_B(x) \leq \alpha(T) + 2$$

where T is an induced subgraph of H such that: $V_T = E(S)$ and S is an odd subset of nodes of the root graph $R(H)$, $v_1, v_2 \in V_T$ and $\alpha(T) = \frac{|S|-1}{2}$.

Therefore, for a fixed gear B, the problem of finding a geared blossom inequality violated by a given fractional point x^* reduces to find an odd subset S of nodes of the root graph $R(H)$ such that $w_1z, w_2z \in E(S)$ and

$$\sum_{e \in E(S) \setminus \{w_1z, w_2z\}} y^*_e > \frac{|S|-1}{2} + 2 - \gamma_B(x^*),$$

where the y variables on the edges of $R(H)$ correspond to the x variables on the nodes of H. Such a problem, since x^* satisfies the edge formulation, can be solved in polynomial time using the standard algorithms defined for the blossom separation problem for the matching problem [23].

4. Conclusions

The gear composition has very interesting consequences on the stable set polytope of *claw-free graphs*, i.e., those graphs such that the neighborhood of each node has no stable set of size three. Claw-free graphs generalize line graphs and, as for line graphs, the problem of optimizing over their stable set polytope is polynomial time solvable [19]. By a well-known result of Grötschel, Lovász and Schrijver (see [15]), this implies that the separation problem for claw-free graphs can be solved in polynomial time. But up to now no explicit set of facet defining inequalities is known despite many research efforts [11, 14, 17, 21, 24] and many disproved conjectures [14].

The recent results of Chudnovsky and Seymour on the structure of quasi-line graphs [6] led to the settlement of a well-known conjecture on the linear description of $STAB(G)$ when G is *quasi-line* [9] (a graph is quasi-line if the neighborhood of each node can be partitioned into two cliques) and revived the interest for the facial structure of the stable set polytope of claw-free graphs.

Claw-free graphs with stability number at least four seem to be a good starting point to face the problem of finding a defining linear system for $STAB(G)$; in fact, a result of Fouquet [10] states that claw-free graphs with $\alpha(G) \geq 4$ do not contain odd antihole $\bar{C}_{2p+1}$ with $p \geq 3$, i.e., claw-free graphs with "large" stability number which are not quasi-line contain only 5-wheels. This result somehow supported the idea that all non-rank facet defining inequalities of $STAB(G)$ are produced by 5-wheels and their neighbors, the so called *lifted* 5*-wheel inequalities*. Recently, Stauffer [25] proposed the following:

Conjecture 4.1. *The stable set polytope of a claw-free but not quasi-line graph G with $\alpha(G) \geq 4$ is given by: non-negativity inequalities, rank inequalities and lifted* 5*-wheel inequalities.*

Now, using the gear composition introduced in Section 2, we can build graphs G having the following properties:

G claw-free and not quasi-line, $\alpha(G) \geq 4$,

G produces non-rank facet defining inequalities with rhs greater than two

Examples of such graphs are shown in Fig. (3)-(5). The geared inequalities produced by these graphs are all counterexamples for Conjecture 4.1. This implies that rank inequalities and lifted 5-wheels inequalities are not enough to describe $STAB(G)$ when G is claw-free, not quasi-line and with $\alpha(G) \geq 4$. Nevertheless the decomposition theorem for claw-free graphs of Chudnovsky and Seymour [5] strongly suggests us that the inequalities presented in this paper suffice to give a linear description for $STAB(G)$. Thus we can formulate the following:

Conjecture 4.2. *The stable set polytope of a claw-free but not quasi-line graph G with $\alpha(G) \geq 4$ is given by*

- *non-negativity inequalities*
- *rank inequalities*
- *lifted 5-wheel inequalities*
- *lifted geared inequalities.*

References

[1] F. Barahona and A.R. Mahjoub. Compositions of graphs and polyhedra I: balanced induced subgraphs and acyclic subgraphs. *SIAM J. on Discr. Math.*, 7:344–358, 1994.

