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MOD-2 CUTS GENERATION YIELDS THE CONVEX HULL OF BOUNDED INTEGER FEASIBLE SETS

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Abstract

This paper focuses on the outer description of the convex hull of all integer solutions to a given system of linear inequalities. It is shown that if the given system contains lower and upper bounds for the variables, then the convex hull can be produced by iteratively generating so-called mod-2 cuts only. This fact is surprising and might even be counterintuitive, since many integer rounding cuts exist that are not mod-2, i.e., representable as the zero- one-half combination of the given constraint system. The key, however, is that in general many more rounds of mod-2 cut generation are necessary to produce the final description compared to the traditional integer rounding procedure.

Key words: Integer Programming, Mod-2 cuts, Convex Hull.

1. Introduction

One of the fundamental results in the theory of linear integer programming states that the convex hull of all integer points in the intersection of finitely many rational halfspaces is a polyhedron. This polyhedron that we denote by $\mathcal{P}_I$ in the following can be described by linear inequalities that one obtains in finitely many steps by integer rounding [5]. A single step of the integer rounding procedure consists of taking all inequalities $a^T x \leq \beta$ with $a \in \mathbb{Z}^n$ that are valid for a relaxation $\mathcal{P} = \mathcal{P}^0$ of $\mathcal{P}_I$ and adding the constraint $a^T x \leq \lfloor \beta \rfloor$ to obtain the next relaxation $\mathcal{P}^1$ to which we refer as the first closure of $\mathcal{P}$.

It has been recently shown in [4] that optimizing over the first closure of a polyhedron is $\mathcal{NP}$-hard. This explains that one cannot expect to turn this nice concept of integer rounding into an effective and stand-alone algorithmic tool. The question emerges whether instead of considering the first closure of a polyhedron one can resort to a weaker relaxation that is algorithmically more tractable. One relaxation that appears particularly appealing for many combinatorial optimization problems is defined as the closure of a polyhedron associated with a special family of rounding cuts. These cuts have been introduced in [1] and are referred to as mod-2 cuts.

More precisely, if $\mathcal{P} = \{x \in \mathbb{R}^n | Ax \leq b\}$ with $A \in \mathbb{Z}^{m \times n}$, then a mod-2 cut is an inequality of the form $\frac{1}{2}u^T Ax \leq \lfloor \frac{1}{2}u^T b \rfloor$ where $u_i \in \{0,1\}$ for all $i = 1, \ldots, m$ and $\frac{1}{2}u^T A \in \mathbb{Z}^m$, i.e., $u^T A \equiv 0 \mod 2$.

Among the many important examples of mod-2 cuts we mention the blossom inequalities for the matching problem, the comb inequalities for the traveling salesman problem, the odd-cycle inequalities for the stable set problem or for the set covering problem, and the cycle inequalities for the max cut problem.

Although the problem of separating mod-2 cuts is $\mathcal{NP}$-hard in general, it can be solved in polynomial time if the constraint matrix meets certain properties (see [1]). Interestingly, [2] showed that there is a polynomial time algorithm for separating a subclass of mod-k cuts for any prime number k.

These results suggest that mod-2 cuts are an interesting object to study into further depth. Our paper contributes to this topic by showing that under mild assumptions a description of the integer polyhedron can be obtained by iteratively generating mod-2 cuts only.

In the remainder of this paper we will focus on bounded integer programming problems in inequality form. We will, in addition, assume that lower and upper bounds for the variables are available. More precisely, for $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$ and $v \in \mathbb{Z}^n$, the feasible set of integer points is described as

$$\{x \in \mathbb{Z}^n : Ax \leq b, \ -Ix \leq 0, \ Ix \leq v\}.$$

We define $\mathcal{P}_I = conv(\{x \in \mathbb{Z}^n : Ax \leq b, \ -Ix \leq 0, \ Ix \leq v\})$.

Definition 1.1. *Let $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$, $v \in \mathbb{Z}^n$,*

$$\tilde{A} = \begin{pmatrix} A \\ -I \\ I \end{pmatrix} \text{ and } \tilde{b} = \begin{pmatrix} b \\ 0 \\ v \end{pmatrix}.$$

We denote an initial system with

$$S = S^{(0)} = (\tilde{A}, \tilde{b}).$$

The first mod-2 closure of the system S is

$$S^{(1)} = \begin{pmatrix} \tilde{A}, & \tilde{b} & \\ \frac{1}{2}u^{\mathrm{T}}\tilde{A}, & \lfloor\frac{1}{2}u^{T}\tilde{b}\rfloor & \textit{for all } u \in \{0,1\}^{m+2n} \textit{ s.t. } u^{T}\tilde{A} \in \mathbb{Z}^{n} \end{pmatrix}.$$

For $t \in \mathbb{Z}_+$, $t \geq 2$, we define recursively $S^{(t)} = (S^{(t-1)})^{(1)}$ to be t-th mod-2 closure of S.

Given any system $S = (A, b)$, then $\mathcal{P}(S) = \{x \in \mathbb{R}^n : Ax \leq b\}$ is the corresponding polyhedron.

The main result of this paper is a proof of the fact that by generating mod-2 cuts iteratively we can produce the convex hull of the integer programming problem.

Theorem. There exists $t \in \mathbb{Z}_+$ such that $\mathcal{P}(S^{(t)}) = \mathcal{P}_I$.

Our proof requires to make use of properties of the mod-2 closure that we summarize in Section 2. Section 3 is devoted to the proof of the main theorem.

2. Properties of the mod-2 closure

This section develops structural properties of mod-2 closures of polyhedra. Starting with a system $S^{(0)}$ introduced in Definition 1.1 the iterative application of mod-2 cuts provides a second copy of the inequality system $Ax \leq b$. For this to be true it is essential that explicit upper bounds on the variables are part of the system $S^{(0)}$ as we show in a subsequent example.

Lemma 2.1. *Let $S^{(0)}$ be a system as introduced in Definition 1.1. There exists $t \in \mathbb{Z}_+$ such that $\begin{pmatrix} A, b \\ A, b \end{pmatrix}$ is part of the system $S^{(t)}$.*

Proof: Let $a^T x \leq \beta$ be an inequality of the system $Ax \leq b$. We want to prove that after a finite number of iterations, t say, $\begin{pmatrix} a^T x \leq \beta \\ a^T x \leq \beta \end{pmatrix}$ is part of the system $S^{(t)}$.

The system $S^{(1)}$ contains the inequality

$$\sum_{\substack{i=1 \\ a_i \text{ even}}}^{n} \frac{a_i}{2} x_i + \sum_{\substack{i=1 \\ a_i \text{ odd}}}^{n} \frac{a_i - 1}{2} x_i \leq \left\lfloor \frac{\beta}{2} \right\rfloor. \tag{1}$$

Then the system $S^{(2)}$ contains two copies of inequality (1). If we consider the original inequality $a^T x \leq \beta$, the two copies of inequality (1) and the upper bounds constraints $x_i \leq v_i$ for all i such that a_i is odd, and sum them up with multipliers $\frac{1}{2}$, we derive that an inequality of the form $a^T x \leq \beta + \delta$ where $\delta \in \mathbb{Z}_+$ is contained in $S^{(3)}$.

In subsequent rounds we generate mod-2 cuts with multipliers $\frac{1}{2}$ from:

$$\begin{aligned} a^T x &\leq \beta \\ a^T x &\leq \beta + \delta. \end{aligned}$$

This gives

$$a^T x \leq \beta + \left\lfloor \frac{1}{2}\delta \right\rfloor.$$

Setting $\delta := \lfloor \frac{1}{2}\delta \rfloor$, the argument applies iteratively and shows that after $\lceil \log_2(\delta) \rceil$ steps a second copy of $a^T x \leq \beta$ is included in some system $S^{(t)}$. □

Our next example illustrates that upper bounds on the variables are needed for Lemma 2.1 to be true.

Example 2.1. Consider the feasible set described as

$$\{(x_1, x_2) \in \mathbb{Z}_+^2 | -3x_1 + 5x_2 \leq 8\}, \tag{2}$$

where the inequality $-2x_1 + 3x_2 \leq 4$ can be derived from multiplying $-3x_1 + 5x_2 \leq 8$ by $3/5$, adding $1/5$ times the inequality $-x_1 \leq 0$ and rounding the right-hand-side. One may observe that, using only lower bounds and the initial inequality, it is not possible to derive a copy of $-3x_1 + 5x_2 \leq 8$. The reason is that both numbers -3 and 5 are odd. Therefore, all the inequalities belonging to any mod-2 closure attain a ratio of the two coefficients strictly less than $-3/5$. □

Resorting to Lemma 2.1 we are now ready to prove that every mod-k cut with k prime can be obtained by generating mod-2 cuts iteratively.

Lemma 2.2. *Let $S^{(0)}$ be a system as introduced in Definition 1.1. Let $a^T x \leq \beta$ be a mod-k cut for $\mathcal{P}(S)$, i.e., $a^T = \frac{1}{k} u^T \tilde{A} \in \mathbb{Z}^n$ and $\beta = \lfloor \frac{1}{k} u^T \tilde{b} \rfloor$ with $u \in \{0, 1, \ldots, k-1\}^{m+2n}$. There exists a number $t \in \mathbb{Z}_+$ such that $a^T x \leq \beta$ is part of the system $S^{(t)}$.*

Proof: W.l.o.g. we may assume that k is prime. By Lemma 2.1, there exists a number $t' \in \mathbb{Z}_+$ such that $u^T \tilde{A} x \leq u^T \tilde{b}$ is part of the system $S^{(t')}$. The inequality $u^T \tilde{A} x \leq u^T \tilde{b}$ can be represented as

$$k a^T x \leq k\beta + r \tag{3}$$

where $r \in \{0, 1, \ldots, k-1\}$.