[2] F. Barahona and A.R. Mahjoub. Compositions of graphs and polyhedra II: stable sets. *SIAM J. on Discr. Math.*, 7:359–371, 1994.

[3] E. Burlet and J. Fonlupt. Polyhedral consequence of the amalgam operation. *Discrete Mathematics*, 130:39–55, 1994.

[4] E. Cheng and W. H. Cunningham. Wheel inequalities for stable set polytopes. *Mathematical Programming*, 77:389–421, 1997.

[5] M. Chudnovsky and P. Seymour. Claw-free graphs IV: Decomposition theorem. *J. Comb. Th. B*. To appear.

[6] M. Chudnovsky and P. Seymour. Claw-free graphs VII: Quasi-line graphs, 2004. Manuscript.

[7] V. Chvátal. On certain polytopes associated with graphs. *J. Comb. Th. B*, 18:138–154, 1975.

[8] J. Edmonds. Maximum matching and a polyhedron with $0, 1$ vertices. *J. Res. of Nat. Bureau of Stand. B*, 69B:125–130, 1963.

[9] F. Eisenbrand, G. Oriolo, G. Stauffer, and P. Ventura. The stable set polytope of quasi-line graphs. *Combinatorica*. To appear.

[10] J.L. Fouquet. A strengthening of Ben Rebea's lemma. *J. Comb. Th. B*, 59:35–40, 1993.

[11] A. Galluccio and A. Sassano. The rank facets of the stable set polytope for claw-free graphs. *J. Comb. Th. B*, 69:1–38, 1997.

[12] A. M. H. Gerards and A. Schrijver. Matrices with the Edmonds-Johnson property. *Combinatorica*, 6:365–379, 1986.

[13] A.M.H. Gerards. A min-max relation for stable sets in graphs with no odd-K_4. *J. Comb. Th. B*, 47:330–348, 1989.

[14] R. Giles and L.E. Trotter. On stable set polyhedra for $K_{1,3}$-free graphs. *J. Comb. Th. B*, 31:313–326, 1981.

[15] M. Grötschel, L. Lovasz, and A. Schrijver. *Geometric algorithms and combinatorial optimization*. Springer Verlag, Berlin, 1988.

[16] A. S. LaPaugh and C.H. Papadimitriou. The even-path problem for graphs and digraphs. *Networks*, 14:507–513, 1984.

[17] T.M. Liebling, G. Oriolo, B. Spille, and G. Stauffer. On non-rank facets of the stable set polytope of claw-free graphs and circulant graphs. *Math. Methods of Oper. Research*, 2004.

[18] A.R. Mahjoub. On the stable set polytope of a series-parallel graph. *Mathematical Programming*, 40:53–57, 1988.

[19] G.J. Minty. On maximal independent sets of vertices in claw-free graphs. *J. Comb. Th. B*, 28:284–304, 1980.

[20] G. Nemhauser and L.E. Trotter Jr. Properties of vertex packing and independence system polyhedra. *Mathematical Programming*, 6:48–61, 1974.

[21] G. Oriolo. Clique family inequalities for the stable set polytope for quasi-line graphs. *Discrete Applied Mathematics*, 132:185–201, 2003.

[22] M. W. Padberg. On the facial structure of vertex packing polytope. *Mathematical Programming*, 5:199–215, 1973.

[23] M. W. Padberg and M. R. Rao. Odd minimum cut-sets and b-matchings. *Mathematics of Operations Research*, 7:67–80, 1982.

[24] W.R. Pulleyblank and B. Shepherd. Formulations of the stable set polytope. In G. Rinaldi and L. Wolsey, editors, *Proceedings Third IPCO Conference*, pages 267–279, 1993.

[25] G. Stauffer. *On the stable set polytope of claw-free graphs*. PhD thesis, EPF Lausanne, 2005.

[26] L.E. Trotter Jr. A class of facet producing graphs for vertex packing polyhedra. *Discrete Mathematics*, 12:373–388, 1975.

[27] L.A. Wolsey. Further facet generating procedures for vertex packing polytopes. *Mathematical Programming*, 11:158–163, 1976.