Then $\lfloor \frac{1}{2} u^T \tilde{A} \rfloor x \leq \lfloor \frac{1}{2} u^T \tilde{b} \rfloor$ is part of $S^{(t+1)}$. The latter inequality dominates $\lfloor k/2 \rfloor a^T x \leq \lfloor \frac{1}{2} u^T \tilde{b} \rfloor$, since $\lfloor k/2 \rfloor a^T x \leq \lfloor k/2 a^T \rfloor = \lfloor \frac{1}{2} u^T \tilde{A} \rfloor$. Therefore, we assume in the following that $\lfloor k/2 \rfloor a^T x \leq \lfloor \frac{1}{2} u^T \tilde{b} \rfloor$ is contained in the system $S^{(t+1)}$.

Let $\pi \in \mathbb{Z}_+$ such that $k - 1 = 2^\alpha \pi$ and π odd. Then after α steps the inequality

$$\pi a^T x \leq \pi \beta + \delta \tag{4}$$

with $\delta = \lfloor \frac{\beta + r}{2^\alpha} \rfloor$ is contained in $S^{(t+1+\alpha)}$.

In the next iterations, by considering mod-2 cuts obtained from inequalities (3) and (4) with multipliers $\frac{1}{2}$, we will produce a mod-2 inequality of the form $\pi' a^T x \leq \pi' \beta + \delta'$, where $\pi' 2^{\alpha'} = (k + \pi)$ with $\alpha' \in \mathbb{Z}_+$, $\pi' \in \mathbb{Z}_+$, π' odd, and $\delta' = \lfloor \frac{(r+\delta)\pi'}{k+\pi} \rfloor$.

The crucial observation here is that if $\delta \geq \pi$, then $\frac{\delta'}{\pi'} + \frac{1}{2k} < \frac{\delta}{\pi}$. In fact, since $r \leq k-1$ and $k > \pi$, we derive this relation with the following simple computations:

$$\begin{aligned} \frac{\delta}{\pi} - \frac{\delta'}{\pi'} = \frac{\delta}{\pi} - \left\lfloor \frac{(r+\delta)\pi'}{k+\pi} \right\rfloor \frac{1}{\pi'} \geq \frac{\delta}{\pi} - \frac{r+\delta}{k+\pi} \geq \\ \geq \frac{\delta}{\pi} - \frac{k-1+\delta}{k+\pi} = \frac{\delta k - k\pi + \pi}{\pi(k+\pi)} \geq \\ \geq \frac{\pi k - k\pi + \pi}{\pi(k+\pi)} = \frac{1}{k+\pi} > \frac{1}{2k}. \end{aligned}$$

Therefore, after a finite number of iterations we will produce a system $S^{(t'')}$ that contains the inequalities

$$\begin{aligned} ka^T x &\leq k\beta + r \quad \text{and} \\ \pi a^T x &\leq \pi\beta + \delta \quad \text{with } \delta/\pi < 1. \end{aligned}$$

By Lemma 2.1 there exists a number $t^{(3)}$ such that $S^{(t^{(3)})}$ contains i copies of $ka^T x \leq k\beta + r$ and j copies of $\pi a^T x \leq \pi\beta + \delta$ where $ik + j\pi = 2^\gamma$, i.e., $ik + j\pi$ is a power of 2.

Then $S^{(t^{(3)}+\gamma)}$ contains the inequality

$$\frac{ik + j\pi}{2^\gamma} a^T x \leq \frac{ik + j\pi}{2^\gamma}\beta + \left\lfloor \frac{ir + j\delta}{2^\gamma} \right\rfloor .$$

Since $r \leq k - 1$ and $\delta \leq \pi - 1$, $\lfloor \frac{ir+j\delta}{2^\gamma} \rfloor = 0$. This completes the proof. □

Example 2.2. Consider the feasible set described as

$$\{(x_1, x_2) \in \mathbb{Z}_+^2 | 7x_1 + 14x_2 \leq 20\}.$$

The inequality $x_1 + 2x_2 \leq 2$ can be derived from multiplying $7x_1 + 14x_2 \leq 20$ by 1/7 and rounding the right-hand-side. Following Lemma 2.2, with one mod-2 operation, we obtain the first inequality of type (4)

$$3x_1 + 6x_2 \leq 10.$$

We then produce the next inequality by using the previous two

$$5x_1 + 10x_2 \leq 15.$$

Iterating the procedure we generate

$$3x_1 + 6x_2 \leq 8,$$

that is another inequality of type (4) where $\pi = 3$, $\beta = 2$, and $\delta = 2$, that is, $\delta/\pi < 1$. Finally, we consider one copy of $7x_1 + 14x_2 \leq 20$ and three copies of $3x_1 + 6x_2 \leq 8$, we divide by 16 (corresponding to 4 consecutive mod-2 operations), and obtain $x_1 + 2x_2 \leq 2$.

□

3. Proof of the main theorem

Theorem 3.1. *Let $S^{(0)}$ be a system as introduced in Definition 1.1. There exists $t \in \mathbb{Z}_+$ such that $\mathcal{P}(S^{(t)}) = \mathcal{P}_I$.*

Proof: It suffices to show that there exists $t' \in \mathbb{Z}_+$ such that the inequalities describing the first Chvátal-Gomory closure $\mathcal{P}^1$ are part of the system $S^{(t')}$. The polyhedron $\mathcal{P}^1$ is described by the Gomory cuts

$$\mathcal{P}^1 = \{x \in \mathbb{R}^n | u^T Ax \leq \lfloor u^T b \rfloor, \text{ for all } u \geq 0, u^T A \in \mathbb{Z}^n\}.$$

Every such inequality $u^T Ax \leq \lfloor u^T b \rfloor$ with $u = (p_1/q_1, \ldots, p_m/q_m)$ and $p_i \in \mathbb{Z}_+$, $q_i \in \mathbb{Z}_+ \setminus \{0\}$ is a mod-k cut with $k = \prod_{i=1}^m q_i$. In fact, there is a finite representation for $\mathcal{P}^1$ (see [6]) as $\mathcal{P}^1 = \{x \in \mathbb{R}^n |\ u^T Ax \leq \lfloor u^T b \rfloor$, for all $u \in H(\mathcal{C})\}$, where $H(\mathcal{C})$ is the Hilbert basis of the cone $\mathcal{C} = \{u^T A | u \in \mathbb{R}_+^m\}$.

By Lemma 2.2 every inequality $u^T Ax \leq \lfloor u^T b \rfloor$ with $u \in H(\mathcal{C})$ is contained in $S^{(t')}$ for some $t' \in \mathbb{Z}_+$. Therefore, there exists $t_1 \in \mathbb{Z}_+$ such that $S^{(t_1)}$ contains all the inequalities $u^T Ax \leq \lfloor u^T b \rfloor$ for all $u \in H(\mathcal{C})$, i.e., $\mathcal{P}(S^{(t_1)}) \subseteq \mathcal{P}^1$.

By a theorem of Chvátal [3], $\mathcal{P}_I = \mathcal{P}^\tau$ for some integer $\tau \in \mathbb{Z}_+$. Therefore, we can repeat the same argument for $\mathcal{P}^2, \ldots, \mathcal{P}^\tau$ by finding systems $S^{(t_2)}, \ldots, S^{(t_\tau)}$ such that $\mathcal{P}(S^{(t_i)}) \subseteq \mathcal{P}^i$ for all $i = 2, \ldots, \tau$. This gives the result. □

Our proof of Theorem 3.1 strongly relies on Lemma 2.1. As Example 2.1 illustrates, Lemma 2.1 is not true if upper bounds on the variables are not present. As a consequence, the proof of Theorem 3.1 does not apply to systems without upper bounds. It is, however, straightforward to extend the proof to the case we allow multipliers $\{0, \frac{1}{2}, 1\}$ for generating cuts as opposed to having $\{0, \frac{1}{2}\}$ multipliers only. We also remark that even though in Example 2.1 Lemma 2.1 is not applicable, one can still show that the inequality $-2x_1 + 3x_2 \leq 4$ is representable as a mod-2 cut in some system $S^{(t)}$. This fact might indicate that even an extension of Theorem 3.1 to the unbounded integer programming case could be true.

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References

[1] A. Caprara and M. Fischetti. $\{0, \frac{1}{2}\}$-Chvátal-Gomory cuts. *Mathematical Programming*, 74:221–235, 1996.

[2] A. Caprara, M. Fischetti, and A. N. Letchford. On the separation of maximally violated mod-k cuts. *Mathematical Programming*, 87(1):37–56, 2000.

[3] V. Chvátal. Edmonds polytopes and a hierarchy of combinatorial problems. *Discrete Mathematics*, 4:305–337, 1973.

[4] F. Eisenbrand. On the membership problem for the elementary closure of a polyhedron. *Combinatorica*, 19(2):297–300, 1999.

[5] R.E. Gomory. Outline of an algorithm for integer solutions to linear programs. *Bulletin of the American Mathematical Society*, 64:275–278, 1958.

[6] A. Schrijver. *Theory of Linear and Integer Programming.* Wiley, 1986.